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Existence of classical solutions for parabolic functional differential equations with initial boundary conditions of Robin type

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Abstract. The paper deals with the initial boundary value problem of Robin type for parabolic functional differential equations. The unknown function is the functional variable in the equation and the partial derivatives appear in the classical sense. A theorem on the existence of a classical solution is proved. Our formulation and results cover differential equations with deviated variables and differential integral problems.

1. Introduction. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions defined on X and taking values in Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let $S \subset \mathbb{R}^n$ be a bounded domain with boundary ∂S of class C^1. Write

$$Q_0 = [-b_0, 0] \times \bar{S}, \quad Q = (0,a] \times \bar{S}, \quad \Omega = Q_0 \cup \bar{Q}, \quad \partial_0 Q = Q_0 \cup ([0,a] \times \partial S)$$

where $a > 0, b_0 \in \mathbb{R}_+ = [0,+\infty)$ and $\bar{S}$ is the closure of S. For each $(t,x) \in [0,a] \times \bar{S}$ we define

$$D[t,x] = \{(\tau,y) \in \mathbb{R}^{n+1} : \tau \le 0,\ (t+\tau, x+y) \in Q_0 \cup Q\}.$$

There is $[c,d]^n \subset \mathbb{R}^n$ such that

$$D[t,x] \subset [-b_0 - a, 0] \times [c,d]^n \quad \text{for } (t,x) \in [0,a) \times \bar{S}.$$

Write $I = [-b_0 - a, 0]$ and $B = I \times [c,d]^n$. For a function $z : Q_0 \cup Q \to \mathbb{R}$ and a point $(t,x) \in [0,a) \times \bar{S}$ we define $z_{(t,x)} : D[t,x] \to \mathbb{R}$ by

$$z_{(t,x)}(\tau,y) = z(t+\tau, x+y), \quad (\tau,y) \in D[t,x].$$

That is, $z_{(t,x)}$ is the restriction of z to the set $(Q_0 \cup Q) \cap ([-b_0,t] \times \mathbb{R}^n)$ shifted to the set $D[t,x]$. Suppose that $\phi_0 : [0,a) \to \mathbb{R}$ and $\phi = (\phi_1,\dots,\phi_n) : Q \to \mathbb{R}^n$ are given functions. Write $\varphi(t,x) = (\phi_0(t), \phi(t,x))$ for $(t,x) \in Q$.

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We assume that $0 \le \phi_0(t) \le t$ and $\phi(t,x) \in S$ for $(t,x) \in Q$. Put $\varXi = Q \times \mathbb{R} \times C(B,\mathbb{R}) \times \mathbb{R}^n$ and suppose that $\psi : Q_0 \to \mathbb{R}$, $\beta, \gamma, \varPsi : [0,a] \times \partial S \to \mathbb{R}$ are given functions. Write

$$\varLambda[z](t,x) = \beta(t,x)z(t,x) + \gamma(t,x)\frac{\partial z(t,x)}{\partial n(x)},$$

where $n(x)$ is the unit outward normal on ∂S at $x \in \partial S$. We write ∂n instead of $\partial n(x)$. We assume that the functions $\beta : [0,a] \times \partial S \to (0,\infty)$ and $\gamma : [0,a] \times \partial S \to \mathbb{R}_+$ are continuous and let $\tilde{B} = \inf_{(x,t)\in[0,a]\times\partial S} \beta(t,x)$.

Suppose that $F : \varXi \to \mathbb{R}$, $a_{ij}, b_i : \bar{Q} \to \mathbb{R}$, $i,j = 1,\dots,n$, are given functions. We assume that F satisfies the *condition* (V) i.e. if for each $(t,x,p,w,q) \in \varXi$ and $\tilde{w} \in C(B,\mathbb{R})$ such that $w(\tau,y) = \tilde{w}(\tau,y)$ for $(\tau,y) \in D[t,x]$ we have $F(t,x,p,w,q) = F(t,x,p,\tilde{w},q)$. Note that the condition (V) means that the value of F at $(t,x,p,w,q) \in \varXi$ depends on (t,x,p,q) and on the restriction of w to the set $D[t,x]$ only. Write

$$L[z](t,x) = \partial_t z(t,x) - \sum_{i,j=1}^{n} a_{ij}(t,x)\partial_{x_i x_j} z(t,x) + \sum_{i=1}^{n} b_i(t,x)\partial_{x_i} z(t,x).$$

We assume that L is *strictly uniformly parabolic* in Q, i.e. there exists some positive constant k such that

$$k^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \le k|\xi|^2.$$

We consider the problem consisting of the functional differential equation

$$L[z](t,x) = F(t,x,z(t,x),z_{\varphi(t,x)},\partial_x z(t,x)) \tag{1.1}$$

and the initial boundary conditions

$$\varLambda[z](t,x) = \varPsi(t,x) \quad \text{on } [0,a] \times \partial S, \qquad z(t,x) = \psi(t,x) \quad \text{on } Q_0. \tag{1.2}$$

The aim of this paper is to give sufficient conditions for the existence of classical solutions to (1.1), (1.2).

Problems of the existence of solutions to parabolic functional differential equations have been considered by many authors under various assumptions. The paper [14] deals with weakly coupled parabolic systems with time delays. It is shown by using upper and lower solutions and by monotone iterative techniques that the corresponding sequences of approximate solutions converge monotonically to a unique solution of the original problem. The given functions in the nonlinear parts of the systems satisfy the Lipschitz condition with respect to the unknown functions and have a mixed quasimonotonicity property. The main difficulty in using monotone iterative methods is to construct lower and upper functions. There is not much literature on general methods for finding such functions. This method also requires assumptions on monotonicity of given functions with respect to functional variables. In our paper we do not need this restriction. Nor do we need assumptions on lower and upper functions. In contrast to many papers ([4], [6], [14], [15]), in our considerations it is important that F depends on $\partial_x z$.

