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Applications of the theory of differential subordination for functions with fixed initial coefficient to univalent functions

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Abstract. By using the theory of first-order differential subordination for functions with fixed initial coefficient, several well-known results for subclasses of univalent functions are improved by restricting the functions to have fixed second coefficient. The influence of the second coefficient of univalent functions becomes evident in the results obtained.

1. Introduction and preliminaries. It is well-known that the second coefficient of univalent functions influences many properties. For example, a bound for the second coefficient of univalent functions yields growth and distortion estimates as well as the Koebe constant. Various subclasses of univalent functions with fixed second coefficients were investigated beginning with Gronwall [3]. For a brief survey of these developments as well as for some radius problem, see [1]. The necessary modifications to the theory of differential subordination to handle problems for functions with second coefficients are recently carried out in [2]. Using the results in [2], the influence of the second coefficient in certain differential implications associated with starlike and convex functions with fixed second coefficients is investigated in this paper.

Let p be an analytic function in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\psi(r, s)$ be a complex function defined in a domain of $\mathbb{C}^2$. Consider a class of functions Ψ, and two subsets Ω and Δ in $\mathbb{C}$. Given any two of these objects, the aim of the theory of first-order differential subordination is to determine the third so that the following differential implication is satisfied:

$$\psi \in \Psi \text{ and } \{\psi(p(z), zp'(z)) : z \in \mathbb{D}\} \subset \Omega \;\Rightarrow\; p(\mathbb{D}) \subset \Delta.$$

Furthermore, the problem is to find "smallest" such Δ and "largest" such Ω. In [2], the authors proposed a new methodology by making appropriate modifications and improvements to Miller and Mocanu's theory (see [4, 5]

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and their monograph [6]) of second-order differential subordination and gave interesting applications of the newly formulated theory to the classes of normalized convex and starlike functions with fixed second coefficient.

Let $\mathcal{H}_\beta[a,n]$ consist of all analytic functions p of the form

$$p(z) = a + \beta z^n + p_{n+1} z^{n+1} + \cdots,$$

where $\beta \in \mathbb{C}$ is fixed. Without loss of generality, we assume that β is a positive real number.

DEFINITION 1.1 ([5, Definition 1, p. 158]). Let Q be the class of functions q that are analytic and injective in $\overline{\mathbb{D}} \setminus E(q)$, where

$$E(q) := \{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(q)$.

DEFINITION 1.2 ([2, Definition 3.1, p. 616]). Let Ω be a domain in $\mathbb{C}$, $n \in \mathbb{N}$ and $\beta > 0$. Let $q \in Q$ be such that $|q'(0)| \geq \beta$. The class $\Psi_{n,\beta}(\Omega, q)$ consists of *β-admissible functions* $\psi : \mathbb{C}^2 \to \mathbb{C}$ satisfying the following conditions:

(i) $\psi(r,s)$ is continuous in a domain $D \subset \mathbb{C}^2$,
(ii) $(q(0),0) \in D$ and $\psi(q(0),0) \in \Omega$,
(iii) $\psi(q(\zeta), m\zeta q'(\zeta)) \notin \Omega$ whenever $(q(\zeta), m\zeta q'(\zeta)) \in D$, $\zeta \in \partial\mathbb{D} \setminus E(q)$ and

$$m \geq n + \frac{|q'(0)| - \beta}{|q'(0)| + \beta}.$$

We write $\Psi_{1,\beta}(\Omega, q)$ as $\Psi_\beta(\Omega, q)$.

THEOREM 1.3 ([2, Theorem 3.1, p. 617]). *Let $q \in Q$, $q(0) = a$, $\beta > 0$ with $|q'(0)| \geq \beta$, and $\psi \in \Psi_{n,\beta}(\Omega, q)$ with associated domain D. Let $p \in \mathcal{H}_\beta[a,n]$. If $(p(z), zp'(z)) \in D$ for $z \in \mathbb{D}$ and*

$$\psi(p(z), zp'(z)) \in \Omega \quad (z \in \mathbb{D}),$$

then $p \prec q$.

