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The Lax–Phillips infinitesimal generator and the scattering matrix for automorphic functions

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Abstract. We study the infinitesimal generator of the Lax–Phillips semigroup of the automorphic scattering system defined on the Poincaré upper half-plane for $\mathrm{SL}_2(\mathbb{Z})$. We show that its spectrum consists only of the poles of the resolvent of the generator, and coincides with the poles of the scattering matrix, counted with multiplicities. Using this we construct an operator whose eigenvalues, counted with algebraic multiplicities (i.e. dimensions of generalized eigenspaces), are precisely the non-trivial zeros of the Riemann zeta function. We give an operator model on $L^2(\mathbb{R})$ of this generator as explicit as possible. We obtain a condition equivalent to the Riemann hypothesis in terms of cyclic vectors for a weak resolvent of the scattering matrix.

1. Introduction. Since the scattering theoretic view of the theory of automorphic functions was suggested by Gelfand [Ge] in 1962, Pavlov and Faddeev [PavFa] showed in 1972 that the Lax–Phillips scattering theory, applied to the non-Euclidean wave equation, is a natural tool in the theory of automorphic functions. This was taken up and further studied by Lax and Phillips themselves and culminated in their monograph [LP1] and its important supplement [LP2].

In [LP1], the poles of the scattering matrix of the non-Euclidean wave equation on the Poincaré upper half-plane are related to the poles of the resolvent (and so to the eigenvalues) of the infinitesimal generator of the Lax–Phillips semigroup. See also [LP2, Cor. 4.3]. In this paper we study this relation in more detail.

To make the paper as self-contained as possible, we begin in §2 with a description of the Lax–Phillips scattering theory for automorphic functions on the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. Then in §§3 and 4 we develop a general spectral theory of discrete-time and continuous-time Lax–Phillips scattering, respectively, except that a unitary factor of the scattering matrix we

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have in mind consists of the non-trivial zeros of the Riemann zeta-function. The only result we use from scattering theory is a translation representation theorem (see e.g. Theorem 1.1 of Ch. II in [LP3]) for the discrete scattering system. In §§5 and 6 we treat scattering theory in the setting of automorphic functions.

In §2, we briefly review the Lax–Phillips scattering theory for automorphic functions. For more details, see [LP1], [LP2], Epilogue of [LP3] and the references there. In the general case of a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, the behavior of the eigenvalues for cusp forms is complicated. However in our case (and in most of [LP1]), where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, it is well-known (see Motohashi [Mo], Zagier [Z]) that the eigenvalues λ_j of the non-Euclidean Laplacian Δ for the cusp forms ψ_j are all real and $\lambda_j > 1/4$. In §2 we apply this fact to describe explicitly the subspaces $\mathcal{P}$ and $\mathcal{I}$ introduced in [LP1, 2]. We also use the Eisenstein transform to explain that the energy form E is positive definite on the subspace corresponding to the continuous spectrum of Δ.

In §3 we study discrete-time Lax–Phillips scattering systems corresponding to their continuous-time counterparts. We define the subspace K of the whole Hilbert space H (H is a subspace of the L^2-space of functions on the fundamental domain $\mathfrak{F}$ for $\mathrm{SL}_2(\mathbb{Z})$) on which the Lax–Phillips semigroup and generator will be defined. This space is defined by using a unitary causal factor $\mathcal{S}_{d0}(z)$ of the discrete scattering matrix. We decompose $\mathcal{S}_{d0}(z)$ into a weak resolvent form. This procedure allows one to *carve out* an operator A_d on a Hilbert space K. We show in §6 that this A_d has a cyclicity property. We base our development on the methods of shift operator realization of linear systems from Helton [H] and [U2].

In §4 we define our Lax–Phillips semigroup and study some spectral properties of its infinitesimal generator A_c. We call this generator the Lax–Phillips (infinitesimal) generator, as in the title of the paper. We show that the operator A_d carved out from the discrete scattering matrix is related to A_c by the Cayley transform. Using this relation, we prove that the resolvent of A_c is meromorphic. Then we show that the spectrum of A_c corresponds precisely (i.e. counted with multiplicities) to the poles of the unitary causal factor $\mathcal{S}_{c0}(s)$ of the continuous scattering matrix, corresponding to K.

In [LP1, 2], the original Lax–Phillips generator denoted by B'' acting on K'' is defined. There the meromorphy in $\mathbb{C}$ of its resolvent is proved by showing compactness of the resolvent. It turns out that our space K is obtained by discarding a one-dimensional non-essential generalized eigenspace of B''. Actually $\sigma(B'') \setminus \sigma(A_c) = \{-1/2\}$. So the meromorphy of B'''s resolvent proved in [LP1, 2] follows from that of A_c's resolvent. Our proof of Theorem 4.2(i) gives a simple proof of this.

In §4 we also construct an operator acting on K with eigenvalues corresponding precisely to the non-trivial zeros of the Riemann zeta-function. For discussions of this kind of operator, see e.g. Lax and Phillips [LP2, §6] and Patterson [Pat, §5.18].

The underlying mechanism of pole correspondence between the scattering matrix and the resolvent of the Lax–Phillips generator is the cyclicity of the two vectors in the weak resolvent decomposition of the scattering matrix. This has been shown in [U1] and [U3]. In this paper we show this pole correspondence directly in Lemma 4.1 used to prove Theorem 4.2(ii). The cyclicity will be used in §6.

In §5 we represent A_c acting on K as minus the left L^2-derivative restricted to the subspace $\mathcal{K}_c$ of $L^2(\mathbb{R}_-)$, using the explicit formulas for translation representations for the scattering system in §2 obtained in [LP2]. Here $\mathcal{K}_c$ is the image of K under the outgoing representation. We give an expression of $\mathcal{K}_c$ as explicit as possible.

In §6 we show cyclicity of the two vectors in the weak resolvent decomposition of $\mathcal{S}_{d0}(z)$. Using these and the operator model in §5, we formulate a condition equivalent to the Riemann hypothesis in terms of cyclic vectors. To deduce this condition, we use a result on absence of zero-pole cancellation in cascade connection of dynamical systems in Hilbert space [U3].

Notations. I denotes the identity operator; $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty,0]$; P_W is the orthogonal projection onto the Hilbert space W; $S_d\colon \ell^2 \to \ell^2$ is the (discrete) scattering operator; $\mathcal{S}_d\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the (discrete) scattering matrix; $S_c\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the (continuous) scattering operator; $\mathcal{S}_c\colon L^2(i\mathbb{R}) \to L^2(i\mathbb{R})$ is the (continuous) scattering matrix.

2. Lax–Phillips scattering theory on the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$**.** Consider the fundamental domain $\mathfrak{F} = \Gamma\backslash\mathfrak{H}$ of $\Gamma := \mathrm{SL}_2(\mathbb{Z})$, where $\mathfrak{H} = \{w = x + iy;\ y > 0\}$ is the Poincaré upper half-plane, and Γ is the modular group defined by

$$\Gamma = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix};\ a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\},$$

$$\mathfrak{H} \ni w \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} w := \frac{aw+b}{cw+d} \in \mathfrak{H}.$$

$\mathfrak{F}$ has a cusp at $i\infty$.

Let $L^2(\mathfrak{F})$ be the Hilbert space defined by

$$L^2(\mathfrak{F}) = \left\{ u = u(w), w \in \mathfrak{F};\ \|u\|_{L^2(\mathfrak{F})} := \sqrt{\langle u, u\rangle_{L^2(\mathfrak{F})}} < \infty \right\}$$

with the inner product

$$\langle u, v\rangle_{L^2(\mathfrak{F})} = \int_{\mathfrak{F}} u\overline{v}\, \frac{dx\,dy}{y^2}.$$

It is well-known that $L^2(\mathfrak{F})$ has the following spectral decomposition with respect to the non-Euclidean Laplacian $\varDelta = -y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ ($-\varDelta$ is called the *Laplace–Beltrami operator*):

$$L^2(\mathfrak{F}) = L^2_d(\mathfrak{F}) \oplus L^2_c(\mathfrak{F}), \qquad L^2_d(\mathfrak{F}) = \mathbb{C} \oplus {}^\circ L^2(\mathfrak{F}),$$
$${}^\circ L^2(\mathfrak{F}) = \text{cl span}\{\psi_j;\ j \in \mathbb{N}\}.$$

Here cl means topological closure. Each subspace is an invariant subspace of $L^2(\mathfrak{F})$ with respect to $\varDelta$. ψ_j is a cusp form and it is known (Motohashi [Mo], Zagier [Z]) that $\varDelta\psi_j = (1/4 + \kappa_j^2)\psi_j$ with $\kappa_j > 0$. Let

$$L = -\varDelta + 1/4.$$

Then $L\psi_j = -\kappa_j^2\psi_j$. Therefore the only non-negative eigenvalue of L is $1/4$ with a constant eigenfunction c: $Lc = (1/4)c$. Note that $\|c\|_{L^2(\mathfrak{F})} = \sqrt{\pi/3}\,|c|$ ([Mo]).

The *Eisenstein series* of two variables $E(z, s)$ on $\mathfrak{H}$ is by definition

$$E(z, 1/2 + s) = \sum_{\gamma\in\varGamma_\infty\backslash\varGamma} [\Im\gamma(z)]^{1/2+s}.$$

Here for $\gamma \in \varGamma_\infty$, $\gamma(z) = z + n$ for some $n \in \mathbb{Z}$. For convenience the second variable s is shifted by $1/2$. $E(z, 1/2 + s)$ is an automorphic function, that is, $E(\gamma(z), 1/2 + s) = E(z, 1/2 + s)$ for all $\gamma \in \varGamma$, thus $E(z, 1/2 + s)$ can be viewed as a function on $\mathfrak{F}$.

The Eisenstein series is a (non-L^2-)eigenfunction of $\varDelta$:

$$(\varDelta - 1/4)E(z, 1/2 + i\xi) = -(i\xi)^2 E(z, 1/2 + i\xi)$$

for all $\xi \in \mathbb{C}$.

The *Eisenstein transform* Eis: $L^2_c(\mathfrak{F}) \to L^2(\mathbb{R}_+)$ is defined by

$$\mathrm{Eis}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathfrak{F}} f(z)E(z, 1/2 - i\xi)\, d\mu, \qquad d\mu = \frac{dx\,dy}{y^2},$$

for $f = f(z) \in L^2_c(\mathfrak{F})$ (Lang [La], Motohashi [Mo]). This transform is unitary (see §4 for the inner product of $L^2(\mathbb{R}) \supset L^2(\mathbb{R}_+)$) and the inverse is given by

$$\mathrm{Eis}^{-1}[w](z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty w(\xi)E(z, 1/2 + i\xi)\, d\xi$$

for $w = w(\xi) \in L^2(\mathbb{R}_+)$ ([Mo]).

Note that by the above eigenfunction property of the Eisenstein series,

$$(\Delta - 1/4)\mathrm{Eis}^{-1}[w](z) = \frac{1}{\sqrt{2\pi}}\,(\Delta - 1/4)\int_0^\infty w(\xi)E(z,1/2+i\xi)\,d\xi$$
$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty w(\xi)(\Delta - 1/4)E(z,1/2+i\xi)\,d\xi$$
$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty w(\xi)\xi^2 E(z,1/2+i\xi)\,d\xi$$

for $w \in C_{\mathrm{c}}^\infty(\mathbb{R}_+)$ (the space of compactly supported C^∞-functions). Note also that $\mathrm{Eis}^{-1}[C_{\mathrm{c}}^\infty(\mathbb{R}_+)]$ is dense in $L_c^2(\mathfrak{F}) \supset \mathrm{dom}(\Delta)$. So we see that $-L = \Delta - 1/4$ is still positive definite on $L_c^2(\mathfrak{F})$. Consequently, the energy form E defined below is positive definite on $L_c^2(\mathfrak{F}) \times L_c^2(\mathfrak{F})$.