Monotone iterative methods have been applied in [5], [3], [6] to studying the existence of solutions to parabolic functional differential problems. The results presented in those papers can be characterized as follows: theorems have simple assumptions and their proofs are very natural; unfortunately, the class of functional differential equations covered is rather small. The results given in [5], [3], [6] are not applicable to differential integral equations of Volterra type or to equations with deviated variables.

Initial boundary value problems for parabolic functional differential equations lead to integral functional equations. Classical solutions of the latter are considered to be generalized solutions of the original problems. The paper [16] gives sufficient conditions for the existence of generalized solutions of the Cauchy problem for semilinear parabolic systems with functionals.

Classical solutions of initial boundary value problem of the Dirichlet type have been considered in [17].

The present paper is a continuation of [5], [17] and generalizes some results presented in those papers.

We now give examples of equations which can be obtained from (1.1) by specializing F.

Example 1.1. Suppose that $f : Q \times \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}$ is a given function. Write

$$F(t,x,p,w,q) = f\Big(t,x,p,\int_{D[t,x]} w(\tau,y)\,dy\,d\tau, q\Big) \quad \text{on } \Xi.$$

Then (1.1) reduces to the differential integral equation

$$L[z](t,x) = f(t,x,z(t,x),\int_{D[t,x]} z(\phi_0(t)+\tau,\phi(t,x)+y)\,dy\,d\tau, \partial_x z(t,x)).$$

Example 1.2. For the above f we put

$$F(t,x,p,w,q) = f(t,x,p,w(0,0_{[n]}),q) \quad \text{on } \Xi, \tag{1.3}$$

where $0_{[n]} = (0,\dots,0) \in \mathbb{R}^n$. Then (1.1) reduces to the equation with deviated variables

$$L[z](t,x) = f(t,x,z(t,x),z(\varphi(t,x)),\partial_x z(t,x)). \tag{1.4}$$

We will write $\mathrm{CLS}(F,\Psi,\psi)$ for the set of classical solutions of (1.1), (1.2). A function $\varrho \in C(\mathbb{R}_+,\mathbb{R}_+)$ is called a *modulus* if ϱ is nondecreasing and $\varrho(0^+) = 0$. Let $C(B,\mathbb{R},r) = \{w \in C(B,\mathbb{R}) : \|w\|_B \le r\}$ where $\|\cdot\|_B$ is the supremum norm in the space $C(B,\mathbb{R})$. For any $A \subset \mathbb{R}^{1+n}$ we write $A_t = \{(s,x) \in A : -b_0 \le s \le t\}$.

Let the ordinary differential equation

$$\omega'(t) = \sigma(t, \omega(t)) \tag{1.5}$$

be defined in a region D and let $(t_0, \eta) \in D$. A solution ω of (1.5), passing through the point (t_0, η) and defined in some interval $\Delta^+ = [t_0, a_1)$, is called a *right-hand maximum* (resp. *minimum*) *solution* of (1.5) in Δ^+ passing through (t_0, η) if for every solution $\tilde{\omega}$ of (1.5) passing through (t_0, η) and defined in an interval $\tilde{\Delta}^+ = [t_0, a_2)$, we have

$$\tilde{\omega}(t) \le \omega(t) \quad (\text{resp. } \tilde{\omega}(t) \ge \omega(t)) \quad \text{for } t \in \Delta^+ \cap \tilde{\Delta}^+.$$

Let $\eta \ge 0$. We will write $\sigma \in O_\eta$ if $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing with respect to both variables and the right-hand maximum solution of the problem

$$\omega'(t) = \sigma(t, \omega(t)), \quad \omega(0) = \eta, \tag{1.6}$$

exists in $[0, a]$. We will denote this solution by $\omega(\cdot, \eta)$.

For $\sigma \in O_\eta$, we will denote by X_σ the set of all functions $F : \Xi \to \mathbb{R}$ such that

(i) for every $(t, x, u, w) \in Q \times \mathbb{R} \times C(B, \mathbb{R})$

$$F(t, x, u, w, 0_{[n]}) \operatorname{sgn}(u) \le \sigma(t, \max\{|u|, \|w\|_B\});$$

(ii) for every $r > 0$ there exists a modulus ϱ_r such that

$$|F(t, x, u, w, p) - F(t, x, u, w, 0_{[n]})| \le \varrho_r(\|p\|)$$

in $Q \times [-r, r] \times C(B, \mathbb{R}, r) \times \mathbb{R}^n$.

The remark below is a reformulation of Lemma 4.1 in [13].

Remark 1.3. Suppose that $F \in X_\sigma$, $z \in \mathrm{CLS}(F, \Psi, \psi)$ and $\|\psi\| \le \eta$, $\|\Psi\|_{[0,t] \times \partial S} \le \tilde{B}\omega(t, \eta)$. Then

$$\|z\|_{Q_t} \le \omega(t, \eta) \le \tilde{r} = \omega(a, \eta), \quad t \in [0, a].$$

Let $(Y, \|\cdot\|)$ be a normed space and $r \ge 0$. We define $I_r : Y \to Y$ by

$$I_r(x) = \begin{cases} x & \text{if } \|x\| \le r, \\ \frac{x}{\|x\|} r & \text{if } \|x\| > r. \end{cases} \tag{1.7}$$

We see at once that

$$\|I_r(x)\| = \min\{\|x\|, r\}, \qquad \|I_r(x) - I_r(y)\| \le 2\|x - y\| \quad \text{in } Y,$$

which follows from (1.7).

Let $I_r^* : \mathbb{R} \to \mathbb{R}$ and $I_r : C(B, \mathbb{R}) \to C(B, \mathbb{R})$ be defined by (1.7). For a function $F : \Xi \to \mathbb{R}$ we define $F_r : \Xi \to \mathbb{R}$ by

$$F_r(t, x, u, w, p) = F(t, x, I_r^*(u), I_r(w), p). \tag{1.8}$$

The remark below is a consequence of Remark 1.3 and definition (1.7).