The special case of Δ being a half-plane is important in our investigation. Let $\Delta = \{w : \operatorname{Re} w > 0\}$. The function

$$q(z) = \frac{a + \overline{a}z}{1 - z} \quad (z \in \mathbb{D}),$$

where $\operatorname{Re} a > 0$, is univalent in $\overline{\mathbb{D}} \setminus \{1\}$ and satisfies $q(\mathbb{D}) = \Delta$, $q(0) = a$ and $q \in Q$. Let $\Psi_{n,\beta}(\Omega, a) := \Psi_{n,\beta}(\Omega, q)$ and for $\Omega = \Delta$ denote the class by $\Psi_{n,\beta}\{a\}$. Set $\Psi_\beta\{a\} := \Psi_{1,\beta}\{a\}$. The class $\Psi_{n,\beta}(\Omega, a)$ consists of those functions $\psi : \mathbb{C}^2 \to \mathbb{C}$ that are continuous in a domain $D \subset \mathbb{C}^2$ with $(a,0) \in D$ and $\psi(a,0) \in \Omega$, and that satisfy the admissibility condition:

(1.1) $\psi(i\rho,\sigma)\notin\Omega$ whenever $(i\rho,\sigma)\in D$ and

$$\sigma\le -\frac{1}{2}\left(n+\frac{2\operatorname{Re}a-\beta}{2\operatorname{Re}a+\beta}\right)\frac{|a-i\rho|^2}{\operatorname{Re}a},$$

where $\rho\in\mathbb{R}$ and $n\ge 1$.

If $a=1$, then (1.1) simplifies to

(1.2) $\psi(i\rho,\sigma)\notin\Omega$ whenever $(i\rho,\sigma)\in D$ and

$$\sigma\le -\frac{1}{2}\left(n+\frac{2-\beta}{2+\beta}\right)(1+\rho^2),$$

where $\rho\in\mathbb{R}$, and $n\ge 1$. In this particular case, Theorem 1.3 becomes

THEOREM 1.4 ([2, Theorem 3.4, p. 620]). *Let* $p\in\mathcal{H}_\beta[a,n]$ *with* $\operatorname{Re}a>0$ *and* $0<\beta\le 2\operatorname{Re}a$.

- (i) *Let* $\psi\in\Psi_{n,\beta}(\Omega,a)$ *with associated domain* D. *If* $(p(z),zp'(z))\in D$ *and* $\psi(p(z),zp'(z))\in\Omega$ $(z\in\mathbb{D})$, *then* $\operatorname{Re}p(z)>0$ $(z\in\mathbb{D})$.
- (ii) *Let* $\psi\in\Psi_{n,\beta}\{a\}$ *with associated domain* D. *If* $(p(z),zp'(z))\in D$ *and* $\operatorname{Re}\psi(p(z),zp'(z))>0$ $(z\in\mathbb{D})$, *then* $\operatorname{Re}p(z)>0$ $(z\in\mathbb{D})$.

2. Applications in univalent function theory. Let $\mathcal{A}_n$ be the class consisting of all analytic functions f defined on $\mathbb{D}$ of the form $f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\cdots$, and $\mathcal{A}:=\mathcal{A}_1$. The class $\mathcal{S}^*(\alpha)$ of *starlike functions of order* α, $0\le\alpha<1$, consists of all functions $f\in\mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>\alpha\quad(z\in\mathbb{D}).$$

Similarly, the class $\mathcal{C}(\alpha)$ of *convex functions of order* α, $0\le\alpha<1$, consists of all functions $f\in\mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\quad(z\in\mathbb{D}).$$

When $\alpha=0$, these classes are respectively denoted by $\mathcal{S}^*$ and $\mathcal{C}$.

Let $\mathcal{A}_{n,b}$ denote the class of all functions $f\in\mathcal{A}_n$ of the form

$$f(z)=z+bz^{n+1}+a_{n+2}z^{n+2}+\cdots,$$

where b is fixed. We write $\mathcal{A}_{1,b}$ as $\mathcal{A}_b$.

There are many differential inequalities in classical analysis for which the differential operator is required to have positive real part. A typical example is the Marx–Strohhäcker result, which states that if $f\in\mathcal{A}$, then

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)>0\ (z\in\mathbb{D})\ \Rightarrow\ \operatorname{Re}\frac{zf'(z)}{f(z)}>\frac{1}{2}\ (z\in\mathbb{D}).$$

A natural problem is to extend this result by finding a domain D containing the right half-plane so that

$$\frac{zf''(z)}{f'(z)}+1 \in D\ (z\in\mathbb{D}) \ \Rightarrow\ \operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2}\ (z\in\mathbb{D}).$$

The domain D cannot be taken as the half-plane $\{w\in\mathbb{C} : \operatorname{Re} w > \alpha\}$, with $\alpha < 0$, for functions $f \in \mathcal{A}$. (For a counterexample, see [9].) However, it is possible to take such a D for functions $f \in \mathcal{A}_b$. To prove this, we shall need the following lemma proved by Ozaki.

