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Doubly warped product Finsler manifolds with some non-Riemannian curvature properties

by Esmaeil Peyghan (Arak), Akbar Tayebi (Qom) and
Behzad Najafi (Tehran)

Abstract. We consider doubly warped product (DWP) Finsler manifolds with some non-Riemannian curvature properties. First, we study Berwald and isotropic mean Berwald DWP-Finsler manifolds. Then we prove that every proper Douglas DWP-Finsler manifold is Riemannian. We show that a proper DWP-manifold is Landsbergian if and only if it is Berwaldian. Then we prove that every relatively isotropic Landsberg DWP-manifold is a Landsberg manifold. We show that a relatively isotropic mean Landsberg warped product manifold is a weakly Landsberg manifold. Finally, we show that there is no locally dually flat proper DWP-Finsler manifold.

1. Introduction. The study of relativity theory demands a wider class of manifolds and the idea of doubly warped products was introduced and studied by many authors. Recent studies show that the notion of doubly warped product manifolds has an important role in Riemannian geometry and its applications [A], [BEP], [BP], [G], [Mu1], [Mu2], [U]. For example, Beem–Powell studied this product for Lorentzian manifolds [BP]. Then in [A], Allison considered global hyperbolicity of doubly warped products and null pseudo convexity of Lorentzian doubly warped products.

On the other hand, Finsler geometry is dedicated to classical and generalized Finsler geometries. It studies manifolds whose tangent spaces carry a norm varying smoothly with the base point. Indeed, Finsler geometry is just Riemannian geometry without the quadratic restriction. Thus it is natural to extend the construction of warped product manifolds to Finsler geometry. In the first step, Asanov generalized the Schwarzschild metric to the Finslerian setting and obtained some models of relativity theory described through warped products of Finsler metrics [As1], [As2]. In [Koz], Kozma–Peter–Varga defined their warped product for Finsler metrics and concluded

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that completeness of a doubly warped product can be related to completeness of its components.

Let (M_1, F_1) and (M_2, F_2) be two Finsler manifolds and $f_1 : M_1 \to \mathbb{R}^+$ and $f_2 : M_2 \to \mathbb{R}^+$ be two smooth functions. Let $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ be the natural projection maps. The product manifold $M_1 \times M_2$ endowed with the metric $F : TM_1^\circ \times TM_2^\circ \to \mathbb{R}$ given by

$$F(v_1, v_2) = \sqrt{f_2^2(\pi_2(v_2))F_1^2(v_1) + f_1^2(\pi_1(v_1))F_2^2(v_2)} \tag{1.1}$$

is considered, where $TM_1^\circ = TM_1 \setminus \{0\}$ and $TM_2^\circ = TM_2 \setminus \{0\}$. The metric defined above is a Finsler metric. The product manifold $M_1 \times M_2$ with this metric will be called the *doubly warped product* of the manifolds M_1 and M_2, denoted ${}_{f_2}M_1 \times {}_{f_1}M_2$. If either $f_1 = 1$ or $f_2 = 1$, then ${}_{f_2}M_1 \times {}_{f_1}M_2$ becomes a warped product of Finsler manifolds M_1 and M_2. If $f_1 = f_2 = 1$, then we have a product manifold. If neither f_1 nor f_2 is constant, then we have a proper DWP-manifold.

Let (M, F) be a Finsler manifold. The second and third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ are symmetric trilinear forms $\mathbf{g}_y$ and $\mathbf{C}_y$ on T_xM, called the *fundamental tensor* and *Cartan torsion*, respectively. The rate of change of $\mathbf{C}_y$ along geodesics is the *Landsberg curvature* $\mathbf{L}_y$ on T_xM [Ba], [BCS].The metric F is said to be a *relatively isotropic Landsberg metric* if $\mathbf{L} + cF\mathbf{C} = 0$, where $c = c(x)$ is a scalar function on M. Set $\mathbf{I}_y := \sum_{i=1}^n \mathbf{C}_y(e_i, e_i, \cdot)$ and $\mathbf{J}_y := \sum_{i=1}^n \mathbf{L}_y(e_i, e_i, \cdot)$, where $\{e_i\}$ is an orthonormal basis for $(T_xM, \mathbf{g}_y)$. Then $\mathbf{I}_y$ and $\mathbf{J}_y$ are called the *mean Cartan torsion* and *mean Landsberg curvature*, respectively. The metric F is said to be a *relatively isotropic mean Landsberg metric* if $\mathbf{J} + cF\mathbf{I} = 0$, where $c = c(x)$ is a scalar function on M [ChSh].

The geodesic curves of a Finsler manifold (M, F) are determined by the system of second order differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the *spray coefficients* of F. A Finsler metric F is called a *Berwald metric* if the G^i are quadratic in $y \in T_xM$ for any $x \in M$, and a *Douglas metric* if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k + P(x, y)y^i$ [BM], [NST1]. Taking the trace of the Berwald curvature yields the *mean Berwald curvature* $\mathbf{E}$. The metric F is said to be an *isotropic mean Berwald metric* if $\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h}$, where $\mathbf{h} = h_{ij}dx^i \otimes dx^j$ is the angular metric and $c = c(x)$ is a scalar function on M [NST2].

This paper is arranged as follows: In Section 2, we recall some basic concepts of Finsler manifolds. In Sections 3 and 4, we study doubly warped product Finsler metrics (DWP-Finsler metrics) with vanishing Berwald curvature and isotropic mean Berwald curvature, respectively. In Section 5, we prove that every proper Douglas DWP-Finsler manifold is Riemannian.

In Section 6, we show that a proper DWP-Finsler manifold is a Landsberg manifold if and only if it is a Berwald manifold. Then we prove that every relatively isotropic Landsberg DWP-Finsler manifold is a Landsberg manifold. In Section 7, we prove that a relatively isotropic mean Landsberg warped product manifold is a weakly Landsberg manifold. Finally in Section 8, we show that there is no locally dually flat proper DWP-Finsler manifold.

2. Preliminaries. Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A *Finsler metric* on M is a function $F : TM \to [0,\infty)$ which has the following properties:

(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the quadratic form $\mathbf{g}_y$ on T_xM is positive definite, where

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}[F^2(y+su+tv)]\Big|_{s,t=0}, \qquad u,v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}[\mathbf{g}_{y+tw}(u,v)]\Big|_{t=0}, \qquad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the *Cartan torsion*. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian. For $y \in T_xM_0$, define the mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$, $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ and $u = u^i\frac{\partial}{\partial x^i}\big|_x$. By Deicke's Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$ [BCS], [Sh1].

Given a Finsler manifold (M,F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in standard coordinates (x^i,y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4}g^{il}(y)\bigg\{\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\bigg\}, \qquad y \in T_xM.$$

$\mathbf{G}$ is called the *spray* associated to (M,F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}\big|_x$ and

$\mathbf{E}_y(u,v) := E_{jk}(y)u^j v^k$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \qquad E_{jk} := \frac{1}{2} B^m{}_{jkm}.$$

$\mathbf{B}$ and $\mathbf{E}$ are called the *Berwald curvature* and *mean Berwald curvature*, respectively. Then F is called a *Berwald metric* and *weakly Berwald metric* if $\mathbf{B} = 0$ and $\mathbf{E} = 0$, respectively [Sh1]. It is proved that on a Berwald space, parallel translation along any geodesic preserves the Minkowski functionals [Ich]. Thus Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space.