REMARK 1.4. Suppose that $F \in X_\sigma$, $\|\Psi\|_{[0,t]\times\partial S} \le \tilde{B}\omega(t,\eta)$, $\|\psi\| \le \eta$. Then

(i) $F_{\tilde{r}} \in X_\sigma$;
(ii) $\mathrm{CLS}(F,\Psi,\psi) = \mathrm{CLS}(F_{\tilde{r}},\Psi,\psi)$.

Let $A \subset \mathbb{R}^{1+n}$ be a bounded domain and $0 < \alpha < 1$. We will denote by $C^{\alpha/2,\alpha}(A,\mathbb{R})$ the space of all continuous functions $f : A \to \mathbb{R}$ with the finite norm

$$\|f\|_{C^{\alpha/2,\alpha}(A,\mathbb{R})} = \|f\|_A + H_t^{\alpha/2}[f] + H_x^{\alpha}[f]$$

where

$$H_t^{\alpha/2}[f] = \sup\left\{\frac{|f(t,x) - f(\tilde{t},x)|}{|t-\tilde{t}|^{\frac{\alpha}{2}}} : (t,x),(\tilde{t},x) \in A,\ t \ne \tilde{t}\right\},$$

$$H_x^{\alpha}[f] = \sup\left\{\frac{|f(t,x) - f(t,\tilde{x})|}{\|x-\tilde{x}\|^{\alpha}} : (t,x),(t,\tilde{x}) \in A,\ x \ne \tilde{x}\right\},$$

and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. Moreover, set

$$H^{\alpha/2,\alpha}[f] = H_t^{\alpha/2}[f] + H_x^{\alpha}[f].$$

For A and α as above, let $C^{1+\alpha/2,2+\alpha}(A,\mathbb{R})$ denote the space of all continuous functions $f : A \to \mathbb{R}$ satisfying the conditions:

(i) the partial derivatives $\partial_x f = (\partial_{x_1} f, \ldots, \partial_{x_n} f)$, $\partial_{xx} f = [\partial_{x_i x_j} f]_{i,j=1}^n$, $\partial_t f$ exist on A and are continuous,
(ii) the following norm is finite:

$$\|f\|_{C^{1+\alpha/2,2+\alpha}(A,\mathbb{R})} = \|f\|_A + \|\partial_t f\|_A + \sum_{i=1}^n \|\partial_{x_i} f\|_A + \sum_{i,j=1}^n \|\partial_{x_i x_j} f\|_A + H^{\alpha/2,\alpha}[\partial_t f] + \sum_{i,j=1}^n H^{\alpha/2,\alpha}[\partial_{x_i x_j} f].$$

In a similar way we define the space $C^{(1+\alpha)/2,1+\alpha}(A,\mathbb{R})$, $0 < \alpha < 1$. Let $C^{1,2}(A,\mathbb{R})$ be the space of all continuous functions $f : A \to \mathbb{R}$ satisfying the conditions:

(i) $\partial_t f$, $\partial_x f$, $\partial_{xx} f$ exist and are continuous on A,
(ii) the following norm is finite:

$$\|f\|_{C^{1,2}(A,\mathbb{R})} = \|f\|_A + \|\partial_t f\|_A + \sum_{i=1}^n \|\partial_{x_i} f\|_A + \sum_{i,j=1}^n \|\partial_{x_i x_j} f\|_A.$$

Let $C^{0,1}_*(\Omega,\mathbb{R}) = C(\Omega,\mathbb{R}) \cap C^{0,1}(\bar{Q},\mathbb{R})$ and $\|f\|^{0,1}_* = \|f\|_\Omega + \|\partial_x f\|_{\bar{Q}}$. Write

$$C^{(1+\alpha)/2,1+\alpha}_{\mu,\tilde{\mu}}(\Omega,\mathbb{R}) = C^{\mu,\tilde{\mu}}(\Omega,\mathbb{R}) \cap C^{(1+\alpha)/2,1+\alpha}(\bar{Q},\mathbb{R}),$$
$$C^{1+\alpha/2,2+\alpha}_{\mu,\tilde{\mu}}(\Omega,\mathbb{R}) = C^{\mu,\tilde{\mu}}(\Omega,\mathbb{R}) \cap C^{1+\alpha/2,2+\alpha}(\bar{Q},\mathbb{R})$$

with norms

$$\|f\|^{(1+\alpha)/2,1+\alpha}_{\mu,\tilde{\mu}} = \max\{\|f\|_{C^{\mu,\tilde{\mu}}(\Omega,\mathbb{R})}, \|f\|_{C^{(1+\alpha)/2,1+\alpha}(\bar{Q},\mathbb{R})}\},$$
$$\|f\|^{1+\alpha/2,2+\alpha}_{\mu,\tilde{\mu}} = \max\{\|f\|_{C^{\mu,\tilde{\mu}}(\Omega,\mathbb{R})}, \|f\|_{C^{1+\alpha/2,2+\alpha}(\bar{Q},\mathbb{R})}\}.$$

Let $L^q(A,\mathbb{R})$, $q \geq 1$, be the Banach space consisting of all equivalence classes of Lebesgue measurable functions f defined on A into $\mathbb{R}$ with the finite norm

$$\|f\|_{L^q(A,\mathbb{R})} = \Big(\int_A |f(\tau,y)|^q \, dy \, d\tau\Big)^{1/q}.$$

We denote by $W^{1,2}_q(A,\mathbb{R})$ the Banach space consisting of all $f \in L^q(A,\mathbb{R})$ having generalized derivatives $\partial_t f$, $\partial_x f$, $\partial_{xx} f = [\partial_{x_i x_j} f]^n_{i,j=1}$ such that the following norm is finite:

$$\|f\|_{W^{1,2}_q(A,\mathbb{R})} = \|f\|_{L^q(A,\mathbb{R})} + \|\partial_t f\|_{L^q(A,\mathbb{R})} + \sum_{i=1}^n \|\partial_{x_i} f\|_{L^q(A,\mathbb{R})} + \sum_{i,j=1}^n \|\partial_{x_i x_j} f\|_{L^q(A,\mathbb{R})}.$$

For nonintegral α, the Banach space $W^{\alpha/2,\alpha}_q(A,\mathbb{R})$ is defined analogously (see [11]).