Lemma 2.1 ([8]). *If $f\in\mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > -\frac{1}{2} \qquad (z\in\mathbb{D}),$$

then f is univalent in $\mathbb{D}$.

Theorem 2.2. *If $f\in\mathcal{A}_b$ with $|b|\le 1$, then*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > \frac{|b|-1}{2(|b|+1)}\ (z\in\mathbb{D}) \ \Rightarrow\ \operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2}\ (z\in\mathbb{D}).$$

Proof. If we set

$$\alpha := \frac{|b|-1}{2(|b|+1)} \tag{2.1}$$

then $\alpha\in[-1/2,0]$. Define the function $p:\mathbb{D}\to\mathbb{C}$ by

$$p(z) := 2\frac{zf'(z)}{f(z)} - 1.$$

From Lemma 2.1 it follows that f is univalent and hence $p(z) = 1+2bz+\cdots$ is analytic in $\mathbb{D}$. Thus $p\in\mathcal{H}_{2b}[1,1]$ and

$$\frac{zf''(z)}{f'(z)}+1-\alpha = \frac{p(z)+1}{2}+\frac{zp'(z)}{p(z)+1}-\alpha = \psi(p(z),zp'(z)) \quad (z\in\mathbb{D}), \tag{2.2}$$

where

$$\psi(r,s) := \frac{r+1}{2}+\frac{s}{r+1}-\alpha,$$

and α is given by (2.1). The function ψ is continuous in the domain $D = (\mathbb{C}\backslash\{-1\})\times\mathbb{C}$, $(1,0)\in D$ and $\operatorname{Re}\psi(1,0) = 1-\alpha > 0$, as $\alpha\in[-1/2,0]$. We need to show that the admissibility condition (1.2) is satisfied. Since

$$\psi(i\rho,\sigma) = \frac{i\rho+1}{2}+\frac{\sigma}{1+\rho^2}(1-i\rho)-\alpha,$$

we have

$$\operatorname{Re}\psi(i\rho,\sigma) = \frac{1}{2}+\frac{\sigma}{1+\rho^2}-\alpha \le \frac{1}{2}-\frac{1}{2}\left(1+\frac{2-2|b|}{2+2|b|}\right)-\alpha = \frac{|b|-1}{2(|b|+1)}-\alpha = 0$$

whenever $\rho \in \mathbb{R}$ and

$$\sigma \leq -\frac{1}{2}\left(1+\frac{2-\beta}{2+\beta}\right)(1+\rho^2), \qquad \beta = 2|b|.$$

Thus $\psi \in \Psi_{2|b|}\{1\}$.

From the hypothesis and (2.2), we obtain

$$\operatorname{Re}\psi(p(z), zp'(z)) > 0 \qquad (z \in \mathbb{D}).$$

Therefore, by applying Theorem 1.4(ii), we conclude that $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$. This is equivalent to

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2} \qquad (z \in \mathbb{D}). \ \blacksquare$$

REMARK 2.3. For $|b| = 1$, Theorem 2.2 reduces to [6, Theorem 2.6a, p. 57]. Also, if $|b| = 0$ then $f \in \mathcal{A}_2$ and $\alpha = -1/2$. Therefore, Theorem 2.2 reduces to [6, Theorem 2.6i, p. 68] in this case.

THEOREM 2.4. *If $f \in \mathcal{A}_b$ with $|b| \leq 1$, then*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > \frac{|b|-1}{|b|+1} \ (z \in \mathbb{D}) \ \Rightarrow \ \operatorname{Re}\sqrt{f'(z)} > \frac{1}{2} \ (z \in \mathbb{D}),$$

where the branch of the square root is so chosen that $\sqrt{1} = 1$.

Proof. Set

$$\alpha := \frac{|b|-1}{|b|+1}. \tag{2.3}$$

Then $\alpha \in [-1, 0]$. Define the function $p : \mathbb{D} \to \mathbb{C}$ by $p(z) := 2\sqrt{f'(z)} - 1$. From the hypothesis it follows that the function $p(z) = 1 + 2bz + \cdots$ is analytic in $\mathbb{D}$. Thus $p \in \mathcal{H}_{2b}[1, 1]$ and

$$\frac{zf''(z)}{f'(z)} + 1 - \alpha = 1 + \frac{2zp'(z)}{p(z)+1} - \alpha = \psi(p(z), zp'(z)) \qquad (z \in \mathbb{D}) \tag{2.4}$$

where

$$\psi(r, s) := 1 + \frac{2s}{r+1} - \alpha,$$

and α is given by (2.3). The function ψ is continuous in the domain $D = (\mathbb{C}\backslash\{-1\}) \times \mathbb{C}$, $(1, 0) \in D$ and $\operatorname{Re}\psi(1, 0) = 1 - \alpha > 0$, as $\alpha \in [-1, 0]$.

