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On the attractors of Feigenbaum maps

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Abstract. A solution of the Feigenbaum functional equation is called a Feigenbaum map. We investigate the likely limit set (i.e. the maximal attractor in the sense of Milnor) of a non-unimodal Feigenbaum map, prove that it is a minimal set that attracts almost all points, and then estimate its Hausdorff dimension. Finally, for every $s \in (0,1)$, we construct a non-unimodal Feigenbaum map with a likely limit set whose Hausdorff dimension is s.

1. Introduction. In order to explain the famous phenomenon of quantitative universality of approaching chaos via period doubling bifurcation, Mitchell J. Feigenbaum [F2] introduced in 1978 the Feigenbaum functional equation $g^2(-\lambda x) + \lambda g(x) = 0$. This attracted many scholars, and a lot of research on the existence and dynamical properties of solutions of this equation was undertaken. In [HWL], we discussed the dynamical properties of a class of unimodal Feigenbaum maps, estimated the Hausdorff dimension of the likely limit set (the maximal attractor in the sense of Milnor) for a unimodal Feigenbaum map, and proved that for every $s \in (0,1)$, there always exists a unimodal Feigenbaum map such that the Hausdorff dimension of the likely limit set is s. We also discussed the kneading sequences of unimodal Feigenbaum maps.

In this article, we study a class of non-unimodal Feigenbaum maps similar to [HWL], show that their likely limit sets are minimal sets, and then consider the Hausdorff dimension of the likely limit set. We prove that for every $s \in (0,1)$ there exists a non-unimodal Feigenbaum map with likely limit set whose Hausdorff dimension is exactly s.

The main results are Theorems 3.1 and 3.3.

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2. Basic definitions and preparations. X is always a compact metric space. Let $x \in X$ and $A \subset X$. Define the *distance* between x and A by

$$\rho(x, A) = \inf\{\rho(x, a) : a \in A\}.$$

If $A \subset X$ and $\varepsilon > 0$, define the *ε-neighbourhood* of A by

$$A_\varepsilon = \{x \in X : \rho(x, A) < \varepsilon\}.$$

Let $\mathcal{B}$ be the class of non-empty closed bounded subsets of X. Define the *Hausdorff metric* d on $\mathcal{B}$ by

$$d(A, B) = \sup\{\rho(a, B), \rho(b, A) : a \in A,\, b \in B\}.$$

$|E|$ denotes the diameter of a subset of X, i.e., $|E| = \sup\{\rho(x, y) : x, y \in E\}$.

If $E \subset \bigcup_i U_i$ and $0 < |U_i| \le \delta$ for each i, we say that $\{U_i\}$ is a *δ-cover* of E.

Let $E \subset X$ and $0 \le s < \infty$. For $\delta > 0$, define

$$\mathcal{H}^s_\delta(E) = \inf \sum_{i=1}^{\infty} |U_i|^s,$$

where the infimum is over all (countable) δ-covers $\{U_i\}$ of E.

The *Hausdorff s-dimensional outer measure* of E is defined by

$$\mathcal{H}(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E).$$

Then there is a unique value, $\dim E$, called the *Hausdorff dimension* of E, such that

$$\mathcal{H}^s(E) = \infty \quad \text{if } 0 \le s < \dim E, \qquad \mathcal{H}^s(E) = 0 \quad \text{if } \dim E < s < \infty.$$

A mapping $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is called a *contraction* if there exists $c < 1$ such that $|\psi(x) - \psi(y)| \le c|x - y|$ for all $x, y \in \mathbb{R}^n$. We call the infimum of such c the *ratio* of the contraction.

We call a set $E \subset \mathbb{R}^n$ *invariant* for a set of contractions $\psi_1, \ldots, \psi_m$ if

$$E = \bigcup_{j=1}^{m} \psi_j(E).$$

We will use the following lemmas.