Let $S \subset \mathbb{R}^n$ be a bounded domain. We will say that ∂S is of class $C^{2+\alpha}$, $0 < \alpha < 1$, if for every $x \in \partial S$ there exist a neighborhood U_x of x and $i \in \{1,\ldots,n\}$ such that $\partial S \cap U_x$ can be represented in the form

$$x_i = h(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n), \quad (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \in \tilde{U}_x$$

where $\tilde{U}_x \subset \mathbb{R}^{n-1}$ is an open set and $h \in C^{2+\alpha}(\tilde{U}_x,\mathbb{R})$.

2. Existence of solution for mixed problems. We will say that problem (1.1), (1.2) *satisfies the compatibility conditions* if

$$L[\Psi](0,x) = F(0,x,\Psi(0,x),\Psi_{\varphi(0,x)},\partial_x\Psi(0,x)) \quad \text{for } x \in \partial S, \tag{2.1}$$

$$\beta(0,x)u(0,x) + \gamma(0,x)\frac{\partial u(0,x)}{\partial n} = \Psi(0,x) \quad \text{for } x \in \partial S. \tag{2.2}$$

We give sufficient conditions for the existence of a solution to (1.1), (1.2).

Assumption $\mathbf{H}[F,\Psi,\psi]$. Suppose that $1/2 \leq \mu < (1+\alpha)/2$, $\alpha < \tilde{\mu} \leq 1$ are given constants and:

1) there exists $\sigma \in O_\eta$ such that $F \in X_\sigma$;
2) there exists a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that
$$|F(t,x,u,w,p)| \le \rho(\max\{|u|, \|w\|_B\})(1+\|p\|^2) \quad \text{on } \varXi;$$
3) for every $r,q,L \ge 0$ there exists a constant $C(r,q,L) \ge 0$ such that
$$|F(t,x,u,w,p) - F(t,x,\bar{u},\bar{w},\bar{p})| \le C(r,q,L)[|u-\bar{u}|^\alpha + \|w-\bar{w}\|_B^\alpha + \|p-\bar{p}\|]$$
on $Q \times [-r,r] \times C(B,\mathbb{R},q) \times \{p \in \mathbb{R}^n : \|p\| \le L\}$;
4) for every $r,q,L \ge 0$ there exists a constant $H(r,q,L) \ge 0$ such that
$$|F(t,x,u,w,p) - F(\bar{t},\bar{x},u,w,p)| \le H(r,q,L)[|t-\bar{t}|^{\alpha/2} + |x-\bar{x}|^\alpha] \tag{2.3}$$
on $Q \times [-r,r] \times C(B,\mathbb{R},q) \times \{p \in \mathbb{R}^n : \|p\| \le L\}$;
5) $\varPsi \in C^{(1+\alpha)/2,1+\alpha}([0,a]\times\partial S,\mathbb{R})$, $\psi \in C^{\mu,\tilde{\mu}}(Q_0,\mathbb{R})$, $\psi(0,\cdot) \in C^{2+\alpha}(S,\mathbb{R})$ and $\|\varPsi\|_{[0,t]\times\partial S} \le \tilde{B}\omega(t,\eta)$, $\|\psi\| \le \eta$;
6) there exists $\bar{\varPsi} \in C^{1+\alpha/2,2+\alpha}_{\mu,\tilde{\mu}}(\varOmega,\mathbb{R})$ such that $\varLambda[\bar{\varPsi}]_{|[0,a)\times\partial S} = \varPsi$ and $\bar{\varPsi}_{|Q_0} = \psi$.

Assumption $\mathbf{H}[\varphi]$. Suppose that:

1) $\phi_0 \in C([0,a],\mathbb{R}_+)$, $\phi \in C(\bar{Q},\mathbb{R}^n)$ and $\phi_0(t) \le t$ for $t \in [0,a]$, $\phi(t,x) \in \bar{S}$ for $(t,x) \in \bar{Q}$;
2) there is $C_0 \ge 0$ such that
$$|\phi_0(t) - \phi_0(\bar{t})| \le C_0|t-\bar{t}|, \qquad t,\bar{t} \in [0,a],$$
$$\|\phi(t,x) - \phi(\bar{t},\bar{x})\| \le C_0[|t-\bar{t}| + \|x-\bar{x}\|], \qquad (t,x),(\bar{t},\bar{x}) \in \bar{Q}.$$

Define
$$F_{\bar{\varPsi}}(t,x,u,w,p) = F(t,x,u+\bar{\varPsi}(t,x), w+\bar{\varPsi}_{\varphi(t,x)}, p+\partial_x\bar{\varPsi}(t,x)) - L[\bar{\varPsi}](t,x),$$
where $\bar{\varPsi}$ is given by condition 6) of Assumption $\mathbf{H}[F,\varPsi,\psi]$.

Remark 2.1. Suppose that Assumption $\mathbf{H}[F,\varPsi,\psi]$ is satisfied and $a_{ij}, b_i \in C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$, $1 \le i,j \le n$. Then $z \in \mathrm{CLS}(F,\varPsi,\psi)$ if and only if $z - \bar{\varPsi} \in \mathrm{CLS}(F_{\bar{\varPsi}},0,0)$.