A Finsler metric F is said to be an *isotropic mean Berwald metric* if its mean Berwald curvature is of the form

$$E_{ij} = \frac{1}{2}(n+1)cF^{-1}h_{ij}, \tag{2.1}$$

where $h_{ij} = g_{ij} - F^{-2}y_i y_j$ is the angular metric and $c = c(x)$ is a scalar function on M [ChSh].

Define $\mathbf{D}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by

$$\mathbf{D}_y(u,v,w) := D^i{}_{jkl}(y)u^i v^j w^k \frac{\partial}{\partial x^i}\bigg|_x$$

where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\left\{ E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + \frac{E_{jk}}{\partial y^l} y^i \right\}.$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y \in TM_0}$ the *Douglas curvature*. A Finsler metric with $\mathbf{D} = 0$ is called a *Douglas metric*. The notion of Douglas metrics was proposed by Bácsó–Matsumoto as a generalization of Berwald metrics [BM].

There is another extension of Berwald curvature. For a tangent vector $y \in T_xM_0$, define $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^i v^j w^k$, where

$$L_{ijk} := -\frac{1}{2} y_l B^l{}_{ijk}.$$

The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the *Landsberg curvature*. A Finsler metric is called a *Landsberg metric* if $\mathbf{L} = 0$. The quantity $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along geodesics. A Finsler metric F is said to be a *relatively isotropic Landsberg metric* if

$$\mathbf{L} = cF\mathbf{C}$$

for some scalar function $c = c(x)$ on M [ChSh].

Taking the trace of the Landsberg curvature yields the *mean Landsberg curvature* $\mathbf{J}_y : T_xM \to \mathbb{R}$, defined by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := g^{jk} L_{ijk}.$$

A Finsler metric is called a *weakly Landsberg metric* if $\mathbf{J} = 0$. The quantity $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of $\mathbf{I}$ along geodesics. A Finsler metric F is said to be a *relatively isotropic mean Landsberg metric* if

$$\mathbf{J} = cF\mathbf{I}$$

for some scalar function $c = c(x)$ on M [ChSh]. It is obvious that every relatively isotropic Landsberg metric is a relatively isotropic mean Landsberg metric.

A Finsler metric $F = F(x, y)$ on a manifold M is said to be *locally dually flat* if at any point there is a standard coordinate system (x^i, y^i) in TM that satisfies

$$(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}. \tag{2.2}$$

In this case, the coordinate system (x^i) is called an *adapted* local coordinate system [Am], [amna]. It is easy to see that every locally Minkowskian metric satisfies (2.2), hence is locally dually flat [Sh2]. But the converse is not true, generally.

3. Berwaldian DWP-Finsler manifolds. In this section, we study DWP-Finsler manifolds with vanishing Berwald curvature.

LEMMA 3.1. *Every proper DWP-Finsler manifold $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ with vanishing Berwald curvature is a Riemannian manifold.*

Proof. The Berwald curvature of $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ is as follows:

$$\mathbf{B}^k_{ijl} = B^k_{ijl} - \frac{1}{4f_2^2} \frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^l} \frac{\partial f_1^2}{\partial x^h} F_2^2, \tag{3.1}$$

$$\mathbf{B}^k_{i\beta l} = -\frac{1}{4f_2^2} \frac{\partial^2 g^{kh}}{\partial y^l \partial y^i} \frac{\partial f_1^2}{\partial x^h} \frac{\partial F_2^2}{\partial v^\beta}, \tag{3.2}$$

$$\mathbf{B}^k_{\alpha\beta l} = -\frac{1}{f_2^2} \frac{\partial f_1^2}{\partial x^h} \frac{\partial g^{kh}}{\partial y^l} g_{\alpha\beta}, \tag{3.3}$$

$$\mathbf{B}^k_{\alpha\beta\lambda} = -\frac{1}{f_2^2} \frac{\partial f_1^2}{\partial x^h} g^{kh} C_{\alpha\beta\lambda}, \tag{3.4}$$

$$\mathbf{B}^\gamma_{\alpha\beta\lambda} = B^\gamma_{\alpha\beta\lambda} - \frac{1}{4f_1^2} \frac{\partial^3 g^{\gamma\nu}}{\partial v^\beta \partial v^\alpha \partial v^\lambda} \frac{\partial f_2^2}{\partial u^\nu} F_1^2, \tag{3.5}$$

$$\mathbf{B}^\gamma_{i\beta\lambda} = -\frac{1}{4f_1^2} \frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha} \frac{\partial F_1^2}{\partial y^i}, \tag{3.6}$$

$$\mathbf{B}^\gamma_{ij\lambda} = -\frac{1}{2f_1^2} g_{ij} \frac{\partial g^{\alpha\gamma}}{\partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha}, \tag{3.7}$$

$$\mathbf{B}^{\gamma}_{ijk} = -\frac{1}{f_1^2} C_{ijk} g^{\alpha\gamma} \frac{\partial f_2^2}{\partial u^\alpha}. \tag{3.8}$$

If $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ is Berwaldian, then $\mathbf{B}^d_{abc} = 0$. By (3.4), we get

$$C_{\alpha\beta\lambda} g^{kh} \frac{\partial f_1^2}{\partial x^h} = 0. \tag{3.9}$$

Multiplying (3.9) with g_{kr} implies that

$$C_{\alpha\beta\lambda} \frac{\partial f_1^2}{\partial x^r} = 0. \tag{3.10}$$

By (3.10), if f_1 is not constant then we get $C_{\alpha\beta\lambda} = 0$, i.e., (M_2, F_2) is Riemannian. In a similar way, from (3.8) we conclude that if f_2 is not constant then (M_1, F_1) is Riemannian. ∎

THEOREM 3.2. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a DWP-Finsler manifold.*

(i) *If f_1 is constant and f_2 is not constant, then $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ is a Berwald manifold if and only if M_1 is Riemannian, M_2 is a Berwald manifold and*

$$\frac{\partial g^{\alpha\gamma}}{\partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha} = 0.$$

(ii) *If f_2 is constant and f_1 is not constant, then $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ is a Berwald manifold if and only if M_2 is Riemannian, M_1 is Berwaldian and*

$$\frac{\partial g^{ij}}{\partial y^k} \frac{\partial f_1^2}{\partial x^i} = 0.$$

Proof. Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a Berwaldian manifold with f_1 constant on M_1. Then from (3.8) we get $C_{ijk} = 0$, i.e., (M_1, F_1) is Riemannian. Also, (3.7) gives

$$\frac{\partial g^{\alpha\gamma}}{\partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha} = 0.$$

Differentiating this equation with respect to v^β implies that

$$\frac{\partial^2 g^{\alpha\gamma}}{\partial v^\lambda \partial v^\beta} \frac{\partial f_2^2}{\partial u^\alpha} = 0,$$

and consequently

$$\frac{\partial^3 g^{\alpha\gamma}}{\partial v^\lambda \partial v^\beta \partial v^\mu} \frac{\partial f_2^2}{\partial u^\alpha} = 0.$$

Then (3.5) reduces to $B^{\gamma}_{\alpha\beta\lambda} = 0$, i.e., (M_2, F_2) is Berwaldian. In a similar way, we can prove the converse of this assertion. The proof of (ii) is similar and we omit it. ∎

By a similar argument, we obtain

Corollary 3.3. *Let $(M_1 \times {}_{f_1}M_2, F)$ be a proper WP-Finsler manifold. Then $(M_1 \times {}_{f_1}M_2, F)$ is Berwaldian if and only if M_2 is Riemannian, M_1 is Berwaldian and*

$$C^{ij}{}_k \frac{\partial f_1}{\partial x^i} = 0, \tag{3.11}$$

where $C^{ij}{}_k = -2\frac{\partial g^{ij}}{\partial y^k}$ is the Cartan tensor.