Lemma 2.1 (see [H]). *Let $\varphi_1, \ldots, \varphi_m$ be contractions on $\mathbb{R}^n$ with contraction ratios $r_j < 1$. Then there exists a unique non-empty compact set E such that*

$$E = \varphi(E) = \bigcup_{i=1}^{m} \varphi_i(E).$$

Further, if F is any non-empty compact subset of $\mathbb{R}^n$, the iterates $\varphi^k(F)$ converge to E in the Hausdorff metric as $k \to \infty$.

LEMMA 2.2 (see [F1]). *Let* $\{\varphi_i\}_{i=1}^m$ *be contractions on* $\mathbb{R}$ *for which the open set condition holds, i.e., there is an open interval* V *such that*

(1) $\varphi(V) = \bigcup_{i=1}^m \varphi_i(V) \subset V$,
(2) $\varphi_1(V), \dots, \varphi_m(V)$ *are pairwise disjoint.*

Moreover, suppose that for each i, *there exist* q_i, r_i *such that*

$$q_i|x-y| \le |\varphi_i(x) - \varphi_i(y)| \le r_i|x-y|$$

for all $x, y \in \overline{V}$. *Then* $s \le \dim E \le t$, *where* s *and* t *are defined by* $\sum_{i=1}^m q_i^s = 1 = \sum_{i=1}^m r_i^s$.

DEFINITION 2.3. Let M be a compact manifold (possibly with boundary), and f a continuous map of M into itself. The *likely limit set* $\Lambda = \Lambda(f)$ of f is the smallest closed subset of M with $\omega(x, f) \subset \Lambda$ for Lebesgue almost every $x \in M$ ($\omega(x, f)$ denotes the ω-limit set of x under f).

As described in [M], the likely limit set always exists and it is the unique maximal attractor which contains all the others (in the sense of Milnor).

Let $I = [0, 1]$. A set $E \subset I$ is called a *minimal set* of f if $E \neq \emptyset$ and $\omega(x, f) = E$ for any $x \in E$. As is well known, the minimal set is a non-empty, closed and invariant subset of f, and it has no proper subset with these three properties (see [BC]). Therefore, if E is a minimal set with $\omega(x, f) \subset E$ for almost all $x \in I$, then $E = \Lambda(f)$.

In 1978, Mitchell J. Feigenbaum [F2] put forward the *Feigenbaum functional equation*

$$\begin{cases} g^2(-\lambda x) = -\lambda g(x), \\ g(0) = 1, \quad -1 \le g(x) \le 1, \quad x \in [-1, 1], \end{cases} \tag{2.1}$$

where $\lambda \in (0, 1)$ is to be determined.

In 1985, L. Yang and J. Z. Zhang [YZ] proposed another Feigenbaum type functional equation,

$$\begin{cases} f^2(\lambda x) = \lambda f(x), \\ f(0) = 1, \quad 0 \le f(x) \le 1, \quad x \in [0, 1]. \end{cases} \tag{2.2}$$

where $\lambda \in (0, 1)$ is to be determined.

There is a close link between solutions of these two types of equations.

LEMMA 2.4.

(1) *If* $g(x)$ *is a non-unimodal solution of* (2.1), *then* $f(x) = |g(x)|$ ($x \in [0, 1]$) *is a non-unimodal solution of* (2.2).
(2) *If* $f(x)$ *is a non-unimodal solution of* (2.2), *then there is* $\mu \in (0, 1)$ *such that* $f(\mu) = 0$ *is the minimum value, and* $g(x) = (\mathrm{sgn}(\mu - |x|))f(|x|)$ *is a non-unimodal solution of* (2.1).

Definition 2.5. A continuous solution of (2.2) is said to be *a non-unimodal Feigenbaum map* if $f|_{[\lambda,1]}$ is *univallecular* (i.e., there exists $\mu \in (\lambda, 1)$ such that $f|_{[\lambda,\mu]}$ strictly decreases and $f|_{[\mu,1]}$ strictly increases), but f is non-univallecular.