We define the Nemytskiĭ operator for problem (1.1), (1.2). Put
$$\mathbb{F}[z](t,x) = F(t,x,z(t,x),z_{\varphi(t,x)},\partial_x z(t,x)), \quad (t,x) \in Q,$$
where $z \in C^{0,1}_*(\varOmega,\mathbb{R})$.

We will need the following lemmas.

Lemma 2.2. *Suppose that Assumptions $\mathbf{H}[F,\varPsi,\psi]$, $\mathbf{H}[\varphi]$ are satisfied and let $\mathbb{F}$ be the Nemytskiĭ operator for (1.1), (1.2). Then*

(i) $\mathbb{F} : C^{0,1}_*(\varOmega,\mathbb{R}) \to C(\bar{Q},\mathbb{R})$ *is continuous and bounded;*
(ii) $\mathbb{F}(C^{(1+\alpha)/2,1+\alpha}_{\mu,\tilde{\mu}}(\varOmega,\mathbb{R})) \subseteq C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$.

Proof. (i) Let $\|u-\bar{u}\|_*^{0,1} \le 1$. Then

$$\begin{aligned}
&|\mathbb{F}[u](t,x) - \mathbb{F}[\bar{u}](t,x)| \\
&= |F(t,x,u(t,x),u_{\varphi(t,x)},\partial_x u(t,x)) - F(t,x,\bar{u}(t,x),\bar{u}_{\varphi(t,x)},\partial_x \bar{u}(t,x))| \\
&\le C(r,q,L)[|u(t,x)-\bar{u}(t,x)|^\alpha + \|u_{\varphi(t,x)} - \bar{u}_{\varphi(t,x)}\|_B^\alpha + \|\partial_x u(t,x) - \partial_x \bar{u}(t,x)\|],
\end{aligned}$$

which shows that $\mathbb{F}$ is continuous. Let $\|u\|_*^{0,1} \le r$. Since

$$\begin{aligned}
|\mathbb{F}[u](t,x)| &= |F(t,x,u(t,x),u_{\varphi(t,x)},\partial_x u(t,x))| \\
&\le |F(t,x,u(t,x),u_{\varphi(t,x)},\partial_x u(t,x)) - F(t,x,0,0,0_{[n]})| + |F(t,x,0,0,0_{[n]})| \\
&\le C(r,L)[\|u\|_\Omega^\alpha + \|u_{\varphi(t,x)}\|_B^\alpha + \|\partial_x u\|_Q] + \|F(\cdot,\cdot,0,0,0_{[n]})\|_Q,
\end{aligned}$$

$\mathbb{F}$ is bounded.

(ii) Let $u \in C^{(1+\alpha)/2,1+\alpha}_{\mu,\tilde{\mu}}(\Omega,\mathbb{R})$. We claim that $\mathbb{F}[u]$ is Hölder continuous of exponent α with respect to x and is Hölder continuous of exponent $\alpha/2$ with respect to t. Indeed, put $r = \|u\|^{(1+\alpha)/2,1+\alpha}_{\mu,\tilde{\mu}}$. Note that the functions $u_{\varphi(t,x)}$ and $u_{\varphi(t,\bar{x})}$, where $(t,x),(t,\bar{x}) \in Q$, have different domains. Therefore we need the following construction. Write $Y = [-b_0,a] \times [\tilde{c},\tilde{d}]$ where $\tilde{c} = (\tilde{c}_1,\dots,\tilde{c}_n)$, $\tilde{d} = (\tilde{d}_1,\dots,\tilde{d}_n)$, $\tilde{c}_i = c_i - |d_i - c_i|$, $\tilde{d}_i = d_i + |d_i - c_i|$ for $i = 1,\dots,n$. There is $\tilde{u} : Y \to \mathbb{R}$ such that $\tilde{u} \in C^{(1+\alpha)/2,1+\alpha}_{\mu,\tilde{\mu}}(Y,\mathbb{R})$ and $\tilde{u}(t,x) = u(t,x)$ for $(t,x) \in Q_0 \cup \bar{Q}$. Then the function $\tilde{u}_{\varphi(t,x)}$ is defined on B for $(t,x) \in \bar{Q}$. It follows from Assumptions $\mathbf{H}[F,\Psi,\psi]$ and $\mathbf{H}[\varphi]$ that

$$\begin{aligned}
&|\mathbb{F}[u](t,x) - \mathbb{F}[u](t,\bar{x})| \\
&= |F(t,x,u(t,x),u_{\varphi(t,x)},\partial_x u(t,x)) - F(t,\bar{x},u(t,\bar{x}),u_{\varphi(t,\bar{x})},\partial_x u(t,\bar{x}))| \\
&\le |F(t,x,u(t,x),\tilde{u}_{\varphi(t,x)},\partial_x u(t,x)) - F(t,x,u(t,\bar{x}),\tilde{u}_{\varphi(t,\bar{x})},\partial_x u(t,\bar{x}))| \\
&\quad + |F(t,x,u(t,\bar{x}),\tilde{u}_{\varphi(t,\bar{x})},\partial_x u(t,\bar{x})) - F(t,\bar{x},u(t,\bar{x}),\tilde{u}_{\varphi(t,\bar{x})},\partial_x u(t,\bar{x}))| \\
&\le C(r,L)[|u(t,x)-u(t,\bar{x})|^\alpha + \|\tilde{u}_{\varphi(t,x)} - \tilde{u}_{\varphi(t,\bar{x})}\|_B^\alpha + \|\partial_x u(t,x) - \partial_x u(t,\bar{x})\|] \\
&\quad + H(r,L)\|x-\bar{x}\|^\alpha \\
&\le C(r,L)[\|\partial_x u\|_Q^\alpha \|x-\bar{x}\|^\alpha + [H_x^{\tilde{\mu}}[\tilde{u}]]^\alpha \|\phi(t,x) - \phi(t,\bar{x})\|^{\tilde{\mu}\alpha} + H_x^\alpha[\partial_x u]\|x-\bar{x}\|^\alpha] \\
&\quad + H(r,L)\|x-\bar{x}\|^\alpha \\
&\le C(r,L)[\|\partial_x u\|_Q^\alpha \|x-\bar{x}\|^\alpha + 2[H_x^{\tilde{\mu}}[\tilde{u}]]^\alpha C_0^{\tilde{\mu}\alpha} \|x-\bar{x}\|^{\tilde{\mu}\alpha} + H_x^\alpha[\partial_x u]\|x-\bar{x}\|^\alpha] \\
&\quad + H(r,L)\|x-\bar{x}\|^\alpha,
\end{aligned}$$