4. Isotropic mean Berwald DWP-manifolds. In this section, we study DWP-Finsler metrics with isotropic mean Berwald curvature. First, we compute the mean Berwald curvature of a DWP-Finsler manifold.

Lemma 4.1. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a DWP-Finsler manifold. Then the mean Berwald curvature of F is as follows:*

$$\mathbf{E}_{\alpha\beta} = E_{\alpha\beta} - \frac{1}{8f_1^2}\frac{\partial^3 g^{\gamma\nu}}{\partial v^\beta \partial v^\alpha \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}F_1^2 - \frac{1}{4f_2^2}g_{\alpha\beta}\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_1^2}{\partial x^h}, \tag{4.1}$$

$$\mathbf{E}_{ij} = E_{ij} - \frac{1}{8f_2^2}\frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^k}\frac{\partial f_1^2}{\partial x^h}F_2^2 - \frac{1}{4f_1^2}g_{ij}\frac{\partial g^{\alpha\gamma}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\alpha}, \tag{4.2}$$

$$\mathbf{E}_{i\beta} = -\frac{1}{4f_2^2}\frac{\partial^2 g^{kh}}{\partial y^k \partial y^i}\frac{\partial f_1^2}{\partial x^h}v_\beta - \frac{1}{4f_1^2}\frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\alpha}y_i, \tag{4.3}$$

where E_{ij} and $E_{\alpha\beta}$ are mean Berwald curvatures of (M_1, F_1) and (M_2, F_2), respectively.

Theorem 4.2. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a proper DWP-Finsler manifold. Then F is a weakly Berwald metric if and only if F_1 and F_2 are weakly Berwald metrics and*

$$\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_1^2}{\partial x^h} = \frac{\partial g^{\gamma\nu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu} = 0. \tag{4.4}$$

Proof. Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a weakly Berwald manifold. Then we have $\mathbf{E}_{\alpha\beta} = \mathbf{E}_{ij} = \mathbf{E}_{i\beta} = 0$. Using (4.3), we get

$$\frac{1}{f_2^2}\frac{\partial^2 g^{kh}}{\partial y^j \partial y^k}\frac{\partial f_1^2}{\partial x^h}v_\beta = -\frac{1}{f_1^2}\frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\alpha}y_j. \tag{4.5}$$

Contracting (4.5) with y^j gives

$$\frac{1}{f_2^2}\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_1^2}{\partial x^h}v_\beta = \frac{1}{f_1^2}\frac{\partial^2 g^{\nu\gamma}}{\partial v^\beta \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}F_1^2. \tag{4.6}$$

Differentiating (4.6) with respect v^α implies that

$$\frac{1}{f_2^2}\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_1^2}{\partial x^h}g_{\alpha\beta} = \frac{1}{f_1^2}\frac{\partial^3 g^{\nu\gamma}}{\partial v^\alpha \partial v^\beta \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}F_1^2. \tag{4.7}$$

In a similar way, one can obtain

$$\frac{1}{f_1^2}\frac{\partial g^{\gamma\alpha}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\alpha}g_{ij}=\frac{1}{f_2^2}\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}\frac{\partial f_1^2}{\partial x^h}F_2^2. \tag{4.8}$$

Substituting (4.7) into (4.1) and plugging (4.8) into (4.2), we have

$$E_{\alpha\beta}=\frac{3}{8f_1^2}\frac{\partial^3 g^{\nu\gamma}}{\partial v^\alpha\partial v^\beta\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}F_1^2, \tag{4.9}$$

$$E_{ij}=\frac{3}{8f_2^2}\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}\frac{\partial f_1^2}{\partial x^h}F_2^2. \tag{4.10}$$

Since $E_{\alpha\beta}$ is a function of (u^α, v^α), by differentiating (4.9) with respect y^h we deduce that

$$\frac{\partial^3 g^{\nu\gamma}}{\partial v^\alpha\partial v^\beta\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}y_h=0,$$

and consequently

$$\frac{\partial^3 g^{\nu\gamma}}{\partial v^\alpha\partial v^\beta\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}=0. \tag{4.11}$$

Putting (4.11) into (4.9) gives $E_{\alpha\beta}=0$. A similar argument yields $E_{ij}=0$. Further, from (4.11) and (4.7) we derive that

$$\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_1^2}{\partial x^h}=0.$$

Also, contracting (4.11) with $v^\alpha v^\beta$ implies that

$$\frac{\partial g^{\gamma\nu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}=0.$$

Thus we have (4.4).

Conversely, suppose (M_1, F_1) and (M_2, F_2) are weakly Berwald manifolds and (4.4) holds. Then $E_{ij}=E_{\alpha\beta}=0$. Equation (4.4) gives

$$\frac{\partial^2 g^{kh}}{\partial y^j\partial y^k}\frac{\partial f_1^2}{\partial x^h}=\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}\frac{\partial f_1^2}{\partial x^h}=\frac{\partial^2 g^{\nu\gamma}}{\partial v^\beta\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}=\frac{\partial^3 g^{\nu\gamma}}{\partial v^\alpha\partial v^\beta\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}=0.$$

By plugging $E_{ij}=E_{\alpha\beta}=0$ and the above equation into (4.1)–(4.3), we obtain $\mathbf{E}_{\alpha\beta}=\mathbf{E}_{ij}=\mathbf{E}_{i\beta}=0$. This means that $({}_{f_2}M_1\times{}_{f_1}M_2, F)$ is a weakly Berwald manifold. ∎

Now, if f_2 is a constant function on M_2, then (4.7) implies that $E_{\alpha\beta}=0$. Thus we have

Corollary 4.3. *Let $({}_{f_2}M_1\times{}_{f_1}M_2, F)$ be a DWP-Finsler manifold and f_1 be constant on M_1 (resp. f_2 be constant on M_2). Then $({}_{f_2}M_1\times{}_{f_1}M_2, F)$ is weakly Berwald if and only if (M_1, F_1) and (M_2, F_2) are weakly Berwald* *manifolds and*

$$\frac{\partial g^{\gamma\nu}}{\partial v^{\gamma}}\frac{\partial f_2^2}{\partial u^{\nu}}=0 \qquad \Big(\textit{resp.}\ \frac{\partial g^{kh}}{\partial y^{k}}\frac{\partial f_1^2}{\partial x^{h}}=0\Big).$$

COROLLARY 4.4. *Let* $(M_1\times {}_{f_1}M_2,F)$ *be a WP-Finsler manifold. Then* $(M_1\times {}_{f_1}M_2,F)$ *is weakly Berwald if and only if* (M_1,F_1) *and* (M_2,F_2) *are weakly Berwald manifolds and*

$$\frac{\partial g^{kh}}{\partial y^{k}}\frac{\partial f_1^2}{\partial x^{h}}=0.$$