We now show that the admissibility condition (1.2) is satisfied. Since

$$\psi(i\rho, \sigma) = 1 + \frac{2\sigma}{1+\rho^2}(1 - i\rho) - \alpha,$$

we have

$$\operatorname{Re}\psi(i\rho, \sigma) = 1 + \frac{2\sigma}{1+\rho^2} - \alpha \leq 1 - \left(1 + \frac{2-2|b|}{2+2|b|}\right) - \alpha = \frac{|b|-1}{|b|+1} - \alpha = 0,$$

whenever $\rho \in \mathbb{R}$ and

$$\sigma \leq -\frac{1}{2}\left(1+\frac{2-\beta}{2+\beta}\right)(1+\rho^2), \qquad \beta = 2|b|.$$

Thus $\psi \in \Psi_{2|b|}\{1\}$.

From the hypothesis and (2.4), we obtain

$$\operatorname{Re}\psi(p(z), zp'(z)) > 0 \quad (z \in \mathbb{D}).$$

Therefore, by applying Theorem 1.4 (ii), we conclude that $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$. This is equivalent to

$$\operatorname{Re}\sqrt{f'(z)} > 1/2 \quad (z \in \mathbb{D}). \blacksquare$$

Remark 2.5. If $|b| = 1$, then $\alpha = 0$ and Theorem 2.4 reduces to [6, Theorem 2.6a, p. 57].

Theorem 2.6. *If $f \in \mathcal{A}_b$ with $|b| \leq 1$, then*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{|b|}{|b|+1} \ (z \in \mathbb{D}) \ \Rightarrow \ \operatorname{Re}\frac{f(z)}{z} > \frac{1}{2} \ (z \in \mathbb{D}).$$

Proof. Setting

$$\alpha := \frac{|b|}{|b|+1}, \tag{2.5}$$

we see that $\alpha \in [0, 1/2]$. Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) := 2\frac{f(z)}{z} - 1.$$

Since $f \in \mathcal{A}_b$, the function $p(z) = 1 + 2bz + \cdots$ is analytic in $\mathbb{D}$. Thus $p \in \mathcal{H}_{2b}[1, 1]$ and

$$\frac{zf'(z)}{f(z)} - \alpha = 1 + \frac{zp'(z)}{p(z)+1} - \alpha = \psi(p(z), zp'(z)) \quad (z \in \mathbb{D}) \tag{2.6}$$

where

$$\psi(r, s) := 1 + \frac{s}{r+1} - \alpha,$$

and α is given by (2.5). The function ψ is continuous in the domain $D = (\mathbb{C}\backslash\{-1\}) \times \mathbb{C}$, $(1, 0) \in D$ and $\operatorname{Re}\psi(1, 0) = 1 - \alpha > 0$, as $\alpha \in [0, 1/2]$.

We now show that (1.2) is satisfied. Since

$$\psi(i\rho, \sigma) = 1 + \frac{\sigma}{1+\rho^2}(1 - i\rho) - \alpha,$$

we have

$$\operatorname{Re}\psi(i\rho, \sigma) = 1 + \frac{\sigma}{1+\rho^2} - \alpha \leq 1 - \frac{1}{2}\left(1 + \frac{2-2|b|}{2+2|b|}\right) - \alpha = \frac{|b|}{|b|+1} - \alpha = 0$$

whenever $\rho \in \mathbb{R}$ and

$$\sigma \le -\frac{1}{2}\left(1+\frac{2-\beta}{2+\beta}\right)(1+\rho^2), \qquad \beta = 2|b|.$$

Thus $\psi \in \Psi_{2|b|}\{1\}$.

From the hypothesis and (2.6), we obtain

$$\operatorname{Re}\psi(p(z), zp'(z)) > 0 \quad (z \in \mathbb{D}).$$

Therefore, by applying Theorem 1.4(ii), we conclude that $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$. This is equivalent to

$$\operatorname{Re}\frac{f(z)}{z} > \frac{1}{2} \quad (z \in \mathbb{D}). \ ■$$

REMARK 2.7. If $|b| = 1$ then $\alpha = 1/2$ and Theorem 2.6 reduces to [6, Theorem 2.6a, p. 57].