Consider the solutions of the following automorphic wave equation on $\mathfrak{F}$:

$$u_{tt}(w,t) = Lu(w,t), \qquad w = x+iy \in \mathfrak{F},$$

with initial values $u(w,0)$ and $u_t(w,0)$. Rewrite the above wave equation in the first order form as $df/dt = \mathbf{L}f$, where

$$f = f(t) = \{f_1(t), f_2(t)\} = \{u(w,t), u_t(w,t)\} = \begin{bmatrix} u(w,t) \\ u_t(w,t) \end{bmatrix}$$

and

$$\mathbf{L} = \begin{bmatrix} 0 & 1 \\ L & 0 \end{bmatrix}.$$

The operator $\mathbf{L}$ is denoted by A in [LP1, 2].

The bilinear energy form E is given by

$$E(f,g) = \langle f_1, -Lg_1\rangle_{L^2(\mathfrak{F})} + \langle f_2, g_2\rangle_{L^2(\mathfrak{F})}$$

for $f = \{f_1, f_2\}, g = \{g_1, g_2\} \in L^2(\mathfrak{F}) \times L^2(\mathfrak{F})$.

The E-energy form $E(f) := E(f,f)$ for data $f = \{f_1, f_2\}$ will be zero for e.g. $f = \{c, \pm c/2\}$. To avoid this disadvantage of indefiniteness, we introduce another bilinear energy form defined by

$$G(f,g) = E(f,g) + 2K(f,g),$$

where

$$K(f,g) = \int_{\mathfrak{F}_0} \frac{f_1\overline{g}_1}{y^2}\,dx\,dy, \qquad \mathfrak{F}_0 = \mathfrak{F} \cap \{y \le a\},\ a > 1.$$

We denote by H_G the completion with respect to the G-norm defined via $G(f) := G(f,f)$ of the space of C^∞-solutions with compact support in $\mathfrak{F}$. It is known ([LP1]) that the E- and G-forms are equivalent on any closed subspace of H_G on which E is positive. The operator $\mathbf{L}$ generates a one-parameter group $\{U(t)\}_{t\in\mathbb{R}}$ of unitary operators with respect to the indefinite energy form E. We also write $U(t)$ as $e^{t\mathbf{L}}$.

The *incoming* and *outgoing solutions* of the above wave equation are given by

$$u_-(w,t) = y^{1/2}\phi(ye^t) \quad \text{and} \quad u_+(w,t) = y^{1/2}\phi(ye^{-t}),$$

respectively, where ϕ is chosen to be C^∞ and vanishing for $y \leq a$. Then the incoming and outgoing subspaces D_{0-} and D_{0+} (denoted by D_- and D_+ in [LP1, 2]) are defined to be the closure in H_G of the initial data corresponding to the above incoming and outgoing solutions, respectively:

$$D_{0-} = \mathrm{cl}\{\{y^{1/2}\phi(y), y^{3/2}\phi'(y)\}\}, \qquad D_{0+} = \mathrm{cl}\{\{y^{1/2}\phi(y), -y^{3/2}\phi'(y)\}\},$$

where ϕ is chosen as above. Actually, instead of the above $D_{0\pm}$, we will use $D_\pm$ (denoted by $D'_\pm$ or $D''_\pm$ in [LP1, 2]) defined below as our incoming and outgoing subspaces.

In [LP1, 2], the case where L has a finite number of positive eigenvalues λ_j^2, $j = 1, \dots, m$, is treated. Let q_j, $j = 1, \dots, m$, be the corresponding eigenfunctions. As we saw above, in our case of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $m = 1$, $\lambda_1^2 = 1/4$ and $q_1 = c$. Then $\mathbf{L}p_1^\pm = \pm\lambda_1 p_1^\pm$, where $p_1^\pm = \{q_1, \pm\lambda_1 q_1\} = \{c, \pm c/2\}$. Recall that $E(p_1^\pm) = 0$. Let $\mathcal{P} = \mathrm{span}\{p_j^\pm;\, j = 1, \dots, m\} = \mathrm{span}\{p_1^+, p_1^-\} = \mathrm{span}\{p, q\}$, where $p = \{1, 0\}$, $q = \{0, 1\}$. Note that $E(p, q) = G(p, q) = 0$ (i.e. p and q are E- and G-orthogonal). Denote the E-orthogonal complement of $\mathcal{P}$ in H_G by H'_G. Define H'_E to be the quotient space $H'_G/\mathcal{I}$, where $\mathcal{I}$ is the finite-demensional subspace spanned by the null vectors of $\mathbf{L}$. Note that $\mathbf{L}f = 0, f = \{f_1, f_2\} \Leftrightarrow Lf_1 = 0, f_2 = 0$. However, neither the cusp form ψ_j nor the constant c satisfies $Lf_1 = 0$ as we saw above. Hence it turns out that $\mathcal{I} = \{0\}$ and $H'_E = H'_G$ after all. On H'_E, E is positive definite and equivalent to G. Thus $U(t)$ is unitary on H'_E. However the subspaces D_{0-} and D_{0+} do not lie in H'_E. Thus we define two types of incoming and outgoing subspaces in H'_E as follows:

$$D'_\pm = P_{H'_E} D_{0\pm} \quad \text{or} \quad D''_\pm = D_{0\pm} \cap H'_E.$$

Here $P_{H'_E}$ is the E-orthogonal projection onto H'_E from H_G. Define H (denoted by H'_c in [LP1, 2]) to be the E-orthogonal complement of the eigenfunctions in H'_E from the point spectrum of $\mathbf{L}$. It is seen that the eigenfunctions of $\mathbf{L}$ in H'_E are $g_j^\pm = \{\psi_j, \pm i\kappa_j\psi_j\}$ with eigenvalues $\pm i\kappa_j$ for the cusp forms ψ_j. Since H is invariant under $U(t)$, $U(t)$ is also unitary on H. The subspaces $D_\pm = D'_\pm$ or $D''_\pm$ satisfy the following conditions:

(2.1) $$U(t)D_- \subseteq D_-, \quad \forall t \leq 0, \quad \text{and} \quad U(t)D_+ \subseteq D_+, \quad \forall t \geq 0;$$

(2.2) $$\bigcap_{t\leq 0} U(t)D_- = \{0\} = \bigcap_{t\geq 0} U(t)D_+;$$

$$\text{(2.3)} \qquad \mathrm{cl}\Big[\bigcup_{t\ge 0} U(t)D_-\Big] = H = \mathrm{cl}\Big[\bigcup_{t\le 0} U(t)D_+\Big].$$

Note that $D''_+ \perp D''_-$ but D'_+ and D'_- are not orthogonal to each other. We call this the *continuous*(*-time*) *Lax–Phillips* (*automorphic*) *scattering system.*

3. The discrete-time Lax–Phillips scattering system. Recall that $\mathbf{L}$ is the infinitesimal generator of the one-parameter group $\{U(t)\}_{t\in\mathbb{R}}$ of unitary operators acting on H, that is, $\mathbf{L} = \lim_{t\downarrow 0}(U(t)-I)/t$. Then the Cayley transform $V = (I+\mathbf{L})(I-\mathbf{L})^{-1}$ is a unitary operator on H ([LP3, Chap. II, §3]). Moreover it can be shown that the same subspace D_- (or D_+) of the continuous Lax–Phillips scattering system (2.1)–(2.3) will be incoming (or outgoing) for V ([LP3, Lemma 3.2]) in the following sense:

$$\text{(3.1)} \qquad V^nD_- \subset D_-, \quad \forall n\le 0, \quad \text{and} \quad V^nD_+ \subset D_+, \quad \forall n\ge 0;$$

$$\text{(3.2)} \qquad \bigcap_{n\le 0} V^nD_- = \{0\} = \bigcap_{n\ge 0} V^nD_+;$$

$$\text{(3.3)} \qquad \mathrm{cl}\Big[\bigcup_{n\ge 0} V^nD_-\Big] = H = \mathrm{cl}\Big[\bigcup_{n\le 0} V^nD_+\Big].$$

We call this quadruple (V, H, D_-, D_+) the *discrete*(*-time*) *Lax–Phillips* (*automorphic*) *scattering system.*

In the case of the automorphic wave equation on the fundamental domain $\mathfrak{F} = \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ of §1, $\dim(D_+\ominus VD_+) = \dim(D_-\ominus V^{-1}D_-) = \sharp\{\text{cusps}\} = 1$. In this case we have the following discrete incoming and outgoing representations.

First let

$$\ell^2 = \ell^2(-\infty,\infty) = \Big\{u = \sum_{n=-\infty}^{\infty}{}^{\oplus}(u_n)_n;\ \sum_{n=-\infty}^{\infty}|u_n|^2 < \infty\Big\}.$$

Here $(\alpha)_n$ is the vector in ℓ^2 that has $\alpha\in\mathbb{C}$ in the nth place and zeros elsewhere. Thus $\{(1)_n\}_{n\in\mathbb{Z}}$ constitutes an orthonormal basis of ℓ^2. The inner product of ℓ^2 is given by $\langle u, v\rangle_{\ell^2} = \sum_{n=-\infty}^{\infty} u_n\overline{v}_n$ for $u = \sum_{n=-\infty}^{\oplus\infty}(u_n)_n$ and $v = \sum_{n=-\infty}^{\oplus\infty}(v_n)_n$. There exist unitary discrete incoming and outgoing representations ([LP3, Theorem 1.1, pp. 38, 40])

$$\Psi_-\colon H\to\ell^2(-\infty,\infty) \quad \text{and} \quad \Psi_+\colon H\to\ell^2(-\infty,\infty)$$

such that

$$\Psi_-(D_-) = \ell^2(-\infty,-1) =: \ell^2_- \quad \text{and} \quad \Psi_+(D_+) = \ell^2(0,\infty) =: \ell^2_+.$$

Furthermore, in these representations V is transformed into the bilateral shift σ on ℓ^2, that is, $\Psi_-V = \sigma\Psi_-$ and $\Psi_+V = \sigma\Psi_+$. Here for $u = \sum_{n=-\infty}^{\oplus\infty}(u_n)_n \in \ell^2$, $\sigma u = \sum_{n=-\infty}^{\oplus\infty}(u_{n-1})_n$.

The (*discrete*) *scattering operator* $S_d\colon \ell^2 \to \ell^2$ is defined to be $S_d = \Psi_+\Psi_-^{-1}$; it is unitary.

To define the discrete scattering matrix, we introduce the Fourier transform (for the discrete case) $\mathcal{F}_d\colon \ell^2 \to L^2(\mathbb{T})$ defined by

$$\mathcal{F}_d[u](z) = \sum_{n=-\infty}^{\infty} u_n z^n, \qquad z \in \mathbb{T} = \{z;\, |z| = 1\},$$

for $u = \sum_{n=-\infty}^{\oplus\infty}(u_n)_n \in \ell^2$. Here $L^2(\mathbb{T})$ is the Hilbert space of square integrable functions on the unit circle $\mathbb{T}$ equipped with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta = \frac{1}{2\pi i}\oint_{\mathbb{T}} z^{-1} f(z)\overline{g(z)}\,dz = \sum_{n=-\infty}^{\infty} f_n\overline{g}_n$$

for $f(z) = \sum_{n=-\infty}^{\infty} f_n z^n$, $g(z) = \sum_{n=-\infty}^{\infty} g_n z^n$, $z \in \mathbb{T}$. Note that $\{z^n\}_{n=-\infty}^{\infty}$, $z \in \mathbb{T}$, is an orthonormal basis of $L^2(\mathbb{T})$. The operator $\mathcal{F}_d$ is unitary. Then $\mathcal{S}_d = \mathcal{F}_d S_d \mathcal{F}_d^{-1}\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a unitary operator, which we call the (*discrete*) *scattering matrix*.

