

Difference methods for parabolic functional differential problems of the Neumann type

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Abstract. Nonlinear parabolic functional differential equations with initial boundary conditions of the Neumann type are considered. A general class of difference methods for the problem is constructed. Theorems on the convergence of difference schemes and error estimates of approximate solutions are presented. The proof of the stability of the difference functional problem is based on a comparison technique. Nonlinear estimates of the Perron type with respect to the functional variable for given functions are used. Numerical examples are given.

1. Introduction. For any two metric spaces X and Y we denote by $C(X, Y)$ the class of all continuous functions defined on X and taking values in Y . Let $M[n]$ denote the set of all $n \times n$ real matrices. We will use vectorial inequalities, understanding that the same inequalities hold between the corresponding components. Let $E = [0, a] \times [-b, b]$, where $a > 0$, $b = (b_1, \dots, b_n)$, $b_i > 0$ for $1 \leq i \leq n$, and

$$\partial_0 E = [0, a] \times ([-b, b] \setminus (-b, b)).$$

Write $\Sigma = E \times C(E, \mathbb{R}) \times \mathbb{R}^n \times M[n]$ and

$$\partial_0 E_j = \{(t, x) \in \partial_0 E : x_j = b_j\} \cup \{(t, x) \in \partial_0 E : x_j = -b_j\}, \quad 1 \leq j \leq n,$$

and suppose that

$$f : \Sigma \rightarrow \mathbb{R}, \quad \varphi : [-b, b] \rightarrow \mathbb{R}, \quad \varphi_j : \partial_0 E_j \rightarrow \mathbb{R}, \quad 1 \leq j \leq n,$$

are given functions. We consider the functional differential equation

$$(1) \quad \partial_t z(t, x) = f(t, x, z, \partial_x z(t, x), \partial_{xx} z(t, x))$$

together with the initial boundary condition of Neumann type

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- (2) $z(0, x) = \varphi(x)$ for $x \in [-b, b]$,
 (3) $\partial_{x_j} z(t, x) = \varphi_j(t, x)$ for $(t, x) \in \partial_0 E_j$, $1 \leq j \leq n$,

where $\partial_x z = (\partial_{x_1} z, \dots, \partial_{x_n} z)$, and $\partial_{xx} z = [\partial_{x_i x_j} z]_{i,j=1,\dots,n}$.

For $t \in [0, a]$ we write $E_t = [0, t] \times [-b, b]$. The function f is said to satisfy the *Volterra condition* if for each $(t, x, q, s) \in E \times \mathbb{R}^n \times M[n]$ and $z, \bar{z} \in C(E, \mathbb{R})$ such that $z(\tau, y) = \bar{z}(\tau, y)$ for $(\tau, y) \in E_t$ we have $f(t, x, z, q, s) = f(t, x, \bar{z}, q, s)$. Note that the Volterra condition means that the value of f at the point (t, x, z, q, s) of the space Σ depends on (t, x, q, s) and on the restriction of z to the set E_t .

In this paper we assume that f satisfies the Volterra condition and we consider classical solutions of (1)–(3).

We are interested in approximating classical solutions to problem (1)–(3) by solutions of associated difference functional equations and in estimation of the difference between these solutions.

Finite difference approximations of initial boundary value problems for parabolic differential or functional equations were considered by many authors under various assumptions. Difference methods for nonlinear parabolic differential equations with initial boundary conditions of the Dirichlet type were considered in [4], [6], [8]. Numerical treatment of the Cauchy problem can be found in [1], [5], [10], [16].

The paper [3] is concerned with initial boundary value problems of the Neumann type.

Difference methods for nonlinear parabolic equations with nonlinear boundary conditions are investigated in [7], [9], [14], [15].

The papers [11]–[13] deal with numerical methods for the reaction-diffusion-convection equations. The implicit difference methods proposed in those papers have the following property: almost linear parabolic equations without mixed derivatives are transformed into nonlinear systems of difference equations which require some kind of iterative schemes for the computation of numerical solutions. Various monotone iterative methods are presented in [11]–[13].

The main problem in the study of explicit difference methods is to find a difference or difference functional equation which is stable and satisfies consistency conditions with respect to the original problem.

The method of difference inequalities or theorems on linear recurrent inequalities are used in the study of stability of difference schemes. The proofs of convergence are also based on error estimates of approximate solutions to functional difference equations of the Volterra type with initial boundary conditions and with an unknown function of several variables.

It is usually assumed that the given functions have partial derivatives with respect to all variables except for (t, x) . Our assumptions are more

general. We introduce nonlinear estimates of the Perron type with respect to the functional variable. Note that our theorems are new also in the case of parabolic equations without a functional variable.

The paper is organized as follows. In Section 2 we construct a class of difference schemes for (1)–(3). Error estimates for approximate solutions of difference functional problems are proved in Section 3. In Section 4 we prove a theorem on the convergence of a difference method for (1)–(3). This is the main part of the paper. Numerical examples are presented in the last section.