THEOREM 2.8. *If $f \in \mathcal{A}_b$ is locally univalent with $|b| \le 1$, then the following implication holds:*

$$\operatorname{Re}\sqrt{f'(z)} > \sqrt{\frac{1+|b|}{8}} \ (z \in \mathbb{D}) \ \Rightarrow \ \operatorname{Re}\frac{f(z)}{z} > \frac{1}{2} \ (z \in \mathbb{D}),$$

where the branch of the square root is so chosen that $\sqrt{1} = 1$.

Proof. To begin with, note that if we set

$$\alpha := \sqrt{\frac{1+|b|}{8}}, \tag{2.7}$$

then $\alpha \in [1/2\sqrt{2}, 1/2]$. Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) := \frac{2f(z)}{z} - 1.$$

Since $f \in \mathcal{A}_b$, the function $p(z) = 1 + 2bz + \cdots$ is analytic in $\mathbb{D}$. Thus $p \in \mathcal{H}_{2b}[1, 1]$ and

$$\sqrt{f'(z)} - \alpha = \sqrt{\frac{zp'(z) + p(z) + 1}{2}} - \alpha = \psi(p(z), zp'(z)) \quad (z \in \mathbb{D}), \tag{2.8}$$

where

$$\psi(r, s) := \sqrt{\frac{r+s+1}{2}} - \alpha,$$

and α is given by (2.7). The function ψ is continuous in the domain $D = \mathbb{C}^2$, $(1, 0) \in D$ and $\operatorname{Re}\psi(1, 0) = 1 - \alpha > 0$ as $\alpha \in [1/2\sqrt{2}, 1/2]$.

We now show that the following admissibility condition holds:

$$\operatorname{Re}\psi(i\rho, \sigma) = \operatorname{Re}\sqrt{\frac{i\rho + \sigma + 1}{2}} - \alpha \le 0 \tag{2.9}$$

whenever $\rho \in \mathbb{R}$ and

$$\sigma \le -\frac{1}{2}\left(1+\frac{2-\beta}{2+\beta}\right)(1+\rho^2), \qquad \beta = 2|b|.$$

If we let $\zeta = \xi + i\eta = (1+\sigma+i\rho)/2$, and use the conditions on ρ and σ, we obtain

$$\begin{aligned}\xi = \frac{1+\sigma}{2} &\le \frac{1}{2}\left[1-\frac{1}{2}\left(1+\frac{2-2|b|}{2+2|b|}\right)(1+\rho^2)\right]\\ &= \frac{1}{2}\left[1-\frac{1}{1+|b|}(1+\rho^2)\right] = \frac{1}{2(1+|b|)}(|b|-4\eta^2).\end{aligned}$$

This implies that ζ is a point inside the parabola

$$\eta^2 = -\frac{1+|b|}{2}\left[\xi - \frac{|b|}{2(1+|b|)}\right]$$

and

$$\operatorname{Re}\sqrt{\zeta} = \operatorname{Re}\sqrt{\xi+i\eta} = \sqrt{\frac{\xi+\sqrt{\xi^2+\eta^2}}{2}}.$$

Since

$$\xi^2+\eta^2 \le \frac{1}{4(1+|b|)^2}(|b|-4\eta^2)^2+\eta^2 = \frac{1}{4(1+|b|)^2}(1+4\eta^2)(|b|^2+4\eta^2),$$

using the arithmetic and geometric mean inequality we have

$$\sqrt{\xi^2+\eta^2} = \frac{1}{2(1+|b|)}\sqrt{(1+4\eta^2)(|b|^2+4\eta^2)} \le \frac{1}{4(1+|b|)}[1+8\eta^2+|b|^2]$$

so that

$$\xi+\sqrt{\xi^2+\eta^2} \le \frac{1}{2(1+|b|)}(|b|-4\eta^2)+\frac{1}{4(1+|b|)}[1+8\eta^2+|b|^2] = \frac{1+|b|}{4}.$$

Thus

$$\operatorname{Re}\sqrt{\zeta} - \sqrt{\frac{1+|b|}{8}} \le 0.$$

This is exactly the admissibility condition given in (2.9). Thus $\psi \in \Psi_{2|b|}\{1\}$.

From the hypothesis and (2.8), we obtain

$$\operatorname{Re}\psi(p(z), zp'(z)) > 0 \quad (z \in \mathbb{D}).$$

Therefore, by applying Theorem 1.4(ii) we conclude that $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$. This is equivalent to

$$\operatorname{Re}\frac{f(z)}{z} > \frac{1}{2} \quad (z \in \mathbb{D}). \ \blacksquare$$

REMARK 2.9. If $|b| = 1$, then $\alpha = 1/2$ and Theorem 2.8 reduces to [6, Theorem 2.6a, p. 57].