Now, we consider DWP-Finsler manifolds with isotropic mean Berwald curvature. First, as a consequence of Lemma 4.1, we have

LEMMA 4.5. *A DWP-Finsler manifold* $({}_{f_2}M_1\times {}_{f_1}M_2,F)$ *has isotropic mean Berwald curvature if and only if*

$$E_{\alpha\beta}-\frac{1}{8f_1^2}\frac{\partial^3 g^{\gamma\nu}}{\partial v^{\beta}\partial v^{\alpha}\partial v^{\gamma}}\frac{\partial f_2^2}{\partial u^{\nu}}F_1^2-\frac{1}{4f_2^2}g_{\alpha\beta}\frac{\partial g^{kh}}{\partial y^{k}}\frac{\partial f_1^2}{\partial x^{h}} -\frac{n+1}{2}cf_1^2F^{-1}\Big(g_{\alpha\beta}-\frac{f_1^2}{F^2}v_\alpha v_\beta\Big)=0, \tag{4.12}$$

$$E_{ij}-\frac{1}{8f_2^2}\frac{\partial^3 g^{kh}}{\partial y^{i}\partial y^{j}\partial y^{k}}\frac{\partial f_1^2}{\partial x^{h}}F_2^2-\frac{1}{4f_1^2}g_{ij}\frac{\partial g^{\alpha\gamma}}{\partial v^{\gamma}}\frac{\partial f_2^2}{\partial u^{\alpha}} -\frac{n+1}{2}cf_2^2F^{-1}\Big(g_{ij}-\frac{f_2^2}{F^2}y_iy_j\Big)=0, \tag{4.13}$$

$$(n+1)c\frac{f_1^2f_2^2}{F^3}y_iv_\beta-\frac{1}{2f_2^2}\frac{\partial^2 g^{kh}}{\partial y^{k}\partial y^{i}}\frac{\partial f_1^2}{\partial x^{h}}v_\beta-\frac{1}{2f_1^2}\frac{\partial^2 g^{\alpha\gamma}}{\partial v^{\beta}\partial v^{\gamma}}\frac{\partial f_2^2}{\partial u^{\alpha}}y_i=0, \tag{4.14}$$

where $c=c(\mathbf{x})$ *is a scalar function on* M*.*

THEOREM 4.6. *A DWP-Finsler manifold* $({}_{f_2}M_1\times {}_{f_1}M_2,F)$ *with isotropic mean Berwald curvature is a weakly Berwald manifold provided that*

$$\frac{\partial g^{kh}}{\partial y^{k}}\frac{\partial f_1}{\partial x^{h}}=0 \quad \textit{or} \quad \frac{\partial g^{\gamma\nu}}{\partial v^{\gamma}}\frac{\partial f_2}{\partial u^{\nu}}=0.$$

Proof. Suppose that $\frac{\partial g^{kh}}{\partial y^{k}}\frac{\partial f_1}{\partial x^{h}}=0$ and F is an isotropic mean Berwald DWP-Finsler metric. Then by using (4.14), we obtain

$$(n+1)c\frac{f_1^2f_2^2}{F^3}v_\beta=\frac{1}{2f_1^2}\frac{\partial^2 g^{\alpha\gamma}}{\partial v^{\beta}\partial v^{\gamma}}\frac{\partial f_2^2}{\partial u^{\alpha}}.$$

Differentiating the above equation with respect y^j gives

$$\frac{n+1}{F^5}cf_1^2f_2^4v_\beta y_j=0.$$

Thus, $c=0$, so F is a weakly Berwald metric. ∎

5. Douglas DWP-Finsler manifolds. In this section, we study DWP-Finsler manifolds with vanishing Douglas curvature. We prove that every Douglas proper DWP-Finsler manifold is Riemannian. To do this, we need

LEMMA 5.1. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a DWP-Finsler manifold. Then the Douglas curvature of F is as follows:*

$$
\begin{aligned}
(5.1)\qquad \mathbf{D}^k{}_{ijl} = {} & B^k{}_{ijl} - \frac{1}{4f_2^2}\frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^l}\frac{\partial f_1^2}{\partial x^h}F_2^2 \\
& - \frac{2}{n+1}\bigg\{E_{ij}\delta^k_l - \frac{1}{8f_2^2}\frac{\partial^3 g^{sh}}{\partial y^i \partial y^j \partial y^s}\frac{\partial f_1^2}{\partial x^h}F_2^2\delta^k_l \\
& - \frac{1}{4f_1^2}g_{ij}\frac{\partial g^{\alpha\gamma}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\alpha}\delta^k_l + E_{il}\delta^k_j - \frac{1}{8f_2^2}\frac{\partial^3 g^{sh}}{\partial y^i \partial y^l \partial y^s}\frac{\partial f_1^2}{\partial x^h}F_2^2\delta^k_j \\
& - \frac{1}{4f_1^2}g_{il}\frac{\partial g^{\alpha\gamma}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\alpha}\delta^k_j + E_{jl}\delta^k_i - \frac{1}{8f_2^2}\frac{\partial^3 g^{sh}}{\partial y^j \partial y^l \partial y^s}\frac{\partial f_1^2}{\partial x^h}F_2^2\delta^k_i \\
& - \frac{1}{4f_1^2}g_{jl}\frac{\partial g^{\alpha\gamma}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\alpha}\delta^k_i - \frac{1}{4f_1^2}\frac{\partial g_{ij}}{\partial y^l}\frac{\partial g^{\alpha\gamma}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\alpha}y^k \\
& - \frac{1}{8f_2^2}\frac{\partial^4 g^{sh}}{\partial y^l \partial y^i \partial y^j \partial y^s}\frac{\partial f_1^2}{\partial x^h}F_2^2 y^k + \frac{\partial E_{ij}}{\partial y^l}y^k\bigg\},
\end{aligned}
$$

$$
\begin{aligned}
(5.2)\qquad \mathbf{D}^k{}_{i\beta l} = {} & -\frac{1}{4f_2^2}\frac{\partial^2 g^{kh}}{\partial y^l \partial y^i}\frac{\partial f_1^2}{\partial x^h}\frac{\partial F_2^2}{\partial v^\beta} + \frac{2}{n+1}\bigg\{\frac{1}{4f_2^2}\frac{\partial^2 g^{sh}}{\partial y^s \partial y^i}\frac{\partial f_1^2}{\partial x^h}\delta^k_l v_\beta \\
& + \frac{1}{4f_1^2}\frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\alpha}\delta^k_l y_i + \frac{1}{4f_2^2}\frac{\partial^2 g^{sh}}{\partial y^s \partial y^l}\frac{\partial f_1^2}{\partial x^h}\delta^k_i v_\beta \\
& + \frac{1}{4f_1^2}\frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\alpha}\delta^k_i y_l + \frac{1}{4f_2^2}\frac{\partial^3 g^{sh}}{\partial y^l \partial y^s \partial y^i}\frac{\partial f_1^2}{\partial x^h}y^k v_\beta \\
& + \frac{1}{4f_1^2}\frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\alpha}y^k g_{il}\bigg\},
\end{aligned}
$$