Non-unimodal Feigenbaum maps have the following properties.

Lemma 2.6 (see [L]). *Let $f : I \to I$ be a non-unimodal Feigenbaum map. Then*

(1) $f(1) = \lambda, f^2(\lambda) = \lambda^2$.
(2) *If μ is the minimum point of f on $[\lambda, 1]$, then*
   (a) $f(x) = 0 \Leftrightarrow x = \mu$,
   (b) $f(\lambda) > \mu$,
   (c) $f(x) = \lambda x$ *has only one solution* $x = 1$ *on* $[\mu, 1]$.

Conversely, if a continuous univallecular map $f_0 : [\lambda, 1] \to I$ satisfies (1) *and* (2), *then it can be uniquely extended to a non-unimodal Feigenbaum map on I.*

3. Main theorems and their proofs. In this section, we investigate attractors of non-unimodal Feigenbaum maps. We have

Main Theorem 3.1. *Let f be a non-unimodal Feigenbaum map. If $f'(x) < -1$ for $x \in [\lambda, \mu]$, and $f'(x) \geq 1$ for $x \in [\mu, 1]$* (*considering the left or right derivative at the end points*), *where λ, μ are as above, then there exists a set of contractions such that its invariant set is the likely limit set and a minimal set of f.*

Proof. By Lemma 2.6, $f([0, \lambda]) = [\mu, 1]$. To show that $f([\mu, 1]) = [0, \lambda]$, it is enough to show that μ is the minimum value of f on $[0, \lambda]$. We know

$$f(x) = \mu \;\Leftrightarrow\; f^2(x) = 0 \;\Leftrightarrow\; \lambda f(x/\lambda) = 0 \;\Leftrightarrow\; x/\lambda = \mu \;\Leftrightarrow\; x = \lambda\mu.$$

Combining $f(0) = 1 > \mu$ with $f(\lambda) > \mu$, we infer that $f(y) > \mu$ for every $y \in [0, \lambda]$ except $y = \lambda\mu$. This shows that $\lambda\mu$ is the minimum point and μ is the minimum value of f on $[0, \lambda]$.

Define $\varphi_1, \varphi_2 : I \to I$ by setting $\varphi_1(x) = \lambda x$, $\varphi_2(x) = [\mu, 1] \cap f^{-1}(\lambda x)$ for any $x \in I$. It is easy to see that φ_1, φ_2 are both contractions. Since

$$\varphi_1((0, 1)) = (0, \lambda) \subset (0, 1), \quad \varphi_2((0, 1)) = (\mu, 1) \subset (0, 1),$$

and $(0, \lambda) \cap (\mu, 1) = \emptyset$, it follows that $\varphi = \varphi_1 \cup \varphi_2$ satisfies the open set condition. By Lemma 2.1, there exists a unique non-empty compact set E such that

$$E = \varphi(E) = \varphi_1(E) \cup \varphi_2(E).$$

For simplicity, we write $\varphi_{i_1 \cdots i_k}$ for $\varphi_{i_1} \circ \cdots \circ \varphi_{i_k}$.

We will get $E = \Lambda(f)$ by showing that E is a minimal set which attracts almost all points. To this end, we need the following three claims; the proofs are similar to those in [HWL].

CLAIM 1. *For any* $x \in I$, $f \circ \varphi_1(x) = \varphi_2 \circ f(x)$ *and* $f \circ \varphi_2(x) = \varphi_1(x)$.

CLAIM 2. *For any* $k > 0$, $\varphi^k(I) = \bigcup_{i_1,\ldots,i_k=1}^{2} \varphi_{i_1\cdots i_k}(I)$ *is a forward invariant set of* f, *i.e.* $f(\varphi^k(I)) \subset \varphi^k(I)$.