Now set $S_d(1)_0 = \sum_{n=-\infty}^{\oplus\infty}(t_n)_n$. Define a complex-valued function $\mathcal{S}_d(z)$ by $\mathcal{S}_d(z) = \mathcal{F}_d[S_d(1)_0](z) = \sum_{n=-\infty}^{\infty} t_n z^n$.

We collect some known facts in the following lemma with proof.

LEMMA 3.1.

(i) *The discrete scattering operator S_d commutes with σ^n, that is, $\sigma^n S_d = S_d\sigma^n$, for all $n \in \mathbb{Z}$.*
(ii) *For $\mathcal{F}_d[u] \in L^2(\mathbb{T})$, $u \in \ell^2$, we have $(\mathcal{S}_d\mathcal{F}_d[u])(z) = \mathcal{S}_d(z)\cdot\mathcal{F}_d[u](z)$.*
(iii) *The following are equivalent*: (a) $D_- \perp D_+$; (b) $S_d\ell^2_- \subset \ell^2_-$; and (c) $t_n = 0$ *for* $n \geq 1$, *that is*, $\mathcal{S}_d(z) = \sum_{n=-\infty}^{0} t_n z^n$.

Proof. (i) Since $\sigma\Psi_+ = \Psi_+ V$ and $\sigma\Psi_- = \Psi_- V$, we get $\sigma S_d\sigma^{-1} = S_d$. The assertion follows by using this recursively.

(ii) Write $u = \sum_{n=-\infty}^{\oplus\infty}(u_n)_n$. Since $\mathcal{F}_d[\sigma^n u](z) = z^n\mathcal{F}_d[u](z)$ for all $n \in \mathbb{Z}$, we have, by using (i),

$$\begin{aligned}
(\mathcal{S}_d\mathcal{F}_d[u])(z) &= \mathcal{F}_d[S_d u](z) = \mathcal{F}_d\Big[S_d\Big(\sum_{n=-\infty}^{\infty}{}^{\oplus}(u_n)_n\Big)\Big](z)\\
&= \mathcal{F}_d\Big[S_d\Big(\sum_{n=-\infty}^{\infty}{}^{\oplus} u_n\sigma^n(1)_0\Big)\Big](z)\\
&= \mathcal{F}_d\Big[\sum_{n=-\infty}^{\infty} u_n\sigma^n S_d(1)_0\Big](z) = \sum_{n=-\infty}^{\infty} u_n z^n\mathcal{F}_d[S_d(1)_0](z)\\
&= \mathcal{F}_d[S_d(1)_0](z)\cdot\mathcal{F}_d[u](z) = \mathcal{S}_d(z)\cdot\mathcal{F}_d[u](z).
\end{aligned}$$

(iii) Since Ψ_+ is unitary, if $D_- \perp D_+$ then

$$S_d\ell^2_- = S_d\Psi_-(D_-) = \Psi_+\Psi_-^{-1}\Psi_-(D_-) = \Psi_+(D_-) \subset [\Psi_+(D_+)]^\perp = \ell^2_-,$$

from which (a)⇒(b) follows. If $S_d\ell^2_- \subset \ell^2_-$ then $\Psi_+(D_-) = S_d\ell^2_- \subset \ell^2_- = [\Psi_+(D_+)]^\perp$. Thus (b)⇒(a). (b)⇔(c) follows from (ii). ∎

By Lemma 3.1(ii), we also write $\mathcal{S}_d(z)$ for the discrete scattering matrix with a slight notation abuse.

Suppose that the discrete scattering matrix decomposes as $\mathcal{S}_d(z) = \mathcal{S}_{d1}(z)\mathcal{S}_{d0}(z)$, where $\mathcal{S}_{dj}(z)$ $(j = 0, 1)$ are unitary. Then the discrete scattering operator has the corresponding decomposition $S_d = S_{d1}S_{d0}$ with S_{dj} $(j = 0, 1)$ unitary. Let us call S_{d0} (or $\mathcal{S}_{d0}(z)$) *causal* if $S_{d0}\ell^2_- \subset \ell^2_-$. For causal S_{d0}, let $\mathcal{K}_d = \ell^2_- \ominus S_{d0}\ell^2_-$ and $K = \Psi_+^{-1}\mathcal{K}_d$.

If S_{d0} is causal, then $\mathcal{S}_{d0}$ is written as $\mathcal{S}_{d0}(z) = \sum_{n=-\infty}^{0} s_n z^n$ by the same argument of Lemma 3.1(iii) and its proof. Set $\beta_d = \sum_{n=-\infty}^{\oplus -1}(s_n)_n$ and $\delta_d = s_0$. Thus $S_{d0}(1)_0 = \beta_d \oplus (\delta_d)_0$. Let $P_{\ell^2_-}$ be the orthogonal projection of ℓ^2 onto ℓ^2_-.

Lemma 3.2. *Let*

$$\mathcal{X}_d = \mathrm{cl\ span}\{(P_{\ell^2_-}\sigma)^n\beta_d;\ n \ge 0\} := \mathrm{cl}\Big\{\sum_{n=0}^{N}\alpha_n(P_{\ell^2_-}\sigma)^n\beta_d;\ \alpha_n \in \mathbb{C},\ N < \infty\Big\}.$$

Then for causal S_{d0}, we have

$$\mathcal{K}_d = P_{\ell^2_-}S_{d0}\ell^2_+ = \mathcal{X}_d.$$

Proof. We have $\mathcal{K}_d \oplus S_{d0}\ell^2_- \oplus \ell^2_+ = \ell^2$. So, since S_{d0} is unitary, $S_{d0}^{-1}\mathcal{K}_d \oplus \ell^2_- \oplus S_{d0}^{-1}\ell^2_+ = S_{d0}^{-1}\ell^2 = \ell^2$, from which we see $S_{d0}^{-1}\mathcal{K}_d \oplus S_{d0}^{-1}\ell^2_+ = \ell^2_+$. Therefore $\mathcal{K}_d \oplus \ell^2_+ = S_{d0}\ell^2_+$, which proves the first equality.

On the other hand,

$$\begin{aligned}
P_{\ell^2_-}S_{d0}\ell^2_+ &= \Big\{P_{\ell^2_-}S_{d0}\sum_{n=0}^{\infty}{}^{\oplus}(\alpha_n)_n\Big\} \quad \Big(\text{for } \sum_{n=0}^{\infty}{}^{\oplus}(\alpha_n)_n \in \ell^2_+\Big)\\
&= \Big\{P_{\ell^2_-}S_{d0}\sum_{n=0}^{\infty}{}^{\oplus}\alpha_n\sigma^n(1)_0\Big\}\\
&= \Big\{P_{\ell^2_-}\sum_{n=0}^{\infty}{}^{\oplus}\alpha_n\sigma^n S_{d0}(1)_0\Big\} \quad (\text{Lemma 3.1(i)})\\
&= \Big\{\sum_{n=0}^{\infty}\alpha_n(P_{\ell^2_-}\sigma)^n P_{\ell^2_-}S_{d0}(1)_0\Big\}\\
&\qquad\qquad (\text{since } P_{\ell^2_-}\sigma^n = (P_{\ell^2_-}\sigma)^n P_{\ell^2_-},\ \forall n \ge 0)\\
&\subset \mathrm{cl\ span}\{(P_{\ell^2_-}\sigma)^n\beta_d;\ n \ge 0\} = \mathcal{X}_d.
\end{aligned}$$

We also see that $\operatorname{span}\{(P_{\ell^2_-}\sigma)^n\beta_d;\ n \geq 0\} \subset P_{\ell^2_-}S_{d0}\ell^2_+$. Since $P_{\ell^2_-}S_{d0}\ell^2_+ = \ell^2_- \ominus S_{d0}\ell^2_-$ is closed, we have $\mathcal{X}_d \subset P_{\ell^2_-}S_{d0}\ell^2_+$ by taking closure. Thus $\mathcal{X}_d = P_{\ell^2_-}S_{d0}\ell^2_+$. This proves the second equality. ∎

THEOREM 3.3. *There exist a bounded operator $A_d\colon K \to K$, $b_d, c_d \in K$ and $d_d \in \mathbb{C}$ such that $\mathcal{S}_{d0}(z) = \langle (zI - A_d)^{-1}b_d, c_d\rangle_K + d_d$ for $|z| > 1$. Here $\langle\cdot,\cdot\rangle_K$ is the restriction of the inner product on H to K. The function $\mathcal{S}_{d0}(z)$ is holomorphic in $|z| > 1$.*

Proof. Note that $\mathcal{X}_d = \mathcal{K}_d$ is $P_{\ell^2_-}\sigma$-invariant by construction. Thus we can define $\mathfrak{A}_d\colon \mathcal{K}_d \to \mathcal{K}_d$ by $\mathfrak{A}_d x = P_{\ell^2_-}\sigma x$. Set $\gamma_d = P_{\mathcal{K}_d}(1)_{-1}$. Since $\beta_d = \sum^{\oplus -1}_{n=-\infty}(s_n)_n \in \mathcal{K}_d$, we have, for $n \leq -1$,

$$\begin{aligned} s_n &= \langle (P_{\ell^2_-}\sigma)^{-(n+1)}\beta_d, (1)_{-1}\rangle_{\ell^2} = \langle P_{\mathcal{K}_d}(P_{\ell^2_-}\sigma)^{-(n+1)}\beta_d, (1)_{-1}\rangle_{\ell^2} \\ &= \langle (P_{\ell^2_-}\sigma)^{-(n+1)}\beta_d, P_{\mathcal{K}_d}(1)_{-1}\rangle_{\ell^2} = \langle \mathfrak{A}_d^{-(n+1)}\beta_d, \gamma_d\rangle_{\mathcal{K}_d} \end{aligned}$$

($\langle\cdot,\cdot\rangle_{\mathcal{K}_d}$ is the restriction of $\langle\cdot,\cdot\rangle_{\ell^2}$ to $\mathcal{K}_d$). Since $\mathfrak{A}_d$ is obviously a contraction, we see that

$$\begin{aligned} \mathcal{S}_d(z) &= \sum_{n=-\infty}^{0} s_n z^n = \sum_{n=-\infty}^{-1} \langle z^n \mathfrak{A}_d^{-(n+1)}\beta_d, \gamma_d\rangle_{\mathcal{K}_d} + \delta_d \\ &= \langle (zI - \mathfrak{A}_d)^{-1}\beta_d, \gamma_d\rangle_{\mathcal{K}_d} + \delta_d \end{aligned}$$

for $|z| > 1$ and hence that $\mathcal{S}_d(z)$ is holomorphic in $|z| > 1$. Since $\Psi_+|_K\colon K \to \mathcal{K}_d$ is unitary, the weak resolvent form in the theorem can be obtained by setting $A_d = (\Psi_+|_K)^{-1}\mathfrak{A}_d\Psi_+|_K$, $b_d = (\Psi_+|_K)^{-1}\beta_d$, $c_d = (\Psi_+|_K)^{-1}\gamma_d$ and $d_d = \delta_d$. This completes the proof. ∎