3. Two sufficient conditions for starlikeness. In 1989, Nunokawa [7] gave the following sufficient condition for starlikeness: if $f \in \mathcal{A}$, then

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) < \frac{3}{2}\ (z \in \mathbb{D}) \ \Rightarrow\ 0 < \operatorname{Re}\frac{zf'(z)}{f(z)} < \frac{4}{3}\ (z \in \mathbb{D}).$$

We will improve this result for a function $f \in \mathcal{A}_b$.

THEOREM 3.1. *If $f \in \mathcal{A}_b$ satisfies*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) < \frac{3}{2} \qquad (z \in \mathbb{D}),$$

then

$$\left|\frac{zf'(z)}{f(z)}-\alpha\right| < \alpha \qquad (z \in \mathbb{D}),$$

where α is given by

$$\alpha := \frac{3(|b|+6)+\sqrt{9|b|^2+28|b|+4}}{8(|b|+4)}. \tag{3.1}$$

In particular,

$$0 < \operatorname{Re}\frac{zf'(z)}{f(z)} < 2\alpha \qquad (z \in \mathbb{D}).$$

Proof. The hypothesis can be written in terms of subordination as

$$\frac{zf''(z)}{f'(z)}+1 \prec \frac{1-2z}{1-z} \qquad (z \in \mathbb{D}),$$

which gives $|b| \le 1/2$. Also, the constant α given by (3.1) satisfies the equation

$$4(|b|+4)\alpha^2 - 3(|b|+6)\alpha + 5 = 0. \tag{3.2}$$

If $\alpha > 2/3$, then we obtain

$$3\sqrt{9|b|^2+28|b|+4} > 7|b|+10.$$

On solving this, we get $|b| > 1/2$, which is a contradiction. Similarly, if we let $\alpha < 5/8$, then we obtain $|b| < 0$. Thus $\alpha \in [5/8, 2/3]$.

Define the function

$$w = q(z) := \frac{\alpha(1-z)}{(\alpha-1)z+\alpha} \qquad (z \in \mathbb{D}),$$

where α is given by (3.1). As $\alpha \in [5/8, 2/3]$, q is analytic and univalent in $\overline{\mathbb{D}}$. Thus, $q \in Q$. Since $q(-1) = 2\alpha$ and $q(1) = 0$, we see that

$$q(\mathbb{D}) = \{w : |w-\alpha| < \alpha\}.$$

Now, define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) := \frac{zf'(z)}{f(z)}.$$

Since $f \in \mathcal{A}_b$ and f is starlike (univalent), the function $p(z) = 1 + bz + \cdots$ is analytic in $\mathbb{D}$. Thus $p \in \mathcal{H}_b[1,1]$ and

$$\frac{zf''(z)}{f'(z)} + 1 = p(z) + \frac{zp'(z)}{p(z)} = \psi(p(z), zp'(z)) \qquad (z \in \mathbb{D}), \tag{3.3}$$

where

$$\psi(r,s) := r + \frac{s}{r}.$$

We claim that $\psi \in \Psi_b(\Omega, q)$ where $\Omega = \{w : \operatorname{Re} w < 3/2\}$. The function ψ is continuous in the domain $D = (\mathbb{C}\backslash\{0\}) \times \mathbb{C}$, $(1,0) \in D$ and $\operatorname{Re}\psi(1,0) = 1 < 3/2$ so that $\psi(1,0) \in \Omega$. We now show that

$$\operatorname{Re}\psi(q(\zeta), m\zeta q'(\zeta)) \geq 3/2,$$

where $|\zeta| = 1$ and

$$m \geq 1 + \frac{|q'(0)| - |b|}{|q'(0)| + |b|}, \qquad q'(0) = \frac{1-2\alpha}{\alpha}.$$

Since

$$\begin{aligned}\psi(q(\zeta), m\zeta q'(\zeta)) = q(\zeta) + m\frac{\zeta q'(\zeta)}{q(\zeta)} &= \frac{\alpha(1-\zeta)}{(\alpha-1)\zeta+\alpha} + \frac{m(1-2\alpha)\zeta}{(1-\zeta)[(\alpha-1)\zeta+\alpha]} \\ &= -1 + \frac{(m+2)\alpha - \zeta}{(\alpha-1)\zeta+\alpha} - \frac{m}{1-\zeta}, \qquad \zeta \neq 1,\end{aligned}$$

we have

$$\operatorname{Re}\psi(q(\zeta), m\zeta q'(\zeta)) = -1 + \operatorname{Re}\frac{(m+2)\alpha-\zeta}{(\alpha-1)\zeta+\alpha} - m\operatorname{Re}\frac{1}{1-\zeta}, \qquad \zeta \neq 1. \tag{3.4}$$