CLAIM 3. *For any subsets* $\varphi_{i_1\cdots i_k}(I)$ *and* $\varphi_{j_1\cdots j_k}(I)$, *there is* $n > 0$ *such that*

$$f^n \circ \varphi_{i_1\cdots i_k}(I) = \varphi_{j_1\cdots j_k}(I).$$

We will show that (1) and (2) below hold.

(1) *For almost all* $x \in I$, $\omega(x, f) \subset E$.

It is obvious that f has a unique fixed point $e \in (\lambda, \mu)$ on I. Since $f^2(\lambda x) = \lambda f(x)$ implies $f^{2^k}(\lambda^k x) = \lambda^k f(x)$, it follows that $\lambda^k e$ is a fixed point of f^{2^k}. Let

$$A = \bigcup_{k=0}^{\infty} O(\lambda^k e, f),$$

where $O(\lambda^k e, f)$ denotes the orbit of $\lambda^k e$ under f. Then A is a countable set. Let

$$B = \bigcup_{i=0}^{\infty} f^{-i}(A).$$

Because $f^{-1}(x)$ has at most two points for every $x \in I$, and A is countable it follows that B is countable. In particular, B has Lebesgue measure zero.

Let $x \in I - B$. If $x \in \varphi(I)$, then $f^{N_1}(x) \in \varphi(I)$ for $N_1 = 0$. If $x \notin \varphi(I)$, then $x \in (\lambda, \mu)$. Obviously, $x \neq e$. As $f'(y) < -1$ for every $y \in [\lambda, \mu]$, there must be some $N_1 > 0$ such that $f^{N_1}(x) \notin [\lambda, \mu]$, so $f^{N_1}(x) \in \varphi(I)$. Thus there is always $N_1 \geq 0$ such that $f^{N_1}(x) \in \varphi(I)$.

Suppose that, for $k = p$, there is $N_p \geq 0$ such that $f^{N_p}(x) \in \varphi^p(I)$. It is easy to see that $f^{N_p}(x)$ belongs to some $\varphi_{i_1\cdots i_p}(I)$. By Claim 3, there exists $l > N_p$ such that

$$f^l(x) \in \varphi_{11\cdots 1}(I) = [0, \lambda^p].$$

$f^l(x) \neq \lambda^p e$ implies $\lambda^{-p} f^l(x) \neq e$, so there is $N_1 \geq 0$ such that $f^{N_1}(\lambda^{-p} f^l(x)) \in \varphi(I)$. By (2.2), we have

$$\begin{aligned} f^{N_1\cdot 2^p + l}(x) &= f^{N_1\cdot 2^p}(\lambda^p(\lambda^{-p} f^l(x))) = f^{N_1\cdot 2^p} \circ \varphi_{11\cdots 1}(\lambda^{-p} f^l(x)) \\ &= \varphi_{11\cdots 1} \circ f^{N_1}(\lambda^{-p} f^l(x)) \in \varphi_{11\cdots 1} \circ \varphi(I) \subset \varphi^{p+1}(I). \end{aligned}$$

By induction on k, for every $k \geq 1$, there is $N_k \geq 0$ such that

$$f^{N_k}(x) \in \varphi^k(I).$$

Moreover, by Claim 2, for every $n \geq N_k$,

$$f^n(x) \in \varphi^k(I).$$

As $\varphi^k(I) \to E$, $f^n(x) \to E$. Since E is compact, it is a closed set. Therefore, $\omega(x, f) \subset E$.