$$
\begin{aligned}
(5.3)\qquad \mathbf{D}^k{}_{\alpha\beta l} = {} & -\frac{1}{2f_2^2}g_{\alpha\beta}\frac{\partial g^{kh}}{\partial y^l}\frac{\partial f_1^2}{\partial x^h} - \frac{2}{n+1}\bigg\{E_{\alpha\beta}\delta^k_l - \frac{1}{4f_2^2}g_{\alpha\beta}\frac{\partial g^{sh}}{\partial y^s}\frac{\partial f_1^2}{\partial x^h}\delta^k_l \\
& - \frac{1}{8f_1^2}\frac{\partial^3 g^{\gamma\nu}}{\partial v^\beta \partial v^\alpha \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}F_1^2\delta^k_l - \frac{1}{4f_1^2}\frac{\partial^3 g^{\gamma\nu}}{\partial v^\beta \partial v^\alpha \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}y_l y^k \\
& - \frac{1}{4f_2^2}g_{\alpha\beta}\frac{\partial^2 g^{sh}}{\partial y^l \partial y^s}\frac{\partial f_1^2}{\partial x^h}y^k\bigg\},
\end{aligned}
$$

$$
\begin{aligned}
(5.4)\qquad \mathbf{D}^k{}_{\alpha\beta\lambda} = {} & -\frac{1}{f_2^2}C_{\alpha\beta\lambda}g^{kh}\frac{\partial f_1^2}{\partial x^h} - \frac{2}{n+1}\bigg\{\frac{\partial E_{\alpha\beta}}{\partial v^\lambda}y^k \\
& - \frac{1}{8f_1^2}\frac{\partial^4 g^{\gamma\nu}}{\partial v^\lambda \partial v^\beta \partial v^\alpha \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}F_1^2 y^k - \frac{1}{4f_2^2}\frac{\partial g_{\alpha\beta}}{\partial v^\lambda}\frac{\partial g^{sh}}{\partial y^s}\frac{\partial f_1^2}{\partial x^h}y^k\bigg\},
\end{aligned}
$$

(5.5)

$$\mathbf{D}^{\gamma}{}_{\alpha\beta\lambda}=B^{\gamma}_{\alpha\beta\lambda}-\frac{1}{4f_1^2}\,\frac{\partial^3 g^{\gamma\nu}}{\partial v^\beta\partial v^\alpha\partial v^\lambda}\,\frac{\partial f_2^2}{\partial u^\nu}F_1^2+\frac{2}{n+1}\bigg\{\frac{1}{8f_1^2}\,\frac{\partial^3 g^{\mu\nu}}{\partial v^\beta\partial v^\alpha\partial v^\mu}\,\frac{\partial f_2^2}{\partial u^\nu}F_1^2\delta^\gamma_\lambda$$
$$-E_{\alpha\beta}\delta^\gamma_\lambda+\frac{1}{4f_2^2}g_{\alpha\beta}\frac{\partial g^{kh}}{\partial y^k}\,\frac{\partial f_1^2}{\partial x^h}\delta^\gamma_\lambda+\frac{1}{8f_1^2}\,\frac{\partial^3 g^{\mu\nu}}{\partial v^\lambda\partial v^\alpha\partial v^\mu}\,\frac{\partial f_2^2}{\partial u^\nu}F_1^2\delta^\gamma_\beta$$
$$-E_{\alpha\lambda}\delta^\gamma_\beta+\frac{1}{4f_2^2}g_{\alpha\lambda}\frac{\partial g^{kh}}{\partial y^k}\,\frac{\partial f_1^2}{\partial^h}\delta^\gamma_\beta+\frac{1}{8f_1^2}\,\frac{\partial^3 g^{\mu\nu}}{\partial v^\lambda\partial v^\beta\partial v^\mu}\,\frac{\partial f_2^2}{\partial u^\nu}F_1^2\delta^\gamma_\alpha$$
$$-E_{\beta\lambda}\delta^\gamma_\alpha+\frac{1}{4f_2^2}g_{\beta\lambda}\,\frac{\partial g^{kh}}{\partial y^k}\,\frac{\partial f_1^2}{\partial x^h}\delta^\gamma_\alpha+\frac{1}{8f_1^2}\,\frac{\partial^4 g^{\mu\nu}}{\partial v^\lambda\partial v^\beta\partial v^\alpha\partial v^\mu}\,\frac{\partial f_2^2}{\partial u^\nu}F_1^2v^\gamma$$
$$+\frac{1}{4f_2^2}\,\frac{g_{\alpha\beta}}{\partial v^\gamma}\,\frac{\partial g^{kh}}{\partial y^k}\,\frac{\partial f_1^2}{\partial x^h}v^\gamma-\frac{\partial E_{\alpha\beta}}{\partial v^\lambda}v^\gamma\bigg\},$$

(5.6)
$$\mathbf{D}^{\gamma}{}_{i\beta\lambda}=-\frac{1}{4f_1^2}\,\frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta\partial v^\lambda}\,\frac{\partial f_2^2}{\partial u^\alpha}\,\frac{\partial F_1^2}{\partial y^i}+\frac{2}{n+1}\bigg\{\frac{1}{4f_2^2}\,\frac{\partial^2 g^{kh}}{\partial y^k\partial y^i}\,\frac{\partial f_1^2}{\partial x^h}\delta^\gamma_\lambda v_\beta$$
$$+\frac{1}{4f_1^2}\,\frac{\partial^2 g^{\alpha\mu}}{\partial v^\beta\partial v^\mu}\,\frac{\partial f_2^2}{\partial u^\alpha}\delta^\gamma_\lambda y_i+\frac{1}{4f_2^2}\,\frac{\partial^2 g^{kh}}{\partial y^k\partial y^i}\,\frac{\partial f_1^2}{\partial x^h}\delta^\gamma_\beta v_\lambda$$
$$+\frac{1}{4f_1^2}\,\frac{\partial^2 g^{\alpha\mu}}{\partial v^\lambda\partial v^\mu}\,\frac{\partial f_2^2}{\partial u^\alpha}\delta^\gamma_\beta y_i+\frac{1}{4f_2^2}\,\frac{\partial^2 g^{kh}}{\partial y^k\partial y^i}\,\frac{\partial f_1^2}{\partial x^h}g_{\beta\lambda}v^\gamma$$
$$+\frac{1}{4f_1^2}\,\frac{\partial^3 g^{\alpha\mu}}{\partial v^\lambda\partial v^\beta\partial v^\mu}\,\frac{\partial f_2^2}{\partial u^\alpha}v^\gamma y_i\bigg\},$$

(5.7)

$$\mathbf{D}^{\gamma}{}_{ij\lambda}=-\frac{1}{2f_1^2}g_{ij}\frac{\partial g^{\alpha\gamma}}{\partial v^\lambda}\,\frac{\partial f_2^2}{\partial u^\alpha}-\frac{2}{n+1}\bigg\{E_{ij}\delta^\gamma_\lambda-\frac{1}{8f_2^2}\,\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}\,\frac{\partial f_1^2}{\partial x^h}F_2^2\delta^\gamma_\lambda$$
$$-\frac{1}{4f_1^2}g_{ij}\frac{\partial g^{\alpha\mu}}{\partial v^\mu}\,\frac{\partial f_2^2}{\partial u^\alpha}\delta^\gamma_\lambda-\frac{1}{8f_2^2}\,\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}\,\frac{\partial f_1^2}{\partial x^h}\,\frac{\partial F_2^2}{\partial v^\lambda}v^\gamma$$
$$-\frac{1}{4f_1^2}g_{ij}\frac{\partial^2 g^{\alpha\mu}}{\partial v^\lambda\partial v^\mu}\,\frac{\partial f_2^2}{\partial u^\alpha}v^\gamma\bigg\},$$