Moreover, since for $\alpha \in [5/8, 2/3]$, $m \geq 1$,

$$\operatorname{Re}\frac{(m+2)\alpha-\zeta}{(\alpha-1)\zeta+\alpha} \geq (m+2)\alpha + 1, \qquad |\zeta| = 1,$$

and

$$\operatorname{Re}\frac{1}{1-\zeta} = \frac{1}{2}, \qquad |\zeta| = 1,\, \zeta \neq 1,$$

we have

$$\begin{aligned}\operatorname{Re}\psi(q(\zeta), m\zeta q'(\zeta)) &\geq -1 + \frac{2(m+2)\alpha^2 - m\alpha - 1}{2\alpha - 1} - \frac{m}{2} \\ &= (m+2)\alpha - \frac{m}{2} = \frac{2\alpha-1}{2}m + 2\alpha \\ &\geq \frac{2\alpha-1}{2}\left(1 + \frac{(2\alpha-1) - |b|\alpha}{(2\alpha-1)+|b|\alpha}\right) + 2\alpha \\ &= \frac{2(|b|+4)\alpha^2 - 6\alpha + 1}{(2\alpha-1)+\alpha|b|} = \frac{3}{2},\end{aligned}$$

using (3.2). Thus, $\psi \in \Psi_{|b|}(\Omega, q)$ where $\Omega = \{w : \operatorname{Re} w < 3/2\}$.

From the hypothesis and (3.3), we obtain

$$\psi(p(z), zp'(z)) \in \Omega \quad (z \in \mathbb{D}).$$

Therefore, by applying Theorem 1.3, we have

$$p(z) \prec q(z) \quad (z \in \mathbb{D})$$

or equivalently

$$\left| \frac{zf'(z)}{f(z)} - \alpha \right| < \alpha \quad (z \in \mathbb{D}).$$

In particular, the above inequality yields

$$0 < \mathrm{Re}\, \frac{zf'(z)}{f(z)} < 2\alpha \quad (z \in \mathbb{D}). \blacksquare$$

REMARK 3.2. If $|b| = 1/2$ then α given by (3.1) simplifies to $2/3$. Thus Theorem 3.1 reduces to [7, Main Theorem] in this case.

Another familiar implication is the following [6, Theorem 2.6i, p. 68]:

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1 \right) > -\frac{1}{2} \ (z \in \mathbb{D}) \ \Rightarrow \ \mathrm{Re}\, \frac{zf'(z)}{f(z)} > \frac{1}{2} \ (z \in \mathbb{D})$$

for any function $f \in \mathcal{A}_2$. We generalize this result for $f \in \mathcal{A}_{2,b}$.

THEOREM 3.3. *If* $f \in \mathcal{A}_{2,b}$, *then*

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1 \right) > -\frac{1}{2} \ (z \in \mathbb{D}) \ \Rightarrow \ \mathrm{Re}\, \frac{zf'(z)}{f(z)} > \alpha \ (z \in \mathbb{D}),$$

where α *is the smallest positive root of the equation*

$$2\alpha^3 + 2(1 - |b|)\alpha^2 - (2|b| + 7)\alpha + 3 + |b| = 0 \tag{3.5}$$

in the interval $[1/2, 2/3]$.

Proof. First note that in terms of subordination the hypothesis can be written as

$$\frac{zf''(z)}{f'(z)} + 1 \prec \frac{1 + 2z}{1 - z} \quad (z \in \mathbb{D}),$$

which gives $|b| \le 1/2$. Also, the function g defined by

$$g(\alpha) := 2\alpha^3 + 2(1 - |b|)\alpha^2 - (2|b| + 7)\alpha + 3 + |b|$$

is continuous in $[1/2, 2/3]$ and satisfies

$$g\left(\frac{1}{2}\right) = \frac{1}{4}(1 - 2|b|) \ge 0, \quad g\left(\frac{2}{3}\right) = -\frac{1}{27}(5 + 33|b|) \le 0,$$

as $|b| \le 1/2$. Therefore by the Intermediate Value Theorem, there exists a root of $g(\alpha) = 0$ in $[1/2, 2/3]$. (In fact, $\alpha \in [0.5, \sqrt{2.5} - 1] \simeq [0.5, 0.58]$.)

Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) := \frac{zf'(z)}{f(z)} - \alpha, \tag{3.6}$$

where α is the smallest positive root of (3.5). Since $f \in \mathcal{A}_{2,b}$ and f is univalent, the function $p(z) = (1-\alpha) + 2bz^2 + \cdots$ is analytic in $\mathbb{D}$. Thus $p \in \mathcal{H}_{2b}[1-\alpha, 2]$ and as $\alpha \le 2/3$ we readily see that

$$\operatorname{Re} p(0) = 1 - \alpha > 0.$$

From (3.6), we obtain

$$\frac{zf'(z)}{f(z)} = p(z) + \alpha$$

so that

$$\frac{zf''(z)}{f'(z)} + 1 = p(z) + \alpha + \frac{zp'(z)}{p(z)+\alpha} = \psi(p(z), zp'(z)) \qquad (z \in \mathbb{D}),$$

where

$$\psi(r,s) := r + \alpha + \frac{s}{r+\alpha}.$$

We need to apply Theorem 1.4 to conclude that $\operatorname{Re} p(z) > 0$. If we let

$$\Omega = \{w : \operatorname{Re} w > -1/2\},$$

then by hypothesis, we have

$$\{\psi(p(z), zp'(z)) : z \in \mathbb{D}\} \subset \Omega.$$

To apply Theorem 1.4, we need to show that $\psi \in \Psi_{2,2|b|}(\Omega, 1-\alpha)$. The function ψ is continuous in the domain $D = (\mathbb{C}\backslash\{-\alpha\}) \times \mathbb{C}$, $(1-\alpha, 0) \in D$ and

$$\operatorname{Re} \psi(1-\alpha, 0) = 1 > 0.$$

We now show that the admissibility condition (1.1) is satisfied. Since

$$\psi(i\rho, \sigma) = i\rho + \alpha + \frac{\sigma}{\alpha^2+\rho^2}(\alpha - i\rho),$$

we have

$$\begin{aligned}\operatorname{Re} \psi(i\rho, \sigma) &= \alpha + \frac{\alpha\sigma}{\alpha^2+\rho^2} \\ &\le \alpha - \frac{1}{2}\,\frac{\alpha}{\alpha^2+\rho^2}\left(2 + \frac{2(1-\alpha)-2|b|}{2(1-\alpha)+2|b|}\right)\frac{(1-\alpha)^2+\rho^2}{1-\alpha} \\ &= \alpha - \frac{1}{2}\,\frac{\alpha}{1-\alpha}\,\frac{3(1-\alpha)+|b|}{(1-\alpha)+|b|}\,\frac{(1-\alpha)^2+\rho^2}{\alpha^2+\rho^2}.\end{aligned}$$

Using (3.5) and the monotonicity of the function

$$h(t) = \frac{(1-\alpha)^2+t}{\alpha^2+t}, \qquad t \ge 0,$$

we deduce that

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &\le \alpha - \frac{1}{2}\,\frac{1-\alpha}{\alpha}\,\frac{3(1-\alpha)+|b|}{(1-\alpha)+|b|}\\ &= \frac{(2|b|-1)\alpha^2 - 2\alpha^3 + (6+|b|)\alpha - 3 - |b|}{2\alpha[(1-\alpha)+|b|]} = -\frac{1}{2}\end{aligned}$$

whenever $\rho \in \mathbb{R}$ and

$$\sigma \le -\frac{1}{2}\left(2 + \frac{2\operatorname{Re} p(0) - \beta}{2\operatorname{Re} p(0) + \beta}\right)\frac{|p(0) - i\rho|^2}{\operatorname{Re} p(0)}, \qquad p(0) = 1-\alpha,\ \beta = 2|b|.$$

Thus $\psi \in \Psi_{2,2|b|}(\Omega, 1-\alpha)$. Therefore, by applying Theorem 1.4(i) we conclude that $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$. This is equivalent to

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha \qquad (z \in \mathbb{D}),$$

where α is the smallest positive root of (3.5). ∎

REMARK 3.4. If $|b| = 1/2$ then (3.5) becomes

$$2\alpha^3 + \alpha^2 - 8\alpha + 7/2 = 0,$$

which simplifies to

$$(2\alpha - 1)(\alpha^2 + \alpha - 7/2) = 0.$$

As $\alpha \in [1/2, 2/3]$, we get $\alpha = 1/2$. Thus Theorem 3.3 reduces to [6, Theorem 2.6i, p. 68] in this case.

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