4. The continuous-time Lax–Phillips scattering system. We now recall the continuous incoming and outgoing translation representations for continuous(-time) Lax–Phillips scattering systems. Let $L^2(\mathbb{R})$ be the Hilbert space of square integrable complex-valued functions on $\mathbb{R}$ with the inner product $\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(\tau)\overline{g(\tau)}\,d\tau$ for $f, g \in L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, we define the Fourier transform (for the continuous case) $\mathcal{F}_c\colon L^2(\mathbb{R}) \to L^2(i\mathbb{R})$ by

$$\mathcal{F}_c[f](s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{s\tau} f(\tau)\,d\tau, \qquad s \in i\mathbb{R},\ i = \sqrt{-1}.$$

Here $L^2(i\mathbb{R})$ is the Hilbert space of square integrable complex-valued functions of $i\mathbb{R}$ with the inner product $\langle F, G\rangle_{L^2(i\mathbb{R})} = i^{-1}\int_{-i\infty}^{i\infty} F(s)\overline{G(s)}\,ds = \int_{-\infty}^{\infty} F(-i\xi)\overline{G(-i\xi)}\,d\xi$ for $F, G \in L^2(i\mathbb{R})$. The operator $\mathcal{F}_c$ is unitary with respect to these inner products and its inverse is given by

$$\mathcal{F}_c^{-1}[F](\tau) = \frac{1}{\sqrt{2\pi i}} \int_{-i\infty}^{i\infty} e^{-\tau s} F(s)\, ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\tau\xi} F(-i\xi)\, d\xi.$$

Recall that $L^2(\mathbb{T})$ is the Hilbert space of square integrable functions on $\mathbb{T}$ (see §3). Define the Cayley transform $\mathcal{C}\colon L^2(i\mathbb{R}) \to L^2(\mathbb{T})$ by

$$f(s) \mapsto \frac{2\sqrt{\pi}}{z+1}\, f\left(\frac{z-1}{z+1}\right).$$

Then $\mathcal{C}$ is unitary. Its inverse is given by

$$f(z) \overset{\mathcal{C}^{-1}}{\longmapsto} \frac{1}{\sqrt{\pi}(1-s)}\, f\left(\frac{1+s}{1-s}\right).$$

Now we define the continuous incoming spectral representation $\Phi_-\colon H \to L^2(i\mathbb{R})$ for the continuous Lax–Phillips scattering system by $\Phi_- = \mathcal{C}^{-1}\mathcal{F}_d\Psi_-$. The continuous outgoing spectral representation $\Phi_+\colon H \to L^2(i\mathbb{R})$ is defined by $\Phi_+ = \mathcal{C}^{-1}\mathcal{F}_d\Psi_+$. The continuous incoming and outgoing translation representations $\mathcal{T}_-\colon H \to L^2(\mathbb{R})$ and $\mathcal{T}_+\colon H \to L^2(\mathbb{R})$ are defined respectively by $\mathcal{T}_- = \mathcal{F}_c^{-1}\Phi_-$ and $\mathcal{T}_+ = \mathcal{F}_c^{-1}\Phi_+$.

We recall that $H^2 \subset L^2(\mathbb{T})$ is the Hardy space ($p = 2$) with the orthonormal basis $\{z^n\}_{n=0}^{\infty}$. Let H^2_- be its orthogonal complement (the conjugate Hardy space) in $L^2(\mathbb{T})$ with the orthonormal basis $\{z^n\}_{n=-\infty}^{-1}$. We see that

$$H^2_- \ni z^n \overset{\mathcal{C}^{-1}}{\longmapsto} \frac{(1-s)^{|n|-1}}{\sqrt{\pi}(1+s)^{|n|}} \qquad \text{for } n < 0,$$

$$H^2 \ni z^n \overset{\mathcal{C}^{-1}}{\longmapsto} \frac{(1+s)^{n}}{\sqrt{\pi}(1-s)^{n+1}} \qquad \text{for } n \geq 0.$$

Now define a unitary operator $\mathcal{G}\colon \ell^2 \to L^2(\mathbb{R})$ by $\mathcal{G} = \mathcal{F}_c^{-1}\mathcal{C}^{-1}\mathcal{F}_d$. Then we see that $\mathcal{G}(\ell^2_-) = L^2(\mathbb{R}_-)$ and $\mathcal{G}(\ell^2_+) = L^2(\mathbb{R}_+)$. Since $\mathcal{T}_- = \mathcal{G}\Psi_-$ and $\Psi_-(D_-) = \ell^2_-$, we see that $\mathcal{T}_-(D_-) = L^2(\mathbb{R}_-)$. Similarly, we have $\mathcal{T}_+(D_+) = L^2(\mathbb{R}_+)$.

For the incoming representations, we have the following commutative diagram:

$$\begin{array}{ccccccccccc}
L^2(\mathfrak{F}) \supset H & = & H & \xrightarrow{\Psi_-} & \ell^2 & \xrightarrow{\mathcal{F}_d} & L^2(\mathbb{T}) & \xrightarrow{\mathcal{C}^{-1}} & L^2(i\mathbb{R}) & \xrightarrow{\mathcal{F}_c^{-1}} & L^2(\mathbb{R}) \\
\Big\downarrow{\scriptstyle \frac{I+\mathbf{L}}{I-\mathbf{L}}} & & \Big\downarrow{\scriptstyle V} & & \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle z} & & \Big\downarrow{\scriptstyle \frac{1+s}{1-s}} & & \Big\downarrow{\scriptstyle \frac{1-d^\ell/d\tau}{1+d^\ell/d\tau}} \\
L^2(\mathfrak{F}) \supset H & = & H & \xrightarrow{\Psi_-} & \ell^2 & \xrightarrow{\mathcal{F}_d} & L^2(\mathbb{T}) & \xrightarrow{\mathcal{C}^{-1}} & L^2(i\mathbb{R}) & \xrightarrow{\mathcal{F}_c^{-1}} & L^2(\mathbb{R})
\end{array}$$

Here all the maps are unitary. By replacing Ψ_- by Ψ_+, we get a commutative diagram for the outgoing representations. From this we have similarities between generators $\mathbf{L} \sim s \sim -d^\ell/d\tau$. Here $\sim$ means that two operators are similar, and $d^\ell/d\tau$ is the left L^2-derivative. Thus we also have similarities between one-parameter groups $U(t) = e^{t\mathbf{L}} \sim e^{ts} \sim T(t)$, where $[T(t)f](\tau) = f(\tau - t)$, $f \in L^2(\mathbb{R})$, for each of the incoming and outgoing representations. We therefore have $\mathcal{T}_- U(t) = T(t)\mathcal{T}_-$ and $\mathcal{T}_+ U(t) = T(t)\mathcal{T}_+$.

The (*continuous*) *scattering operator* $S_c \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined to be $S_c = \mathcal{T}_+\mathcal{T}_-^{-1}$. It is obviously unitary. Now the operator $\mathcal{S}_c \colon L^2(i\mathbb{R}) \to L^2(i\mathbb{R})$ is defined to be $\mathcal{S}_c = \mathcal{F}_c S_c \mathcal{F}_c^{-1}$. Since $\mathcal{F}_c$ is unitary with respect to the above inner products, $\mathcal{S}_c$ is also unitary. From the definition we see that $\mathcal{S}_c = \mathcal{C}^{-1}\mathcal{S}_d\mathcal{C}$. Thus we can check that its action is multiplication by the complex-valued function $\mathcal{S}_c(s) = \mathcal{S}_d\big(\frac{1+s}{1-s}\big)$, $s \in i\mathbb{R}$. We call $\mathcal{S}_c(s)$ the (*continuous*) *scattering matrix*. Thus we also see that the action of S_c is realized as a convolution.

Summarizing all the incoming and outgoing representations, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
H & \xrightarrow{\Psi_-} & \ell^2 & \xrightarrow{\mathcal{F}_d} & L^2(\mathbb{T}) & \xrightarrow{\mathcal{C}^{-1}} & L^2(i\mathbb{R}) & \xrightarrow{\mathcal{F}_c^{-1}} & L^2(\mathbb{R}) \\
\| & & \big\downarrow{\scriptstyle S_d} & & \big\downarrow{\scriptstyle \mathcal{S}_d} & & \big\downarrow{\scriptstyle \mathcal{S}_c} & & \big\downarrow{\scriptstyle S_c} \\
H & \xrightarrow{\Psi_+} & \ell^2 & \xrightarrow{\mathcal{F}_d} & L^2(\mathbb{T}) & \xrightarrow{\mathcal{C}^{-1}} & L^2(i\mathbb{R}) & \xrightarrow{\mathcal{F}_c^{-1}} & L^2(\mathbb{R})
\end{array}$$

Here all the maps are unitary.

Suppose that the scattering matrix decomposes as $\mathcal{S}_c(s) = \mathcal{S}_{c1}(s)\mathcal{S}_{c0}(s)$, where $\mathcal{S}_{cj}(s)$ $(j = 0, 1)$ are unitary. Then equivalently $S_c = S_{c1}S_{c0}$ with S_{cj} $(j = 0, 1)$ unitary. Let us say S_{c0} is *causal* if $S_{c0}L^2(\mathbb{R}_-) \subset L^2(\mathbb{R}_-)$. This is equivalent to $S_{d0}\ell^2_- \subset \ell^2_-$ for $S_{d0} = \mathcal{G}^{-1}S_{c0}\mathcal{G}$.

For the above S_{c0}, let $\mathcal{K}_c = L^2(\mathbb{R}_-) \ominus S_{c0}L^2(\mathbb{R}_-)$. Note that $\mathcal{T}_+^{-1}\mathcal{K}_c = K$ $(= \Psi_+^{-1}\mathcal{K}_d)$, defined in §3.

Let P_K be the orthogonal projection of H onto K. Define the *Lax–Phillips semigroup* $\{Z(t)\}_{t\ge 0}$ by $Z(t) = P_K U(t)|_K$ for $t \ge 0$, and let A_c be its infinitesimal generator; that is, $A_c k = \lim_{t\downarrow 0} t^{-1}(P_K U(t) - I)k$ with $\mathrm{dom}(A_c) \subset K$ consisting of those $k \in K$ for which the above limit exists. Note that $A_c = P_K\mathbf{L}|_K$ and $\mathrm{dom}(A_c) = \mathrm{dom}(\mathbf{L}|_K) = \mathrm{dom}(\mathbf{L}) \cap K$. Since $Z(t)$ is strongly continuous (because $U(t)$ is), $\mathrm{dom}(A_c)$ is dense in K (see e.g. [LP3, App. 1]).

Note that if $S_{c1} = 1$ and $S_{c0} = S_c$ itself is causal then $D_- \perp D_+$ and $K = H \ominus (D_- \oplus D_+)$ by Lemma 3.1(iii), and $\{Z(t)\}_{t\ge 0}$ coincides with the original Lax–Phillips semigroup.

Now we go back to the continuous Lax–Phillips automorphic scattering system for $\mathrm{SL}_2(\mathbb{Z})$ described in §2. If $D_\pm = D''_\pm$, then the scattering matrix is

$$\mathcal{S}_c(s) = -\left(\frac{s-1/2}{s+1/2}\right)^2 \frac{\Gamma(1/2)\Gamma(s)\zeta(2s)}{\Gamma(s+1/2)\zeta(2s+1)} = -\frac{s-1/2}{s+1/2}\,\frac{\xi(2s)}{\xi(-2s)},$$

where $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. In [LP1] it is denoted by $\mathcal{S}''(z) = \mathcal{S}_c(iz)$. $\xi(s)$ is an entire function, and its zeros are non-trivial zeros of the Riemann zeta-function $\zeta(s)$. So $\xi(2s)/\xi(-2s)$ has non-real poles in $-1/2 < \Re s < 0$. The Riemann hypothesis is equivalent to saying that all the poles (or zeros) of $\xi(2s)/\xi(-2s)$ lie on $\Re s = -1/4$ (or $1/4$). The case of $D_\pm = D'_\pm$ will be treated in §5.