where $H_x^\alpha[\partial_x u] = \sum_{i=1}^n H_x^\alpha[\partial_{x_i} u]$. Hence $\mathbb{F}[u]$ is Hölder continuous of exponent α with respect to x. We proceed to show that $\mathbb{F}[u]$ is Hölder continuous of exponent $\alpha/2$ with respect to t:

$$
\begin{aligned}
&|\mathbb{F}[u](t,x)-\mathbb{F}[u](\bar t,x)|\\
&=|F(t,x,u(t,x),u_{\varphi(t,x)},\partial_x u(t,x))-F(\bar t,x,u(\bar t,x),u_{\varphi(\bar t,x)},\partial_x u(\bar t,x))|\\
&\le |F(t,x,u(t,x),\tilde u_{\varphi(t,x)},\partial_x u(t,x))-F(t,x,u(\bar t,x),\tilde u_{\varphi(\bar t,x)},\partial_x u(\bar t,x))|\\
&\quad+|F(t,x,u(\bar t,x),\tilde u_{\varphi(\bar t,x)},\partial_x u(\bar t,x))-F(\bar t,x,u(\bar t,x),\tilde u_{\varphi(\bar t,x)},\partial_x u(\bar t,x))|\\
&\le C(r,L)[|u(t,x)-u(\bar t,x)|^\alpha+\|\tilde u_{\varphi(t,x)}-\tilde u_{\varphi(\bar t,x)}\|_B^\alpha+\|\partial_x u(t,x)-\partial_x u(\bar t,x)\|]\\
&\quad+H(r,L)|t-\bar t|^{\alpha/2}\\
&\le C(r,L)[[H_t^{(1+\alpha)/2}[u]]^\alpha|t-\bar t|^{(1+\alpha)/2\alpha}+[H_t^\mu[\tilde u]C_0^\mu|t-\bar t|^\mu+H_x^{\tilde\mu}[\tilde u]C_0^{\tilde\mu}|t-\bar t|]^{\tilde\mu}\\
&\quad+H_t^{(1+\alpha)/2}[\partial_x u]|t-\bar t|^{(1+\alpha)/2}]+H(r,L)|t-\bar t|^{\alpha/2}.
\end{aligned}
$$

This finishes the proof. ∎

LEMMA 2.3. *Suppose that Assumptions* $\mathbf{H}[F,\Psi,\psi]$, $\mathbf{H}[\varphi]$ *are satisfied and* $z\in C_*^{1,2}(\Omega,\mathbb{R})$ *is a solution of* (1.1), (1.2). *Then there exists a constant* $\tilde L$ *such that*

$$\|\partial_x z\|_{\bar Q}\le \tilde L. \tag{2.4}$$

Proof. Without loss of generality we can assume that $\Psi(t,x)=0$ for $(t,x)\in[0,a]\times\partial S$ and $\psi(t,x)=0$ for $(t,x)\in Q_0$. Put $\tilde F: Q\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $\tilde F(t,x,u,p)=F(t,x,u,z_{\varphi(t,x)},p)$. Consider the problem

$$L[u](t,x)=\tilde F(t,x,u(t,x),\partial_x u(t,x)),\qquad (t,x)\in Q, \tag{2.5}$$

$$\Lambda[u](t,x)=0,\quad (t,x)\in[0,a]\times\partial S,\qquad u(t,x)=0,\quad (t,x)\in Q_0. \tag{2.6}$$

Then z is a solution of (2.5), (2.6). It follows from Assumption $\mathbf{H}[F,\Psi,\psi]$ that $\tilde F$ satisfies the hypotheses of [2, Theorem 2.2]. Hence (2.4) is proved. ∎

We can now formulate our main results.

THEOREM 2.4. *Suppose that Assumptions* $\mathbf{H}[F,\Psi,\psi]$, $\mathbf{H}[\varphi]$ *are satisfied and:*

1) $a_{ij},b_i\in C^{\alpha/2,\alpha}(\bar Q,\mathbb{R})$,
2) $S\subset\mathbb{R}^n$ *is a bounded domain and* ∂S *is of class* $C^{2+\alpha}$,
3) *problem* (1.1), (1.2) *satisfies the compatibility conditions* (2.1), (2.2).

Then problem (1.1), (1.2) *has a solution* $z\in C^{1+\alpha/2,2+\alpha}(\Omega,\mathbb{R})$.