(5.8)
$$\mathbf{D}^{\gamma}{}_{ijk}=-\frac{1}{f_1^2}C_{ijk}g^{\alpha\gamma}\frac{\partial f_2^2}{\partial u^\alpha}-\frac{2}{n+1}\bigg\{\frac{\partial E_{ij}}{\partial y^k}v^\gamma$$
$$-\frac{1}{8f_2^2}\,\frac{\partial^4 g^{sh}}{\partial y^k\partial y^i\partial y^j\partial y^s}\,\frac{\partial f_1^2}{\partial x^h}F_2^2v^\gamma-\frac{1}{4f_1^2}\,\frac{\partial g_{ij}}{\partial y^k}\,\frac{\partial g^{\alpha\mu}}{\partial v^\mu}\,\frac{\partial f_2^2}{\partial u^\alpha}v^\gamma\bigg\}.$$

Proof. By lengthy calculations using Lemmas 3.1 and 4.1. ∎

THEOREM 5.2. *Every proper DWP-Finsler manifold $({}_{f_2}M_1\times{}_{f_1}M_2,F)$ with vanishing Douglas curvature is a Riemannian manifold.*

Proof. Suppose that the Douglas curvature of $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ vanishes, i.e., $\mathbf{D}^d_{abc} = 0$. Then by contracting (5.8) with y^k we obtain

$$E_{ij} = \frac{3}{8f_2^2}\frac{\partial^3 g^{sh}}{\partial y^i \partial y^j \partial y^s}\frac{\partial f_1^2}{\partial x^h}F_2^2. \tag{5.9}$$

Since E_{ij} is a function of (x, y), by differentiating the above equation with respect to v^α, we get

$$\frac{\partial^3 g^{sh}}{\partial y^i \partial y^j \partial y^s}\frac{\partial f_1^2}{\partial x^h} = 0. \tag{5.10}$$

Putting the above into (5.9) gives $E_{ij} = 0$. Further, (5.10) implies

$$\frac{\partial^4 g^{sh}}{\partial y^k \partial y^i \partial y^j \partial y^s}\frac{\partial f_1^2}{\partial x^h} = \frac{\partial^2 g^{sh}}{\partial y^j \partial y^s}\frac{\partial f_1^2}{\partial x^h} = \frac{\partial g^{sh}}{\partial y^s}\frac{\partial f_1^2}{\partial x^h} = 0. \tag{5.11}$$

In a similar way, we conclude that $E_{\alpha\beta} = 0$ and

$$\begin{aligned}\frac{\partial^4 g^{\gamma\nu}}{\partial v^\lambda \partial v^\beta \partial v^\alpha \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu} &= \frac{\partial^3 g^{\gamma\nu}}{\partial v^\beta \partial v^\alpha \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu}\\ &= \frac{\partial^2 g^{\gamma\nu}}{\partial v^\alpha \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu} = \frac{\partial g^{\gamma\nu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\nu} = 0.\end{aligned} \tag{5.12}$$

Inserting (5.10)–(5.12) and $E_{ij} = E_{\alpha\beta} = 0$ into (5.4) and (5.8) implies that $C_{ijk} = C_{\alpha\beta\lambda} = 0$. Therefore (M_1, F_1) and (M_2, F_2) are Riemannian, and consequently $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ is Riemannian. ∎

From Theorem 5.2, we obtain

COROLLARY 5.3. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a DWP-Finsler manifold.*

- (i) *If f_2 is constant on M_2, then F is a Douglas metric if and only if F_2 is a Riemannian metric, F_1 is a Berwald metric and $\frac{\partial g^{sh}}{\partial y^s}\frac{\partial f_1^2}{\partial x^h} = 0$.*
- (ii) *If f_1 is constant on M_1, then F is a Douglas metric if and only if F_1 is a Riemannian metric, F_2 is a Berwald metric and $\frac{\partial g^{\gamma\lambda}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\lambda} = 0$.*

Finally, we consider warped product Finsler manifolds with vanishing Douglas curvature:

COROLLARY 5.4. *The WP-Finsler manifold $(M_1 \times {}_{f_1}M_2, F)$ is a Douglas manifold if and only if F_2 is a Riemannian metric, F_1 is a Berwald metric and $\frac{\partial g^{sh}}{\partial y^s}\frac{\partial f_1^2}{\partial x^h} = 0$.*

Proof. By Lemma 5.1. ∎

6. Relatively isotropic Landsberg DWP-Finsler manifolds. In this section, we prove that for a proper DWP-Finsler manifold the notions of being a Landsberg manifold and of being a Berwald manifold are equivalent. Then we study DWP-Finsler metrics with relatively isotropic Landsberg curvature.

LEMMA 6.1. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a DWP-Finsler manifold. Then the Landsberg curvature of F is as follows:*

$$\mathbf{L}_{ijk} = f_2^2 L_{ijk} + \frac{1}{8} f_2^2 y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} F_2^2 + \frac{1}{2} C_{ijk} v^\alpha \frac{\partial f_2^2}{\partial u^\alpha}, \tag{6.1}$$

$$\mathbf{L}_{ij\lambda} = \frac{1}{4} y_l \frac{\partial^2 g^{lh}}{\partial y^i \partial y^j} \frac{\partial f_1^2}{\partial x^h} v_\lambda + \frac{1}{4} g_{ij} v_\gamma \frac{\partial g^{\alpha\gamma}}{\partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha}, \tag{6.2}$$

$$\mathbf{L}_{i\beta\lambda} = \frac{1}{4} y_l \frac{\partial g^{lh}}{\partial y^i} \frac{\partial f_1^2}{\partial x^h} g_{\beta\lambda} + \frac{1}{4} v_\gamma \frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha} y_i, \tag{6.3}$$

$$\mathbf{L}_{\alpha\beta\lambda} = f_1^2 L_{\alpha\beta\lambda} + \frac{1}{8} f_1^2 v_\gamma \frac{\partial^3 g^{\gamma\nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 + \frac{1}{2} C_{\alpha\beta\lambda} y^h \frac{\partial f_1^2}{\partial x^h}. \tag{6.4}$$

Proof. Ise the definition of Landsberg curvature and (3.1)–(3.8). ■

PROPOSITION 6.2. *Every proper DWP-Finsler manifold $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ with vanishing Landsberg curvature is Riemannian.*

Proof. Let the Landsberg curvature tensor of $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be zero. Then by using (6.1) we obtain

$$f_2^2 L_{ijk} + \frac{1}{8} f_2^2 y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} F_2^2 + \frac{1}{2} C_{ijk} v^\alpha \frac{\partial f_2^2}{\partial u^\alpha} = 0. \tag{6.5}$$

Differentiating (6.5) with respect to v^γ implies that

$$\frac{1}{4} f_2^2 y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} v_\gamma + \frac{1}{2} C_{ijk} \frac{\partial f_2^2}{\partial u^\gamma} = 0. \tag{6.6}$$