It is easy to see that K'' in [LP1] is given by $\mathcal{T}_+^{-1}(L^2(\mathbb{R}_-) \ominus S_c L^2(\mathbb{R}_-))$ since S_c itself is causal. The corresponding $Z(t)$ and A_c are denoted by $Z''(t)$ and B'' respectively in [LP1]. It has been shown in [LP1, Th. 6.17] that $(sI - B'')^{-1}$ is meromorphic in the whole complex plane $\mathbb{C}$.

Throughout the rest of this paper, let

$$S_{c0} = \mathcal{F}_c^{-1} \frac{\xi(2s)}{\xi(-2s)} \mathcal{F}_c.$$

For this S_{c0}, let $\mathcal{K}_c = L^2(\mathbb{R}_-) \ominus S_{c0} L^2(\mathbb{R}_-)$ and $K = \mathcal{T}_+^{-1}\mathcal{K}_c$.

For $A_d \colon K \to K$ obtained in Theorem 3.3, we have the following lemma.

LEMMA 4.1.

(i) *$(zI - A_d)^{-1}$ is meromorphic in $\mathbb{C} \cup \{\infty\} \setminus \{-1\}$.*

(ii) *$(zI - A_d)^{-1}$ has a pole of order $m(z_0)$ at z_0 if and only if $\mathcal{S}_{d0}(z)$ has a pole of order $m(z_0)$ at z_0.*

(iii) *$\sigma(A_d) \setminus \{-1\}$ consists only of the poles of the resolvent (thus the eigenvalues) of A_d. The closure of the set of finite linear combinations of generalized eigenvectors (i.e., the vectors from the subspaces* $\mathrm{Ker}\,(z_0 I - A_d)^{m(z_0)}$ *with aforementioned poles z_0) is K.*

Proof. Recall the Hadamard product formula (e.g. [Pat, p. 34])

$$\xi(s) = \frac{1}{2} \prod_{n=0}^{\infty} \left(1 - \frac{s}{\varrho_n}\right)\left(1 - \frac{s}{\overline{\varrho}_n}\right),$$

the product being absolutely convergent for all $s \in \mathbb{C}$. Here $\varrho_n, 0 < \Re \varrho_n < 1$, $\Im \varrho_n > 0$, $n = 0, 1, 2, \ldots$, are the non-trivial zeros counted with multiplicities of the Riemann zeta-function in $\{s;\ \Im s \geq 0\}$. Hence

$$\mathcal{S}_{c0}(s) = \frac{\xi(2s)}{\xi(-2s)} = \prod_{n=0}^{\infty} \frac{(1 - 2s/\varrho_n)(1 - 2s/\overline{\varrho}_n)}{(1 + 2s/\varrho_n)(1 + 2s/\overline{\varrho}_n)}.$$

However, it is easy to check that if $\lambda = \varrho_n/2$ then

$$\frac{(1 - s/\lambda)(1 - s/\overline{\lambda})}{(1 + s/\lambda)(1 + s/\overline{\lambda})} = \frac{|\alpha|}{\alpha} \frac{\alpha - z}{1 - \overline{\alpha} z} \cdot \frac{|\overline{\alpha}|}{\overline{\alpha}} \frac{\overline{\alpha} - z}{1 - \alpha z},$$

where

$$s = \frac{z^{-1} - 1}{z^{-1} + 1} = \frac{1 - z}{1 + z}, \qquad \alpha = \frac{1 - \lambda}{1 + \lambda} \in \{z \in \mathbb{C};\ |z| < 1\}.$$

Since $\overline{\xi(\bar{s})} = \xi(s)$, $\mathcal{S}_{c0}(s)\overline{\mathcal{S}_{c0}(s)} = 1$ for all $s \in i\mathbb{R}$. Thus $\mathcal{S}_{d0}(z^{-1}) \in H^\infty$ and is inner. Since the decomposition of an inner function into Blaschke and singular parts is unique,

$$\mathcal{S}_{d0}(z^{-1}) = \mathcal{S}_{c0}\left(\frac{z^{-1}-1}{z^{-1}+1}\right) = \mathcal{S}_{c0}\left(\frac{1-z}{1+z}\right)$$

consists only of the Blaschke product.

$\mathcal{K}_d = \ell^2_- \ominus S_{d0}\ell^2_-$ is isometrically isomorphic to $M_d := H^2 \ominus \mathcal{S}_{d0}(z^{-1})H^2$ via $u \mapsto z^{-1}\mathcal{F}_d[u](z^{-1})$. It is known that $M_d \subset L^2(\mathbb{T})$ is $P_{H^2}z^{-1}$-invariant (e.g. Radjavi and Rosenthal [RaRo], Nikol'skiĭ [N]). Let $\alpha_d := P_{H^2}z^{-1}|_{M_d} = P_{M_d}z^{-1}|_{M_d}$. Then we have $A_d \sim \mathfrak{A}_d = P_{\mathcal{K}_d}\sigma|_{\mathcal{K}_d} \sim \alpha_d$. ($\mathfrak{A}_d$ is defined in the proof of Theorem 3.3.)

It is known that $z_0 \in \sigma(\alpha_d)$ if and only if $\bar{z}_0$ is a zero of order m_0 of the Blaschke product $\mathcal{S}_{d0}(z^{-1})$, and in that case $\dim \operatorname{Ker}(z_0 - \alpha_d)^{m_0} = \dim \operatorname{span}\{(1 - z_0 z)^{-\nu};\ 1 \le \nu \le m_0\} = m_0$, which is the algebraic multiplicity and the Riesz index (e.g. Theorem 3.14 of [RaRo] and its proof). The operator α_d is bounded and $\sigma(\alpha_d)$ accumulates at -1. So it is easy to see that $(z - \alpha_d)^{-1}$ is meromorphic in $\mathbb{C} \cup \{\infty\} \setminus \{-1\}$, and that z_0 is a pole of order m_0 of $(z - \alpha_d)^{-1}$ if and only if z_0 is a pole of order m_0 of $\mathcal{S}_{d0}(z)$. It is known that the generalized eigenvectors of α_d span M_d densely if a singular part does not exist (see Nikol'skiĭ [N, p. 83]). Thus the proof is complete. ∎

From the above proof, we see that the dimension of K'' in [LP1] is 1 greater than that of our K. The main result of this section is the following theorem.

Theorem 4.2.

(i) *$(sI - A_c)^{-1}$ is meromorphic in the whole complex plane $\mathbb{C}$.*
(ii) *$(sI - A_c)^{-1}$ has a pole of order $m(s_0)$ at s_0 if and only if $\mathcal{S}_{c0}(s)$ has a pole of order $m(s_0)$ at s_0.*
(iii) *The closure of the set of finite linear combinations of generalized eigenvectors (i.e., the vectors from $\operatorname{Ker}(s_0 I - A_c)^{m(s_0)}$ with aforementioned poles s_0) is K.*

From (i) of the above theorem, we also see that $(sI - B'')^{-1}$ is meromorphic in $\mathbb{C}$ since B'' acts on a space of dimension only 1 greater than that of K. To prove this theorem, we need (i) of the following lemma. (ii) will be used in §6.

Lemma 4.3.

(i) *$(sI - A_c)^{-1}$ has a pole of order m at $s_0 = (z_0 - 1)/(z_0 + 1) \neq \infty$ if and only if $(zI - A_d)^{-1}$ has a pole of order m at $z_0 \neq -1$.*
(ii) *$\sigma_c(A_d) = \{-1\}$, that is, -1 is in the continuous spectrum of A_d.*

Proof. Recall that $V = (I + \mathbf{L})(I - \mathbf{L})^{-1}$. From this $V(I - \mathbf{L}) = I + \mathbf{L}$ on $\operatorname{dom}(\mathbf{L})$, and so

$$P_K V|_K - P_K V\mathbf{L}|_K = I_K + P_K\mathbf{L}|_K \quad \text{on } \operatorname{dom}(\mathbf{L}|_K).$$

Let L_1 be the infinitesimal generator of $T(t)$. Recall $L_1 = -d^{\ell}/d\tau$. Let $\mathcal{V} = (I + L_1)(I - L_1)^{-1}$, which is a unitary operator on $L^2(\mathbb{R})$ since $L_1 \sim \mathbf{L}$. First note that $\mathcal{V}L^2(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ [this follows from $\mathcal{V} \sim \sigma$, $L^2(\mathbb{R}_+) \xrightarrow{\sim} \ell^2_+$ ($\xrightarrow{\sim}$ denotes isometric isomorphism) and $\sigma\ell^2_+ \subset \ell^2_+$] and $\mathcal{K}_c \subset L^2(\mathbb{R}_-)$. Hence we have

$$P_{\mathcal{K}_c}\mathcal{V}L_1k = P_{\mathcal{K}_c}\mathcal{V}P_{L^2(\mathbb{R}_-)}L_1k + P_{\mathcal{K}_c}\mathcal{V}P_{L^2(\mathbb{R}_+)}L_1k = P_{\mathcal{K}_c}\mathcal{V}P_{L^2(\mathbb{R}_-)}L_1k$$

for $k \in \operatorname{dom}(L_1)$.

Note that $\mathcal{K}_c = \mathcal{G}\mathcal{K}_d$, $\mathcal{K}_d = \ell^2_- \ominus S_{d0}\ell^2_-$ for $S_{d0} = \mathcal{G}^{-1}S_{c0}\mathcal{G}$ and $\mathcal{G} = \mathcal{F}_c^{-1}\mathcal{C}^{-1}\mathcal{F}_d \colon \ell^2 \to L^2(\mathbb{R})$. However, $\mathcal{K}_d = P_{\ell^2_-}S_{d0}\ell^2_+$ (Lemma 3.2). If $y = P_{\ell^2_-}x$ for some $x \in \ell^2$ then $\mathcal{G}y = P_{\mathcal{G}\ell^2_-}\mathcal{G}x = P_{L^2(\mathbb{R}_-)}\mathcal{G}x$ since $\mathcal{G}$ is unitary. Thus

$$\mathcal{G}P_{\ell^2_-}S_{d0}\ell^2_+ = P_{\mathcal{G}\ell^2_-}\mathcal{G}(\mathcal{G}^{-1}S_{c0}\mathcal{G})\ell^2_+ = P_{L^2(\mathbb{R}_-)}S_{c0}L^2(\mathbb{R}_+).$$

Hence we have $\mathcal{K}_c = P_{L^2(\mathbb{R}_-)}S_{c0}L^2(\mathbb{R}_+)$.

So any $f \in \mathcal{K}_c$ can be written as $f = P_{L^2(\mathbb{R}_-)}S_{c0}g$ for some $g \in L^2(\mathbb{R}_+)$. Since $T(t)$ and S_{c0} carry over to multiplication by e^{ts} and $\mathcal{S}_{c0}(s)$ respectively in the Fourier transforms, they commute: $T(t)S_{c0}g = S_{c0}T(t)g$. So, since $T(t)g \in L^2(\mathbb{R}_+)$ for $t \geq 0$, we have

$$P_{L^2(\mathbb{R}_-)}T(t)P_{L^2(\mathbb{R}_-)}S_{c0}g = P_{L^2(\mathbb{R}_-)}T(t)S_{c0}g = P_{L^2(\mathbb{R}_-)}S_{c0}T(t)g \in \mathcal{K}_c.$$

Hence $\mathcal{K}_c$ is $P_{L^2(\mathbb{R}_-)}T(t)$-invariant for $t \geq 0$.