Proof. In view of Remark 2.1 we may assume that $\Psi(t,x)=0$ for $(t,x)\in[0,a]\times\partial S$ and $\psi(t,x)=0$ for $(t,x)\in Q_0$. Hence condition (2.1) takes the form

$$F(0,x,0,0,0_{[n]})=0\quad\text{for } x\in\partial S. \tag{2.7}$$

Set

$$C_0^{\alpha/2,\alpha}(\bar{Q},\mathbb{R}) = \{g \in C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R}) : g(0,x) = 0 \text{ for } x \in \partial S\},$$
$$C^{l/2,l}(\bar{Q},\mathbb{R},0) = \{g \in C^{l/2,l}(\bar{Q},\mathbb{R}) : g(0,x) = 0,\ x \in S \wedge \Lambda[g]_{|[0,a]\times\partial S} = 0\},$$
$$C^{l/2,l}(\Omega,\mathbb{R},0) = \{g \in C^{l/2,l}(\bar{Q},\mathbb{R}) : g(t,x) = 0 \text{ on } Q_0 \wedge \Lambda[g]_{|[0,a]\times\partial S} = 0\},$$

where $l = 1+\alpha,\ 2+\alpha$. Let us define an operator $V : C_0^{\alpha/2,\alpha}(\bar{Q},\mathbb{R}) \to C^{1+\alpha/2,2+\alpha}(\bar{Q},\mathbb{R},0)$. For $g \in C_0^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$ we denote by Vg a solution of the problem

$$(2.8)\qquad L[z](t,x) = g(t,x), \quad (t,x) \in Q,$$
$$(2.9)\qquad \Lambda[z](t,x) = 0, \quad (t,x) \in [0,a]\times\partial S, \quad z(t,x) = 0, \quad (t,x) \in Q_0.$$

It follows from [10, Ch. IV, Th. 5.3] that there is exactly one solution Vg of problem (2.8), (2.9) and $Vg \in C^{1+\alpha/2,2+\alpha}(\bar{Q},\mathbb{R},0)$. Moreover

$$\|Vg\|_{C^{1+\alpha/2,2+\alpha}(\bar{Q},\mathbb{R})} \le c\|g\|_{C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})}$$

for some $c \ge 0$, which implies that V is continuous. Now we will construct a bounded linear extension of V onto the space $L^q(\bar{Q},\mathbb{R})$ for some $q > 1$. Since $C_0^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$ is dense in $L^q(\bar{Q},\mathbb{R})$ for $g \in L^q(\bar{Q},\mathbb{R})$ there exists a sequence $\{g_i\}_{i=0}^{\infty} \subset C_0^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$ such that $\|g_i - g\|_{L^q(\bar{Q},\mathbb{R})} \to 0$ as $i \to \infty$.

Consider problem (2.8), (2.9) with $g = g_i$ in Q, $i = 0,1,\ldots$. It follows from [10, Ch. IV, Th. 5.3] that there is exactly one solution Vg_i of problem (2.8), (2.9) and $Vg_i \in C^{1+\alpha/2,2+\alpha}(\bar{Q},\mathbb{R},0)$. Since classical solutions of (2.8), (2.9) are also generalized solutions of (2.8), (2.9), we have $Vg_i \in W_q^{1,2}(\bar{Q},\mathbb{R})$ and

$$\|Vg_i - Vg_j\|_{W_q^{1,2}(\bar{Q},\mathbb{R})} \le c_1\|g_i - g_j\|_{L^q(\bar{Q},\mathbb{R})}$$

(see [11, Theorem A.3.3]), which shows that $\{Vg_i\}_{i=0}^{\infty}$ is a Cauchy sequence in $W_q^{1,2}(\bar{Q},\mathbb{R})$. Since $W_q^{1,2}(\bar{Q},\mathbb{R})$ is a Banach space, there exists $\tilde{z} \in W_q^{1,2}(\bar{Q},\mathbb{R})$ such that

$$\lim_{i\to\infty} \|\tilde{z} - Vg_i\|_{W_q^{1,2}(\bar{Q},\mathbb{R})} = 0.$$

Put $V^*g = \tilde{z}$. We see at once that $\tilde{z}$ is independent of the choice of $\{g_i\}_{i=0}^{\infty}$. We claim that $V^* : L^q(\bar{Q},\mathbb{R}) \to W_q^{1,2}(\bar{Q},\mathbb{R})$ is bounded and continuous. Indeed,

$$\|V^*g - V^*\bar{g}\|_{W_q^{1,2}(\bar{Q},\mathbb{R})} \le \|g - \bar{g}\|_{L^q(\bar{Q},\mathbb{R})}$$

and

$$\|Vg_i\|_{W_q^{1,2}(\bar{Q},\mathbb{R})} \le c_1\|g_i\|_{L^q(\bar{Q},\mathbb{R})}.$$

Put $q = (n+2)/(1-\alpha)$ and $\Sigma = (\{0\}\times S)\cup([0,a]\times\partial S)$. Since $W_q^{1,2}(\bar{Q},\mathbb{R})$

is imbedded in $C^{(1+\alpha)/2,1+\alpha}(\bar{Q},\mathbb{R})$ (see [1]), we have

$$\begin{aligned}\|\tilde{z}\|_{C^{0,1}([0,a]\times\partial S,\mathbb{R})} &\le \|\tilde{z}-Vg_i\|_{C^{0,1}([0,a]\times\partial S,\mathbb{R})}\\ &\le \|\tilde{z}-Vg_i\|_{C^{(1+\alpha)/2,1+\alpha}(\bar{Q},\mathbb{R})} \le c_2\|\tilde{z}-Vg_i\|_{W_q^{1,2}(\bar{Q},\mathbb{R})}\end{aligned}$$

and

$$\begin{aligned}\|\tilde{z}\|_{C(\{0\}\times S,\mathbb{R})} &\le \|\tilde{z}-Vg_i\|_{C(\{0\}\times S,\mathbb{R})}\\ &\le \|\tilde{z}-Vg_i\|_{C^{(1+\alpha)/2,1+\alpha}(\bar{Q},\mathbb{R})} \le c_2\|\tilde{z}-Vg_i\|_{W_q^{1,2}(\bar{Q},\mathbb{R})},\end{aligned}$$

for some $c_2 \ge 0$. Hence $\tilde{z}_{|([0,a]\times\partial S)} = 0$, $\partial_x\tilde{z}_{|([0,a]\times\partial S)} = 0$ and $\tilde{z}(0,x) = 0$ for $x \in S$. Therefore $\tilde{z}$ satisfies (2.9).