By differentiating (6.6) with respect v^λ, we have

$$y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} g_{\gamma\lambda} = 0,$$

and consequently

$$y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} = 0.$$

Thus (6.5) reduces to

$$f_2^2 L_{ijk} + \frac{1}{2} C_{ijk} v^\alpha \frac{\partial f_2^2}{\partial u^\alpha} = 0. \tag{6.7}$$

Differentiating (6.7) with respect to v^β gives

$$C_{ijk} \frac{\partial f_2^2}{\partial u^\beta} = 0,$$

and consequently $C_{ijk} = 0$. Thus (M_1, F_1) is a Riemannian manifold. In a similar way, we can conclude that (M_2, F_2) is Riemannian. ■

Using Proposition 6.2 and Lemma 3.1, we get

THEOREM 6.3. *A proper DWP-Finsler manifold is Landsbergian if and only if it is Berwaldian.*

Now, let f_1 be non-constant on M_1 and f_2 be constant on M_2. Then as in the proof of Proposition 6.2, we conclude that (M_2, F_2) is a Riemannain manifold. Also, from (6.7) we conclude $L_{ijk} = 0$, because f_2 is constant. Thus we obtain

THEOREM 6.4. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a proper DWP-Finsler manifold.*

(i) *If f_2 is constant and f_1 is not constant, then $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ is a Landsberg manifold if and only if (M_1, F_1) is a Landsberg manifold, (M_2, F_2) is Riemannian and*

$$y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1}{\partial x^h} = 0. \tag{6.8}$$

(ii) *If f_1 is constant and f_2 is not constant, then $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ is a Landsberg manifold if and only if (M_2, F_2) is a Landsberg manifold, (M_1, F_1) is Riemannian and*

$$v_\gamma \frac{\partial^3 g^{\gamma\nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} \frac{\partial f_2}{\partial u^\nu} = 0. \tag{6.9}$$

Theorem 6.4 yields

COROLLARY 6.5. *A WP-Finsler manifold $(M_1 \times {}_{f_1}M_2, F)$ is a Landsberg manifold if and only if (M_1, F_1) is Landsberg, (M_2, F_2) is Riemannian and*

$$C^h{}_{kj} \frac{\partial f_1}{\partial x^h} = 0. \tag{6.10}$$

Proof. It suffices to show that (6.4) implies (6.10). Multiplying (6.4) with y^i implies that

$$y_l \frac{\partial C^{lh}{}_j}{\partial y^k} \frac{\partial f_1}{\partial x^h} = 0. \tag{6.11}$$

Using $y_l C^{lh}{}_j = 0$ and $\frac{\partial y_l}{\partial y^k} = g_{lk}$, one can obtain (6.10). ∎

Now, we deal with DWP-Finsler manifolds with relatively isotropic Landsberg metric.

THEOREM 6.6. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a DWP-Finsler manifold. Suppose that F is a relatively isotropic Landsberg metric. Then F is a Landsberg metric.*

Proof. Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a relatively isotropic Landsberg manifold. Then by (6.1), we have

$$f_2^2 L_{ijk} + \frac{1}{8} f_2^2 y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} F_2^2 + \frac{1}{2} C_{ijk} v^\alpha \frac{\partial f_2^2}{\partial u^\alpha} = cF f_2^2 C_{ijk}. \tag{6.12}$$

By differentiating (6.12) with respect v^γ and v^λ, one obtains

$$\frac{1}{4}y_l\frac{\partial^3 g^{lh}}{\partial y^i\partial y^j\partial y^k}\,\frac{\partial f_1^2}{\partial x^h}g_{\gamma\lambda}=cF^{-1}\mathbf{h}_{\gamma\lambda}C_{ijk}. \tag{6.13}$$

Contracting (6.13) with $g^{\gamma\lambda}$ implies that

$$\frac{n_2}{4}y_l\frac{\partial^3 g^{lh}}{\partial y^i\partial y^j\partial y^k}\,\frac{\partial f_1^2}{\partial x^h}=(n-1)cF^{-1}C_{ijk}. \tag{6.14}$$

Differentiating (6.14) with respect to v^β gives $(n-1)c(F^{-1})_{v^\beta}C_{ijk}=0$, and so $c=0$. Thus F reduces to a Landsberg metric. ∎

From Proposition 6.2 and Theorem 6.6, we deduce

Corollary 6.7. *Every proper DWP-Finsler manifold with relatively isotropic Landsberg curvature is Riemannian.*

7. Relatively isotropic mean Landsberg DWP-Finsler manifolds. In this section, we consider DWP-Finsler metrics with relatively isotropic mean Landsberg curvature. First, by the definition of mean Landsberg curvature and Lemma 6.1, we get

Lemma 7.1. *Let $({}_{f_2}M_1\times{}_{f_1}M_2,F)$ be a DWP-Finsler manifold. Then the mean Landsberg curvature of F is as follows:*

$$\begin{aligned}\mathbf{J}_i&=\frac{1}{f_2^2}g^{jk}\mathbf{L}_{ijk}+\frac{1}{f_1^2}g^{\beta\lambda}\mathbf{L}_{i\beta\lambda}\\&=J_i+\frac{y_lg^{jk}}{8}\frac{\partial^3 g^{lh}}{\partial y^i\partial y^j\partial y^k}\frac{\partial f_1^2}{\partial x^h}F_2^2+\frac{I_iv^\nu}{2f_2^2}\frac{\partial f_2^2}{\partial u^\nu}+\frac{v_\gamma g^{\beta\lambda}}{4}\frac{\partial^2 g^{\nu\gamma}}{\partial v^\beta\partial v^\lambda}\frac{\partial f_2^2}{\partial u^\nu}y_i,\end{aligned} \tag{7.1}$$

$$\begin{aligned}\mathbf{J}_\alpha&=\frac{1}{f_2^2}g^{jk}\mathbf{L}_{\alpha jk}+\frac{1}{f_1^2}g^{\beta\lambda}\mathbf{L}_{\alpha\beta\lambda}\\&=J_\alpha+\frac{v_\gamma g^{\beta\lambda}}{8}\frac{\partial^3 g^{\gamma\nu}}{\partial v^\alpha\partial v^\beta\partial v^\lambda}\frac{\partial f_2^2}{\partial u^\nu}F_1^2+\frac{I_\alpha y^h}{2f_1^2}\frac{\partial f_1^2}{\partial x^h}+\frac{y_lg^{jk}}{4}\frac{\partial^2 g^{lh}}{\partial y^j\partial y^k}\frac{\partial f_1^2}{\partial x^h}v_\alpha.\end{aligned} \tag{7.2}$$

Theorem 7.2. *Let $({}_{f_2}M_1\times{}_{f_1}M_2,F)$ be a DWP-Finsler manifold.*

(i) *If f_1 is constant and f_2 is not constant, then $({}_{f_2}M_1\times{}_{f_1}M_2,F)$ is a weakly Landsberg manifold if and only if (M_1,F_1) is Riemannian, (M_2,F_2) is weakly Landsbergian and*
$$v_\gamma\frac{\partial^3 g^{\gamma\nu}}{\partial v^\alpha\partial v^\beta\partial v^\lambda}g^{\beta\lambda}\frac{\partial f_2^2}{\partial u^\nu}=0.$$