Therefore, since $\mathcal{K}_c$ is closed, for $k \in \operatorname{dom}(L_1|_{\mathcal{K}_c})$ we have

$$P_{L^2(\mathbb{R}_-)}L_1k = \lim_{t\downarrow 0} P_{L^2(\mathbb{R}_-)}\frac{1}{t}\,(T(t)k - k) \in \mathcal{K}_c.$$

Therefore

$$\begin{aligned} P_{\mathcal{K}_c}\mathcal{V}P_{\mathcal{K}_c}L_1k &= P_{\mathcal{K}_c}\mathcal{V}P_{\mathcal{K}_c}P_{L^2(\mathbb{R}_-)}L_1k + P_{\mathcal{K}_c}\mathcal{V}P_{\mathcal{K}_c}P_{L^2(\mathbb{R}_+)}L_1k \\ &= P_{\mathcal{K}_c}\mathcal{V}P_{\mathcal{K}_c}P_{L^2(\mathbb{R}_-)}L_1k = P_{\mathcal{K}_c}\mathcal{V}P_{L^2(\mathbb{R}_-)}L_1k = P_{\mathcal{K}_c}\mathcal{V}L_1k. \end{aligned}$$

Going back to $\mathbf{L}$, this means that $P_K V\mathbf{L}k = P_K V P_K\mathbf{L}k$ for $k \in \operatorname{dom}(\mathbf{L}|_K)$. Therefore we have $P_K V\mathbf{L}|_K = P_K V|_K P_K\mathbf{L}|_K$. Since $V \sim \sigma$ and $K \xrightarrow{\sim} \mathcal{K}_d$, we have $A_d = P_K V|_K$. Consequently, $A_d - A_dA_c = I + A_c$ on $\operatorname{dom}(A_c) \subset K$.

Since $Z(t)$ is a contraction for all $t \geq 0$, the spectrum $\sigma(A_c)$ of the generator A_c is contained in the left half-plane (see e.g. [LP3, App. 1]). Thus $1 \notin \sigma(A_c)$. Since for $z = (1+s)/(1-s)$,

$$(zI - A_d)^{-1} = \frac{1-s}{2}\,(I - A_c)(sI - A_c)^{-1} = \frac{1}{z+1}\,(I - A_c)(sI - A_c)^{-1},$$

$$\frac{1-s}{2}\,\frac{1}{(s - s_0)^m} = 2^{-m}(z_0 + 1)^m\,\frac{[(z - z_0) + (z_0 + 1)]^{m-1}}{(z - z_0)^m},$$

where $z_0 = (1+s_0)/(1-s_0)$, and $I - A_c$ has a bounded inverse $\frac{1}{2}(I + A_d)$ on K, it follows that $(zI - A_d)^{-1}$ has a pole of order m at $z_0 = (1+s_0)/(1-s_0) \neq \infty, -1$ if and only if $(sI - A_c)^{-1}$ has a pole of order m at $s_0 \neq 1, \infty$. Note that $z = \infty$ (resp. $s = 1$) is not a singular point of the resolvent of A_d (resp. A_c). This completes the proof of (i). Since $I + A_d = 2(I - A_c)^{-1}$, we see that $I + A_d$ is one-to-one, $\mathrm{cl}[\mathrm{Im}(I + A_d)] = K$ but $\mathrm{Im}(I + A_d) \neq K$. Thus -1 is in the continuous spectrum of A_d. ■

Proof of Theorem 4.2. (i) follows from Lemma 4.1(i) and Lemma 4.3(i) and its proof. Given that $(zI - A_d)^{-1}$ is meromorphic in $\mathbb{C} \cup \{\infty\} \setminus \{-1\}$, it can be seen that the equality $\mathcal{S}_{d0}(z) = \langle (zI - A_d)^{-1} b_d, c_d \rangle_K + d_d$ for $|z| > 1$ (Theorem 3.3) extends uniquely to $\mathbb{C} \cup \{\infty\} \setminus \{-1\}$ as a meromorphic function. It is clear that $\mathcal{S}_{c0}(s)$ has a pole of order m at $s_0 \in \mathbb{C}$ if and only if $\mathcal{S}_{d0}(z)$ has a pole of order m at $z_0 = (1+s_0)/(1-s_0)$. Thus (ii) follows from Lemma 4.1(ii) and Lemma 4.3(i).

By Theorem 4.2(i), $s_0 \in \sigma(A_c)$ if and only if s_0 is a pole of $(sI - A_c)^{-1}$. So each $s_0 \in \sigma(A_c)$ is isolated. Since $A_c \sim -P_{\mathcal{K}_c} d^\ell/d\tau|_{\mathcal{K}_c}$, it is easy to see that A_c is a closed operator. So we can use the theorem of Gohberg, Goldberg and Kaashoek [GGK, Th. XV.2.1] (see also [U3, Th. 3.1]) to get $A_c = \mathrm{Diag}(A_c(s_0), A_c(\tau))$, where $A_c(s_0)$ (bounded) is the Riesz projection $(2\pi i)^{-1} \int_\Gamma (sI - A_c)^{-1}\, ds$, where the path of integration Γ is a small circle about s_0 containing no other spectral point of A_c, and $\sigma(A_c(\tau)) = \tau := \mathbb{C} \setminus \{s_0\}$. Let $A_d(z_0) = [I + A_c(s_0)][I - A_c(s_0)]^{-1}$. Note that

$$z_0 I - A_d = \frac{2}{1 - s_0}\,(s_0 I - A_c)(I - A_c)^{-1}, \qquad 1 - s_0 \neq 0.$$

Since $[s_0 I - A_c(s_0)][I - A_c(s_0)]^{-1} = [I - A_c(s_0)]^{-1}[s_0 I - A_c(s_0)]$, we see that $[z_0 I - A_d(z_0)]^m x = 0$ if and only if $[s_0 I - A_c(s_0)]^m x = 0$, $m \geq 0$. This together with Lemma 4.1(iii) proves (iii). ■

The operator $A_c = P_K \mathbf{L}|_K$ has the following properties, where I_K stands for the identity operator on K.

THEOREM 4.4.

(i) *The resolvent of $-2A_c$ is meromorphic in $\mathbb{C}$. So the spectrum of $-2A_c$ consists of eigenvalues of finite algebraic multiplicities.*
(ii) *s_0 is a non-trivial zero of multiplicity m_0 of $\zeta(s)$ if and only if s_0 is an eigenvalue of algebraic multiplicity m_0 of $-2A_c$. The set of the generalized eigenvectors corresponding to each eigenvalue is a basis of K.*
(iii) *The spectrum of $2A_c + \frac{1}{2} I_K$ consists of eigenvalues on the imaginary axis if and only if the Riemann hypothesis is true.*
(iv) *The algebraic multiplicity of any eigenvalue of $2A_c + \frac{1}{2} I_K$ is one if and only if all the non-trivial zeros of $\zeta(s)$ are simple.*

Proof. $\mathcal{S}_{c0}(s)$ has non-real poles in $\{s;\ -1/2 < \Re s < 0\}$ which correspond one-to-one, counted with multiplicities, to the non-trivial zeros of $\zeta(s)$. The critical line $\Re s = 1/2$ of $\zeta(s)$ corresponds to the line $\Re s = -1/4$. Hence assertions (i)–(iv) are immediate consequences of Theorem 4.2(i)–(iii). ∎

5. An operator model for A_c on $L^2(\mathbb{R}_-)$. In [LP2] Lax and Phillips obtained explicit formulas for translation representations for the continuous scattering system in §2 with $H = H'_c$ and $D_\pm = D'_\pm$. In this section, using their representations, we give expressions as explicit as possible for $\mathcal{K}_c$ defined in §4.

Let $u(x,y,t)$ be the solution to the non-Euclidean wave equation with automorphic initial data $f = \{f_1, f_2\}$:

$$u_{tt} = Lu, \qquad u(0) = f_1, \qquad u_t(0) = f_2.$$

Here $f = \{f_1, f_2\}$ is defined on $\mathfrak{H}$ with finite G-norm. The solution $u(x,y,t)$ is periodic in x with period 1 for all $y > 0$. Thus the zero Fourier coefficient $u^{(0)}(y,t) = \int_{-1/2}^{1/2} u(x,y,t)\,dx$, $0 < y < \infty$, satisfies the equation

$$u^{(0)}_{tt} = y^2 u^{(0)}_{yy} + \frac{1}{4}\,u^{(0)} \qquad \text{for all } y > 0.$$

The change of variables $\tau = \log y$, $v = v(\tau,t) = u^{(0)}/\sqrt{y}$ transforms the non-Euclidean wave equation into the classical wave equation $v_{tt} = v_{\tau\tau}$. The initial data goes over into

$$v(\tau,0) = e^{-\tau/2} f_1^{(0)}(e^\tau) \quad \text{and} \quad v_t(\tau,0) = e^{-\tau/2} f_2^{(0)}(e^\tau),$$

where

$$f_i^{(0)}(y) = \int\limits_{-1/2}^{1/2} f_i(x,y)\,dx, \qquad 0 < y < \infty,\ i = 1,2.$$

Since $v_\tau + v_t$ (resp. $v_\tau - v_t$) is a function of $\tau + t$ (resp. $\tau - t$), it can be shown that $T_- : H'_c \to L^2(\mathbb{R})$ defined by

$$\begin{aligned} T_- f &= \frac{1}{\sqrt{2}}\,[v_\tau(-\tau,0) + v_t(-\tau,0)] = \frac{1}{\sqrt{2}}\,[-\partial_\tau(v(-\tau,0)) + v_t(-\tau,0)] \\ &= -\frac{1}{\sqrt{2}}\,[\partial_\tau(e^{\tau/2} f_1^{(0)}(e^{-\tau})) - e^{\tau/2} f_2^{(0)}(e^{-\tau})], \qquad -\infty < \tau < \infty, \end{aligned}$$

for the initial data $f = \{f_1, f_2\}$ is an incoming translation representation. Here $v_\tau(-\tau,0) = (\partial_\tau v)(-\tau,0) = (\partial v/\partial\tau)(-\tau,0)$. Similarly, for $f = \{f_1, f_2\}$,

$$T_+ f = \frac{1}{\sqrt{2}}\,[v_\tau(\tau,0) - v_t(\tau,0)] = \frac{1}{\sqrt{2}}\,[\partial_\tau(e^{-\tau/2} f_1^{(0)}(e^\tau)) - e^{-\tau/2} f_2^{(0)}(e^\tau)]$$

for $-\infty < \tau < \infty$ defines an outgoing translation representation $T_+ : H'_c \to L^2(\mathbb{R})$.

Given $k_- \in L_2(\mathbb{R})$, $h = T_-^{-1}k_- \in H_c'$ is obtained as follows: Set

$$g_0(w) = \{y^{1/2}\phi(y), y^{3/2}\phi'(y)\}, \qquad y = \Im w,$$

where

$$\phi(y) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\log y} k_-(-\sigma)\, d\sigma = \frac{1}{\sqrt{2}} \int_{-\log y}^{\infty} k_-(\tau)\, d\tau.$$

Then define

$$g(w) = \sum_{\Gamma_\infty \backslash \Gamma} g_0(\gamma w).$$

Lastly project g onto H_E': $h = P_{H_E'} g$. Then $h \in H_c'$. Similarly, given k_+ in $L_2(\mathbb{R})$, $h = T_+^{-1}k_+ \in H_c'$ is obtained as follows: Set

$$g_0(w) = \{y^{1/2}\phi(y), -y^{3/2}\phi'(y)\}, \qquad y = \Im w,$$

where

$$\phi(y) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\log y} k_+(\tau)\, d\tau.$$

Then define

$$g(w) = \sum_{\Gamma_\infty \backslash \Gamma} g_0(\gamma w).$$

Lastly project g onto H_E': $h = P_{H_E'} g$. Then $h \in H_c'$.