We proceed to show that $\tilde{z}$ is a classical solution of (1.1), (1.2) if and only if $\tilde{z}$ is a solution of

$$z = (V^*F)z. \tag{2.10}$$

Suppose that $\tilde{z} \in C^{1+\alpha/2,2+\alpha}(\Omega,\mathbb{R},0)$ is a classical solution of (1.1), (1.2). Put $z^* = (V^*\mathbb{F})\tilde{z}$. It follows from Lemma 2.2 and from (2.7) that $\mathbb{F}[\tilde{z}] \in C_0^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$. Hence $z^* = (V\mathbb{F})\tilde{z}$ and z^* is a solution of

$$\begin{cases} L[z](t,x) = \mathbb{F}[\tilde{z}](t,x), & (t,x)\in Q,\\ \Lambda[z](t,x) = 0, & (t,x)\in[0,a]\times\partial \mathrm{S},\\ z(t,x) = 0, & (t,x)\in Q_0.\end{cases} \tag{2.11}$$

But $\tilde{z}$ also satisfies (2.11). Therefore $\tilde{z} = z^*$ by uniqueness. Suppose now that z^* satisfies (2.10). Since $I : C(\bar{Q},\mathbb{R}) \to L^q(\bar{Q},\mathbb{R})$ defined by $Iz = z$ for $z \in C(\bar{Q},\mathbb{R})$ is continuous, and $\tilde{I} : W_q^{1,2}(\bar{Q},\mathbb{R}) \to C^{(1+\alpha)/2,1+\alpha}(\bar{Q},\mathbb{R})$ defined by $\tilde{I}z = z$ for $z \in W_q^{1,2}(\bar{Q},\mathbb{R})$ is continuous, we deduce from Lemma 2.2 that

$$V^*\mathbb{F} : C_*^{0,1}(\Omega,\mathbb{R}) \to C_{\mu,\tilde{\mu}}^{(1+\alpha)/2,1+\alpha}(\Omega,\mathbb{R})$$

is also continuous. Since $z^* = (V^*\mathbb{F})z^*$, $z^* \in C^{(1+\alpha)/2,1+\alpha}(\Omega,\mathbb{R},0)$ and in view of Lemma 2.2, $\mathbb{F}z^* \in C_0^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$. Therefore

$$z^* = (V^*\mathbb{F})z^* = (V\mathbb{F})z^* \in C^{1+\alpha/2,2+\alpha}(\Omega,\mathbb{R},0)$$

and z^* satisfies (1.1), (1.2). Let $C_*^{0,1}(\Omega,\mathbb{R},0) = \{z \in C_*^{0,1}(\Omega,\mathbb{R}) : z_{|Q_0} = 0$ and $\Lambda[z]_{|[0,a]\times\partial S} = 0\}$. The operator $G = V^*\mathbb{F}$ is completely continuous from $C_*^{0,1}(\Omega,\mathbb{R},0)$ into itself, which is clear from Lemma 2.2 and the fact that $C_{\mu,\tilde{\mu}}^{(1+\alpha)/2,1+\alpha}(\Omega,\mathbb{R})$ is compactly imbedded in $C_*^{0,1}(\Omega,\mathbb{R})$ (see [1]).

Let

$$U = \{u \in C_*^{0,1}(\Omega,\mathbb{R},0) : \|u\|_\Omega < \tilde{r}+1,\ \|\partial_x u\|_{\bar{Q}} < \tilde{L}+1\}$$

where $\tilde{r} = \omega(a,\eta)$ is defined in Remark 1.3 and $\tilde{L}$ in Lemma 2.3. We see at once that $0 \in U$ and U is bounded, open subset of $C_*^{0,1}(\Omega,\mathbb{R},0)$. We will show that $u \ne \lambda Gu$ for every $u \in \partial U$, $\lambda \in (0,1)$. On the contrary, suppose that $\lambda G u = u$ for some $u \in \partial U$, $\lambda \in (0,1)$. Then $\lambda(V^*\mathbb{F})u = (V^*\lambda\mathbb{F})u = u$ is a solution of

$$\begin{cases} L[z](t,x) = \lambda\mathbb{F}[\tilde{z}](t,x), & (t,x) \in Q, \\ \Lambda[z](t,x) = 0, & (t,x) \in [0,a] \times \partial S, \\ z(t,x) = 0, & (t,x) \in Q_0. \end{cases}$$

Applying Lemma 2.3 (with $\lambda\mathbb{F}$ instead of $\mathbb{F}$) we find that $\|u\|_{\Omega} \le \tilde{r}$ and $\|\partial_x u\|_{\bar{Q}} < \tilde{L}$ as $\lambda \in (0,1)$. This contradicts the fact that $u \in \partial U$. We conclude from the Leray–Schauder theorem that G has a fixed point, which in view of the first part of the proof is the desired conclusion. ∎

Remark 2.5. Let us consider the functional differential equation

$$L[z](t,x) = F(t,x,z(t,x),z_{(t,x)},\partial_x z(t,x)) \tag{2.12}$$

which is a particular case of (1.1).

Let us note some differences between problems (1.1), (1.2) and (2.12), (1.2). Differential equations with deviated variables are obtained from (2.12) in the following way. Suppose that $f : Q \times \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}$ is a given function. Write

$$F(t,x,u,w,p) = f(t,x,u,w(\varphi(t,x) - (t,x)),p) \quad \text{on } \Xi. \tag{2.13}$$

Then (2.12) reduces to (1.4). Note that Assumption $\mathbf{H}[F,\Psi,\psi]$ is not satisfied for F given by (2.13). More precisely, condition (2.3) is not satisfied on $Q \times [-r,r] \times C(B,\mathbb{R},q) \times \{p \in \mathbb{R}^n : \|p\| \le L\}$.

It is clear that under natural assumptions on f the function F given by (1.3) satisfies Assumption $\mathbf{H}[F,\Psi,\psi]$.

With the above motivation we have considered (1.1), (1.2).

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