(ii) *If f_2 is constant and f_1 is not constant, then $({}_{f_2}M_1\times{}_{f_1}M_2,F)$ is a weakly Landsberg manifold if and only if (M_1,F_1) is weakly Landsbergian, (M_2,F_2) is Riemannian and*
$$y_l\frac{\partial^3 g^{lh}}{\partial y^i\partial y^j\partial y^k}g^{jk}\frac{\partial f_1^2}{\partial x^h}=0.$$

Proof. Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a weakly Landsberg manifold and f_1 be constant on M_1. Then by (7.1) and (7.2), we have

$$J_i + \frac{1}{2f_2^2} I_i v^\nu \frac{\partial f_2^2}{\partial u^\nu} + \frac{1}{4} v_\gamma \frac{\partial^2 g^{\nu\gamma}}{\partial v^\beta \partial v^\lambda} g^{\beta\lambda} \frac{\partial f_2^2}{\partial u^\nu} y_i = 0, \tag{7.3}$$

$$J_\alpha + \frac{1}{8} v_\gamma \frac{\partial^3 g^{\gamma\nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} g^{\beta\lambda} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 = 0. \tag{7.4}$$

By differentiating (7.4) with respect to y^i, we get

$$v_\gamma \frac{\partial^3 g^{\gamma\nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} g^{\beta\lambda} \frac{\partial f_2^2}{\partial u^\nu} = 0. \tag{7.5}$$

Contracting (7.5) with v^α gives

$$v_\gamma \frac{\partial^2 g^{\gamma\nu}}{\partial v^\beta \partial v^\lambda} g^{\beta\lambda} \frac{\partial f_2^2}{\partial u^\nu} = 0. \tag{7.6}$$

Inserting (7.6) into (7.3) implies that

$$J_i + \frac{1}{2f_2^2} I_i v^\nu \frac{\partial f_2^2}{\partial u^\nu} = 0. \tag{7.7}$$

By differentiating (7.7) with respect v^β, we conclude that $I_i = 0$, i.e., (M_1, F_1) is a Riemannian manifold. By inserting (7.5) into (7.2), we get $J_\alpha = 0$, i.e., (M_2, F_2) is a weakly Landsberg manifold. ∎

From Theorem 7.2, we deduce

COROLLARY 7.3. *A proper WP-Finsler manifold $(M_1 \times {}_{f_1}M_2, F)$ is a weakly Landsberg manifold if and only if (M_1, F_1) is weakly Landsberg, (M_2, F_2) is Riemannian and*

$$y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} g^{jk} \frac{\partial f_1^2}{\partial x^h} = 0.$$

Now, we consider DWP-Finsler manifolds with relatively isotropic mean Landsberg curvature.

THEOREM 7.4. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a DWP-Finsler manifold with relatively isotropic mean Landsberg curvature. If f_1 is constant on M_1 (resp. f_2 is constant on M_2), then the DWP-Finsler manifold is a weakly Landsberg manifold.*

Proof. Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a relatively isotropic mean Landsberg manifold and f_1 be constant on M_1. Then by (7.4) we have

$$J_\alpha + \frac{1}{8} v_\gamma \frac{\partial^3 g^{\gamma\nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} g^{\beta\lambda} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 = cFI_\alpha. \tag{7.8}$$

Differentiating (7.8) with respect to y^k implies that

$$\frac{1}{4} v_\gamma \frac{\partial^3 g^{\gamma\nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} g^{\beta\lambda} \frac{\partial f_2^2}{\partial u^\nu} y_k = c \frac{f_2^2 y_k}{F} I_\alpha. \tag{7.9}$$

Contracting (7.9) with y^k gives

$$\frac{1}{4}v_\gamma \frac{\partial^3 g^{\gamma\nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} g^{\beta\lambda} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 = c\frac{f_2^2 F_1^2}{F} I_\alpha. \tag{7.10}$$

By inserting (7.10) into (7.8), it follows that

$$2J_\alpha + c\bigg(\frac{f_2^2 F_1^2}{F} - 2F\bigg) I_\alpha = 0. \tag{7.11}$$

By differentiating (7.11) with respect to y^k, we obtain $c\frac{f_2^4 F_1^2}{F^3} y_k I_\alpha = 0$. Therefore, $c = 0$ and F reduces to a weakly Landsberg metric. ∎

COROLLARY 7.5. *Every WP-manifold $(M_1 \times {}_{f_1}M_2, F)$ with relatively isotropic mean Landsberg curvature is a weakly Landsberg manifold.*

8. Locally dually flat DWP-Finsler manifolds. Dually flat Finsler metrics form a special and useful class of Finsler metrics in Finsler information geometry, which play an important role in studying flat Finsler information structure. In this section, we study locally dually flat DWP-Finsler metrics. We recall that a Finsler metric $F = F(x, y)$ on a manifold M is *locally dually flat* if at any point there is a standard coordinate system (x^i, y^i) in TM such that

$$\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k = 2\frac{\partial F^2}{\partial x^l}. \tag{8.1}$$

In this case, the coordinate system (x^i) is called *adapted.*

THEOREM 8.1. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, F)$ be a DWP-Finsler manifold. Then F is locally dually flat if and only if F_1 and F_2 are locally dually flat and f_1 and f_2 are constant.*

Proof. Let $({}_{f_1}M_1 \times {}_{f_2}M_2, F)$ be a locally dually flat doubly DWP-Finsler manifold. Then

$$f_2^2 \frac{\partial^2 F_1^2}{\partial x^k \partial y^l} y^k + \frac{\partial f_2^2}{\partial u^\alpha} \frac{\partial F_1^2}{\partial y^l} v^\alpha = 2f_2^2 \frac{\partial F_1^2}{\partial x^l} + 2\frac{\partial f_1^2}{\partial x^l} F_2^2, \tag{8.2}$$

$$\frac{\partial f_1^2}{\partial x^k} \frac{\partial F_2^2}{\partial v^\beta} y^k + f_1^2 \frac{\partial^2 F_2^2}{\partial u^\alpha \partial v^\beta} v^\alpha = 2f_1^2 \frac{\partial F_2^2}{\partial u^\beta} + 2\frac{\partial f_2^2}{\partial u^\beta} F_1^2. \tag{8.3}$$

Differentiating (8.2) with respect to v^γ and then with respect to y^k and using non-singularity of g_{ij} yields

$$\frac{\partial f_2}{\partial u^\gamma} = 0,$$

which means that f_2 is constant. Similarly, f_1 is constant. In this case, (8.2) and (8.3) reduce to

$$\frac{\partial^2 F_1^2}{\partial x^k \partial y^l} y^k = \frac{\partial F_1^2}{\partial x^l}, \tag{8.4}$$

$$\frac{\partial^2 F_2^2}{\partial u^\alpha \partial v^\beta} v^\alpha = 2\frac{\partial F_2^2}{\partial u^\beta}. \tag{8.5}$$

Hence F_1 and F_2 are locally dually flat. ∎

From Theorem 8.1, we deduce

COROLLARY 8.2. *There is no locally dually flat proper DWP-Finsler manifold.*

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Esmaeil Peyghan
Department of Mathematics
Faculty of Science
Arak University
Arak 38156-8-8349, Iran
E-mail: epeyghan@gmail.com

Akbar Tayebi
Department of Mathematics
Faculty of Science
University of Qom
Qom, Iran
E-mail: akbar.tayebi@gmail.com

Behzad Najafi
Department of Mathematics
Faculty of Science
Shahed University
Tehran, Iran
E-mail: najafi@shahed.ac.ir

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