Note that in the last step of inversion of $T_\pm$ into an element $h \in H_c'$, it suffices to project g onto $H_E' = H_G'$, not onto H_c'. See p. 277 of [LP2]. Thanks to this, we do not need any knowledge about the cusp forms ψ_j and their eigenvalues.

For the above two translation representations $T_- \colon H_c' \to L^2(\mathbb{R})$ and $T_+ \colon H_c' \to L^2(\mathbb{R})$, one can define the scattering operator $S_c \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $S_c = T_+ T_-^{-1}$. The corresponding scattering matrix $\mathcal{S}_c(s)$ (denoted by $\mathcal{S}'(z) = \mathcal{S}_c(s), s = iz$ in [LP3]) is given by

$$\mathcal{S}_c(s) = -\frac{\Gamma(1/2)\Gamma(s)\zeta(2s)}{\Gamma(s+1/2)\zeta(2s+1)} = \mathcal{S}_{c0}(s)\mathcal{S}_{c1}(s),$$
$$\mathcal{S}_{c0}(s) = \frac{\xi(2s)}{\xi(-2s)}, \qquad \mathcal{S}_{c1}(s) = \frac{1/2+s}{1/2-s}.$$

Note $\mathcal{S}_{c1}(s)$ is different from that in §4.

Take $e_n = e_n(\tau) \in L^2(\mathbb{R}_+)$, $n \geq 0$, that span $L^2(\mathbb{R}_+)$ densely; e.g., we can choose $e_n = e_n(\tau) = \mathcal{F}_c^{-1}\mathcal{C}^{-1}\mathcal{F}_d[(1)_n] = \mathcal{G}[(1)_n] = \mathcal{F}_c^{-1}\mathcal{C}^{-1}[z^n] \in L^2(\mathbb{R}_+)$. Hence by Lemma 3.2, $\kappa_n = \kappa_n(\tau) = P_{L^2(\mathbb{R}_-)} S_{c0} e_n$, $n \geq 0$, span $\mathcal{K}_c$. Note that $\kappa_n = P_{L^2(\mathbb{R}_-)} S_c c_n = P_{L^2(\mathbb{R}_-)} T_+ T_-^{-1} c_n = P_{L^2(\mathbb{R}_-)} T_+ h_n$, where we set $c_n = S_{c1}^{-1} e_n$ and $h_n = T_-^{-1} c_n$. Since

$$e_n(\tau) = \mathcal{F}_c^{-1} \mathcal{C}^{-1}[z^n] = \mathcal{F}_c^{-1} \left[\frac{(1+s)^n}{\sqrt{\pi}(1-s)^{n+1}} \right] \quad \text{for } n \geq 0,$$

e_n is a finite linear combination of

$$\mathcal{F}_c^{-1} \left[\frac{k!}{(1-s)^{k+1}} \right] = \begin{cases} \tau^k e^{-\tau} & (\tau > 0), \\ 0 & (\tau \leq 0), \end{cases}$$

$0 \leq k \leq n$. From e_n, c_n is easy to calculate: $c_n(\tau) = \mathcal{F}_c^{-1}[\mathcal{S}_{c1}(s)^{-1} \mathcal{F}_c[e_n]]$. Note that

$$\mathcal{F}_c^{-1}[\mathcal{S}_{c1}(s)^{-1}] = \mathcal{F}_c^{-1} \left[\frac{1}{s+1/2} - 1 \right] = \sqrt{2\pi}(w(\tau) - \delta(\tau)),$$

where $\delta(\tau)$ is the Dirac function and

$$w(\tau) = \begin{cases} 0 & (\tau > 0), \\ e^{\tau/2} & (\tau \leq 0). \end{cases}$$

Using this as an integral kernel, we have

$$c_n(\tau) = \sqrt{2} \int_{-\infty}^{0} e_n(\tau - \sigma) w(\sigma)\, d\sigma - \sqrt{2}\, e_n(\tau).$$

To obtain $h_n = T_-^{-1} c_n \in H'_c$, we start with a solution of the wave equation given by $u_{0,n}(w,t) = y^{1/2} \phi_n(y e^t)$ for

$$\phi_n(y) = \frac{1}{\sqrt{2}} \int_{-\log y}^{\infty} c_n(\tau)\, d\tau.$$

To make this solution automorphic, we sum over the right cosets $\Gamma_\infty \backslash \Gamma$ as in [LP2, p. 186]:

$$\begin{aligned} u_n(w,t) &= u_{0,n}(w,t) \\ &+ \sum_{\substack{(c,d)=1 \\ 0 \leq d < c}} \sum_{m=-\infty}^{\infty} \left[\frac{y}{\{c(x+m)+d\}^2 + c^2 y^2} \right]^{1/2} \phi_n\left(\frac{y e^t}{\{c(x+m)+d\}^2 + c^2 y^2} \right). \end{aligned}$$

Hence we have

$$g_n = g_n(w) = \{g_{n,1}(w), g_{n,2}(w)\} = \{u_n(w,0), (\partial_t u_n)(w,0)\}$$

and

$$h_n = P_{H'_E} g_n = g_n - \frac{E(g_n,p)}{E(p,p)}\, p - \frac{E(g_n,q)}{E(q,q)}\, q.$$

We recall that $p = \{1,0\}$, $q = \{0,1\}$ span $\mathcal{P}$ and $E(p,q) = 0$. Here

$$E(g_n,p) = -\frac{1}{4} \langle g_{n,1}, 1 \rangle_{L^2(\mathfrak{F})}, \quad E(p,p) = -\frac{\pi}{12},$$

$$E(g_n, q) = \langle g_{n,2}, 1\rangle_{L^2(\mathfrak{F})}, \qquad E(q,q) = \frac{\pi}{3}.$$

Pulling back $\kappa_n = P_{L^2(\mathbb{R}_-)}T_+h_n = P_{\mathcal{K}_c}T_+h_n \in \mathcal{K}_c$ to $K \subset H_c'$, we see that $P_K h_n$ $(n \geq 0)$ span K.

To get $g_n^{(0)} = g_n^{(0)}(y) = \{g_{n,1}^{(0)}(y), g_{n,2}^{(0)}(y)\}$, note that $u_n^{(0)}(y,t) = \int_{-1/2}^{1/2} u_n(x,y,t)\,dx$ is obtained as in [LP2, p. 187]:

$$u_n^{(0)}(y,t) = y^{1/2}\phi_n(ye^t) + y^{1/2}\sum_{m=1}^{\infty}\frac{E(m)}{m}\int_{-\infty}^{\infty}\frac{1}{(r^2+1)^{1/2}}\phi_n\left(\frac{e^t}{ym^2(r^2+1)}\right)dr.$$

Here E is the Euler function, $E(m) = \sum_{(c,m)=1} 1$. Thus

$$g_{n,1}^{(0)}(y) = y^{1/2}\phi_n(y) + y^{1/2}\sum_{m=1}^{\infty}\frac{E(m)}{m}\int_{-\infty}^{\infty}\frac{1}{(r^2+1)^{1/2}}\,\phi_n\left(\frac{1}{ym^2(r^2+1)}\right)dr$$

$$g_{n,2}^{(0)}(y) = y^{3/2}\phi_n(y) + y^{-1/2}\sum_{m=1}^{\infty}\frac{E(m)}{m^3}\int_{-\infty}^{\infty}\frac{1}{(r^2+1)^{3/2}}\,\phi_n'\left(\frac{1}{ym^2(r^2+1)}\right)dr.$$

Hence

$$h_n^{(0)} = g_n^{(0)} - \frac{E(g_n,p)}{E(p,p)}\,p^{(0)} - \frac{E(g_n,q)}{E(q,q)}\,q^{(0)}.$$

Note that $p = p^{(0)}$, $q = q^{(0)}$. Now applying the outgoing representation, using this $h_n^{(0)}$, we obtain $\kappa_n = P_{L^2(\mathbb{R}_-)}T_+h_n$. This construction is summarized in the following theorem. We recall that $d^\ell/d\tau$ is the left L^2-derivative (see §4).

Theorem 5.1. *Using the aforementioned $h_n^{(0)}$, $\mathcal{K}_c \subset L^2(\mathbb{R}_-)$ can be expressed as the closure of the span of*

$$\kappa_n(\tau) = \frac{1}{\sqrt{2}}\left[\frac{d}{d\tau}(e^{-\tau/2}h_{n,1}^{(0)}(e^\tau)) - e^{-\tau/2}h_{n,2}^{(0)}(e^\tau)\right] \quad (-\infty < \tau \leq 0), \qquad n \geq 0.$$

The operator $-d^\ell/d\tau$ restricted to $\mathcal{K}_c$ satisfies all the spectral properties in Theorems 4.2 *and* 4.4, *which A_c satisfies.*

Proof. Since $\mathcal{K}_c$ is $P_{L^2(\mathbb{R}_-)}T(t)$-invariant for $t \geq 0$, as we saw in the proof of Lemma 4.3, we infer that

$$P_{\mathcal{K}_c}\left(-\frac{d^\ell}{d\tau}\right)\bigg|_{\mathcal{K}_c} = P_{L^2(\mathbb{R}_-)}\left(-\frac{d^\ell}{d\tau}\right)\bigg|_{\mathcal{K}_c} = -\frac{d^\ell}{d\tau}\bigg|_{\mathcal{K}_c}.$$

Since this operator is similar to A_c on K, the remaining assertions follow from Theorems 4.2 and 4.4. ∎

6. Cyclic conditions for weak resolvent decomposition. First we recall the notion of a cyclic vector for a bounded operator.

DEFINITION 6.1. Let X be a separable complex Hilbert space. Let $A\colon X \to X$ be a bounded operator on X and $b \in X$. We say that b is *cyclic* for A if $X = \operatorname{cl}\operatorname{span}\{A^n b;\ n \geq 0\}$.

LEMMA 6.2. *For* $A_d\colon K \to K$ *and* $b_d, c_d \in K$ *as in Theorem* 3.3, *the following properties hold*;

(i) b_d *is cyclic for* A_d.

(ii) c_d *is cyclic for* A_d^*. *Here* A_d^* *denotes the adjoint operator of* A_d.

Proof. (i) β_d is obviously cyclic for $\mathfrak{A}_d$ (cf. the proof of Theorem 3.3) by definition. (See Lemma 3.2.) The assertion readily follows from this.