(2) *E is a minimal set of f.*

It is easy to see that the contraction ratio of φ_1 is λ, and the one of φ_2 is not more than λ, thus the ratio of $\varphi_{i_1\cdots i_k}$ is not more than λ^k. When $k \to \infty$, $|\varphi_{i_1\cdots i_k}(I)|$ uniformly converges to zero for $i_r \in \{1, 2\}$, $r = 1, \ldots, k$. Let $x \in I$. For every $y \in E$ and every open set V containing y, there exists some $\varphi_{j_1\cdots j_k}(I) \subset V$. As $\varphi(I) \subset I$, we have $\varphi^{i+1}(I) = \varphi^i \circ \varphi(I) \subset \varphi^i(I)$ for any $i \geq 0$. Hence, it is not difficult to prove that

$$E = \bigcap_{i=0}^{\infty} \varphi^i(I). \tag{3.1}$$

Since $x \in E$ implies $x \in \varphi^k(I)$, it follows that x belongs to some $\varphi_{i_1\cdots i_k}(I)$. By Claim 3, there is $n > 0$ such that

$$f^n(x) \in f^n(\varphi_{i_1\cdots i_k}(I)) = \varphi_{j_1\cdots j_k}(I) \subset V.$$

This shows $y \in \omega(x, f)$. Therefore,

$$E \subset \omega(x, f). \tag{3.2}$$

By (3.1) and Claim 2, $f(E) \subset E$. Moreover, E is a closed set, so

$$\omega(x, f) \subset E. \tag{3.3}$$

By (3.2) and (3.3), we obtain $\omega(x, f) = E$. Because x is arbitrary, E is a minimal set of f.

If we combine (1) with (2), we see that $E = \Lambda(f)$. ∎

Corollary 3.2. *If f is a non-unimodal Feigenbaum map as in Theorem* 3.1, *then*

$$s \leq \dim \Lambda(f) \leq t,$$

where

$$\lambda^s \Big(1 + \inf_{x \in [\mu, 1]} f'(x)\Big)^{-s} = 1 = \lambda^t \Big(1 + \sup_{x \in [\mu, 1]} f'(x)\Big)^{-t}$$

and $\dim(\cdot)$ *denotes Hausdorff dimension.*

Proof. By Theorem 3.1 and Lemma 2.2, we have

$$\varphi_1'(x) = \lambda, \quad \varphi_2'(x) = \lambda(f'_{[\mu,1]}(\varphi_2(x)))^{-1}.$$

So $s \le \dim \Lambda(f) \le t$, where

$$\text{(3.4)} \quad \Big(\inf_{x\in I} |\varphi_1'(x)|\Big)^s + \Big(\inf_{x\in I} |\varphi_2'(x)|\Big)^s = 1 = \Big(\sup_{x\in I} |\varphi_1'(x)|\Big)^t + \Big(\sup_{x\in I} |\varphi_2'(x)|\Big)^t.$$

Then (3.4) becomes

$$\lambda^s\Big(1+\inf_{x\in[\mu,1]} f'(x)\Big)^{-s} = 1 = \lambda^t\Big(1+\sup_{x\in[\mu,1]} f'(x)\Big)^{-t}. \ \blacksquare$$

Main Theorem 3.3. *For every $s \in (0,1)$, there always exists a non-unimodal Feigenbaum map f such that $\dim \Lambda(f) = s$.*

Proof. For any $0 < s < 1$, let

$$\lambda = e^{-\ln 2/s}, \quad \mu = 1-\lambda.$$

Since $\ln 2/s > \ln 2$ implies $e^{\ln 2/s} > 2$, it follows that $\lambda < \mu < 1$ and $\lambda < 1/2$.

Now let $f_0 : [\lambda,1] \to I$ be defined by $f_0(\mu) = 0$, $f_0(1) = \lambda$, $f_0(\lambda) = 1-\lambda+\lambda^2$, and f_0 be linear on $[\lambda,\mu]$ and $[\mu,1]$.

By Lemma 2.4, f_0 can be uniquely extended to a continuous non-univallecular solution of (2.2), denoted by f. A simple calculation gives $f'(x)=1$ for $x \in [\mu,1]$ and $f'(x) = (1-\lambda+\lambda^2)/(2\lambda-1) = -1+(\lambda+\lambda^2)/(2\lambda-1) < -1$ for $x \in [\lambda,\mu]$. By Corollary 3.2, we get $\dim \Lambda(f) = s$. $\blacksquare$

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