(ii) It suffices to show that γ_d is cyclic for $\mathfrak{A}_d^*$. Suppose that there exists a $\xi = \sum_{n=-\infty}^{\oplus -1}(\xi_n)_n \in \mathcal{K}_d \subset \ell^2_-$ such that $\langle \xi, \mathfrak{A}_d^{*k}\gamma_d\rangle_{\mathcal{K}_d} = 0$ for all $k \geq 0$. Then, since

$$\begin{aligned}\langle \xi, \mathfrak{A}_d^{*k}\gamma_d\rangle_{\mathcal{K}_d} &= \langle \mathfrak{A}_d^{k}\xi, \gamma_d\rangle_{\mathcal{K}_d} = \langle \mathfrak{A}_d^{k}\xi, P_{\mathcal{K}_d}(1)_{-1}\rangle_{\ell^2}\\ &= \langle P_{\mathcal{K}_d}\mathfrak{A}_d^{k}\xi, (1)_{-1}\rangle_{\ell^2} = \xi_{-(k+1)},\end{aligned}$$

we have $\xi = 0$. This completes the proof. ∎

Recall that $\mathcal{V} = (I + L_1)(I - L_1)^{-1}$ is a unitary operator on $L^2(\mathbb{R})$ for $L_1 = -d^\ell/d\tau$. Note that

$$\mathfrak{A}_d = P_{\ell^2_-}\sigma|_{\mathcal{K}_d} = P_{\mathcal{K}_d}\sigma|_{\mathcal{K}_d} \sim P_{L^2(\mathbb{R}_-)}\mathcal{V}|_{\mathcal{K}_c} = P_{\mathcal{K}_c}\mathcal{V}|_{\mathcal{K}_c} =: F_{cd}.$$

Let $\beta_c := \mathcal{G}(\beta_d) = \mathcal{G}[P_{\ell^2_-}S_{d0}(1)_0] = P_{L^2(\mathbb{R}_-)}S_{c0}e_0$, where $e_0 = \mathcal{G}[(1)_0]$. Then since $(P_{\ell^2_-}\sigma)^n P_{\ell^2_-} = P_{\ell^2_-}\sigma^n$, $\sigma^n S_d = S_d\sigma^n$ and $\sigma^n(1)_0 = (1)_n$ for $n \geq 0$, we have

$$\begin{aligned}F_{cd}^n\beta_c &= (P_{L^2(\mathbb{R}_-)}\mathcal{V})^n P_{L^2(\mathbb{R}_-)}S_{c0}e_0 = P_{L^2(\mathbb{R}_-)}\mathcal{V}^n S_{c0}e_0\\ &= P_{L^2(\mathbb{R}_-)}S_{c0}\mathcal{V}^n e_0 = P_{L^2(\mathbb{R}_-)}S_{c0}e_n = P_{L^2(\mathbb{R}_-)}S_c c_n = \kappa_n.\end{aligned}$$

Set $\gamma_c = \mathcal{G}[P_{\mathcal{K}_d}(1)_{-1}] = P_{\mathcal{K}_c}e_{-1}$ and $\delta_c = \delta_d$ $(= d_d)$. (See the proof of Theorem 3.3.) Here

$$e_{-1} = \mathcal{G}[(1)_{-1}] = \mathcal{F}_c^{-1}\mathcal{C}^{-1}[z^{-1}] = \mathcal{F}_c^{-1}\left[\frac{1}{\sqrt{\pi}(1+s)}\right] = \begin{cases} 0 & (\tau > 0),\\ e^\tau/\sqrt{\pi} & (\tau \leq 0).\end{cases}$$

Then we see from the proof of Theorem 3.3 that $\mathcal{S}_{d0}(z) = \langle (zI - F_{cd})^{-1}\beta_c, \gamma_c\rangle + \delta_c$. Let $c\colon \mathcal{K}_c \to \mathbb{C}$ be defined by $cx = \langle x, \gamma_c\rangle_{L^2(\mathbb{R})}$, and until the end of this paper let

$$A = F_{cd}, \qquad b = \beta_c, \qquad d = \delta_c.$$

Then $\mathcal{S}_{d0}(z) = c(zI - A)^{-1}b + d$.

Now let $\mathcal{S}_{d2}(z) = 1/(z - z_0)$. Then it is easy to check that $\mathcal{S}_{d2}(z)\mathcal{S}_{d0}(z)$ is expressed as a weak resolvent of an augmented operator A_{aug} of A as follows:

$$\mathcal{S}_{d2}(z)\mathcal{S}_{d0}(z) = c_{\mathrm{aug}}(zI - A_{\mathrm{aug}})^{-1}b_{\mathrm{aug}},$$

where

$$A_{\mathrm{aug}} = \begin{bmatrix} A & 0 \\ c & z_0 \end{bmatrix}, \qquad b_{\mathrm{aug}} = \begin{bmatrix} b \\ d \end{bmatrix}, \qquad c_{\mathrm{aug}} = [0\ 1],$$

$$X_{\mathrm{aug}} = \mathcal{K}_c \oplus \mathbb{C}, \quad A_{\mathrm{aug}} \colon X_{\mathrm{aug}} \to X_{\mathrm{aug}}, \quad b_{\mathrm{aug}} \in X_{\mathrm{aug}}, \quad c_{\mathrm{aug}} \colon X_{\mathrm{aug}} \to \mathbb{C}.$$

This construction has a dynamical system theoretic interpretation, namely a cascade connection of two dynamical systems (see e.g. [U3]).

Note that the zeros of the Riemann zeta-function on the critical line $\Re s = 1/2$ correspond to the zeros of the discrete scattering matrix

$$\mathcal{S}_{d0}(z) = \frac{\xi(2s)}{\xi(-2s)}\bigg|_{s=\frac{z-1}{z+1}} \quad \text{on} \quad C_z = \left\{z;\ z = \frac{1+s}{1-s},\ \Re s = 1/4\right\},$$

and that the zeros of $\zeta(s)$ in the critical strip $0 < \Re s < 1$ correspond to the zeros of $\mathcal{S}_{d0}(z)$ in

$$S_z = \{z;\ z = (1+s)/(1-s),\ 0 < \Re s < 1/2\}.$$

Thus the Riemann hypothesis is true if and only if $\mathcal{S}_{d0}(z)$ has no zero in $S_z \setminus C_z$.

We can restate the Riemann hypothesis in terms of cyclicity as follows.

THEOREM 6.3.

(i) *For a given parameter* $z_0 \in S_z \setminus C_z$, *construct* $(A_{\mathrm{aug}}, b_{\mathrm{aug}})$ *as above. Then* z_0 *is not a zero of* $\mathcal{S}_{d0}(z)$ *if and only if* b_{aug} *is cyclic for* A_{aug}.

(ii) *The Riemann hypothesis is true if and only if for each* $z_0 \in S_z \setminus C_z$, b_{aug} *is cyclic for* A_{aug}.

Proof. First note that $\sigma(A_{\mathrm{aug}}) = \sigma_p(A_{\mathrm{aug}}) \cup \{-1\}$ and $\sigma_p(A_{\mathrm{aug}}) = \{z_0\} \cup \sigma_p(A)$. Here $\sigma_p(A_{\mathrm{aug}})$ consists of the poles of $(zI - A_{\mathrm{aug}})^{-1}$, and -1 is in the continuous spectrum of A_{aug} arising from the continuous spectrum of $A \sim A_d$ (see Lemma 4.3(ii)).

Note that for a fixed $z_0 \in S_z \setminus C_z$, zero-pole cancellation between $\mathcal{S}_{d0}(z)$ and $\mathcal{S}_{d2}(z)$ may occur only at z_0 by construction. So it is easily seen that z_0 is not a zero of $\mathcal{S}_{d0}(z)$ if and only if zero-pole cancellation does not occur at any spectral point z_1 in $\sigma_p(A) \cup \{z_0\}$.

Now it is known [U3, Theorem 4.8, Lemmas 3.3 and 3.7] that such zero-pole cancellation at z_1 does not occur if and only if b_{aug} is cyclic for A_{aug} and $c_{\mathrm{aug}}^* = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ is cyclic for A_{aug}^*, when restricted to the generalized eigenspace corresponding to the spectrum point z_1. In [U3] the notions of local approximate controllability and observability at z_1 are used. Since our operators are all bounded, all subtle conditions to apply Theorem 4.8 in [U3] are satisfied.

Since γ_c $(= c^*)$ is cyclic for A^* by Lemma 6.2, and

$$\begin{bmatrix} A^* & c^* \\ 0 & \overline{z}_0 \end{bmatrix}^0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} A^* & c^* \\ 0 & \overline{z}_0 \end{bmatrix}^{k+1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} A^{*k}\gamma_c + \overline{z}_0 A^{*(k-1)}\gamma_c + \cdots + \overline{z}_0^k \gamma_c \\ \overline{z}_0^{k+1} \end{bmatrix}, \qquad k \geq 0,$$

we see that c_{aug} is also cyclic for A^*_{aug} independently of z_0.

Suppose z_0 is not a zero of $\mathcal{S}_{d0}(z)$. Then zero-pole cancellation does not occur at any spectral point $z_1 \in \sigma_p(A_{\mathrm{aug}})$. If b_{aug} fails to be cyclic for A_{aug}, then one can decompose $(A_{\mathrm{aug}}, b_{\mathrm{aug}})$ into

$$\left(\begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}, \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \right),$$

where b_1 is cyclic for F_{11}. Then $\sigma(F_{22}) \cap \sigma_p(A_{\mathrm{aug}}) = \emptyset$ by Theorem 4.8 in [U3]. Thus $\sigma(F_{22}) = \{-1\}$. (Note that $-1 \notin S_z \setminus C_z$.) It is easily seen that for $k \geq 0$,

$$A^k_{\mathrm{aug}} b_{\mathrm{aug}} = \begin{bmatrix} A & 0 \\ c & z_0 \end{bmatrix}^k \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} A^k b \\ cA^{k-1}b + z_0 cA^{k-2}b + \cdots + z_0^{k-1}cb + z_0^k d \end{bmatrix}.$$

Hence, as b is cyclic for A by Lemma 6.2, the dimension of the space on which F_{22} acts is one. We thus see that $F_{22} = -1$ and $-1 \in \sigma_p(A_{\mathrm{aug}})$, which is a contradiction. Therefore b_{aug} is cyclic for A_{aug}.

The converse can be shown similarly, again using Theorem 4.8 in [U3]. This completes the proof of (i). Assertion (ii) readily follows from (i). ∎

Now, $cA^n b$, $n \geq 0$, in $A^k_{\mathrm{aug}} b_{\mathrm{aug}}$ can be expressed as follows: Recall $F^n_{cd}\beta_c = A^n b = \kappa_n, n \geq 0$. Hence, since $\kappa_n \in \mathcal{K}_c$,

$$\begin{gathered} cA^n b = \langle \kappa_n, \gamma_c \rangle_{L^2(\mathbb{R})} = \langle \kappa_n, P_{\mathcal{K}_c} e_{-1} \rangle_{L^2(\mathbb{R})} = \langle \kappa_n, e_{-1} \rangle_{L^2(\mathbb{R})} \\ (= \langle T_+^{-1}\kappa_n, T_+^{-1}e_{-1} \rangle_{L^2(\mathfrak{F})} = \langle P_K h_n, T_+^{-1} e_{-1} \rangle_{L^2(\mathfrak{F})}). \end{gathered}$$

The constant d is calculated from

$$\begin{aligned} d &= \delta_c = \langle S_{d0}(1)_0, (1)_0 \rangle_{\ell^2} = \langle S_{c0} e_0, e_0 \rangle_{L^2(\mathbb{R})} = \langle T_+ h_0, e_0 \rangle_{L^2(\mathbb{R})} \\ &(= \langle h_0, T_+^{-1} e_0 \rangle_{L^2(\mathfrak{F})}), \end{aligned}$$

where $T_+ h_0$ is given by the right-hand side of the expression $(n = 0)$ with $\tau \geq 0$ in Theorem 5.1, and $e_0 = \mathcal{F}_c^{-1}[1/(\sqrt{\pi}(1-s))] = e^{-\tau}/\sqrt{\pi}$ is supported on $\mathbb{R}_+$.

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