Differential equations with deviated variables and differential integral problems can be obtained from our general model by a natural specification of operators.

2. Discretization of mixed problems. We will denote by $F(X, Y)$ the class of all functions defined on X and taking values in Y , where X and Y are arbitrary sets. For $x, y \in \mathbb{R}^n$ where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, we write $\|x\| = |x_1| + \dots + |x_n|$ and $x * y = (x_1y_1, \dots, x_ny_n)$.

We now formulate a difference problem corresponding to (1)–(3). We define a mesh on E in the following way. Let (h_0, h') where $h' = (h_1, \dots, h_n)$ stand for steps of the mesh. For $h = (h_0, h')$ and $(r, m) \in \mathbb{Z}^{1+n}$ where $m = (m_1, \dots, m_n)$ we define nodal points as follows:

$$t^{(r)} = rh_0, \quad x^{(m)} = m * h', \quad x^{(m)} = (x_1^{(m_1)}, \dots, x_n^{(m_n)}).$$

Let H be the set of all $h = (h_0, h')$ such that there exists $(N_1, \dots, N_n) = N \in \mathbb{N}^n$ satisfying the condition $N * h' = b$. We write $\|h\| = h_0 + h_1 + \dots + h_n$. Let $N_0 \in \mathbb{N}$ be defined by the relation $N_0h_0 \leq a \leq (N_0 + 1)h_0$. For $h \in H$ we put

$$\mathbb{R}_h^{1+n} = \{(t^{(r)}, x^{(m)}) : (r, m) \in \mathbb{Z}^{1+n}\}$$

and

$$\begin{aligned} E_h &= E \cap \mathbb{R}_h^{1+n}, \quad \partial_0 E_h = \partial_0 E \cap \mathbb{R}_h^{1+n}, \\ \partial_0 E_{h,j} &= \partial_0 E_j \cap \mathbb{R}_h^{1+n}, \quad j = 1, \dots, n, \\ E'_h &= \{(t^{(r)}, x^{(m)}) \in E_h : 0 \leq r \leq N_0 - 1\}, \\ \Sigma_h &= E'_h \times F(E_h, \mathbb{R}) \times \mathbb{R}^n \times M[n]. \end{aligned}$$

For $z : E_h \rightarrow \mathbb{R}$ we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$. Put $E_{h,r} = E_h \cap ([0, t^{(r)}] \times \mathbb{R}^n)$, where $0 \leq r \leq N_0$, and

$$\|z\|_{h,r} = \max\{|z^{(\tilde{r},m)}| : (t^{(\tilde{r})}, x^{(m)}) \in E_{h,r}\}, \quad 0 \leq r \leq N_0.$$

Let $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ be the vector with 1 in the i th position. Write

$$J = \{(i, j) : 1 \leq i, j \leq n, i \neq j\}$$

and suppose that we have defined sets $J_+, J_- \subset J$ such that $J_+ \cup J_- = J$, $J_+ \cap J_- = \emptyset$ (in particular, it may happen that $J_+ = \emptyset$ or $J_- = \emptyset$). We assume that $(i, j) \in J_+$ when $(j, i) \in J_+$.

For each $m \in \mathbb{Z}^n$ such that $x^{(m)} \in [-b, b] \setminus (-b, b)$ we consider the class of $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ satisfying the conditions:

- (i) $\|\alpha\| = 1$ or $\|\alpha\| = 2$,
- (ii) if $m = (m_1, \dots, m_n)$ and there is j , $1 \leq j \leq n$, such that $m_j = N_j$ then $\alpha_j \in \{0, 1\}$,
- (iii) if $m = (m_1, \dots, m_n)$ and there is j , $1 \leq j \leq n$, such that $m_j = -N_j$ then $\alpha_j \in \{-1, 0\}$.

The set of $\alpha \in \mathbb{Z}^n$ satisfying the above conditions will be denoted by $A^{(m)}$. Define

$$\begin{aligned} \partial E_h^+ &= \{(t^{(r)}, x^{(m+\alpha)}) : 0 \leq i \leq N_0, (t^{(r)}, x^{(m)}) \in \partial_0 E_h \text{ and } \alpha \in A^{(m)}\}, \\ E_h^+ &= \partial E_h^+ \cup E_h. \end{aligned}$$

Let $z : E_h^+ \rightarrow \mathbb{R}$ and $-N \leq m \leq N$. We define

$$\delta_i^+ z^{(r,m)} = \frac{1}{h_i} (z^{(r,m+e_i)} - z^{(r,m)}), \quad \delta_i^- z^{(r,m)} = \frac{1}{h_i} (z^{(r,m)} - z^{(r,m-e_i)}),$$

where $1 \leq i \leq n$. We apply the difference operators δ_0 and $\delta = (\delta_1, \dots, \delta_n)$ given by

$$\begin{aligned} \delta_0 z^{(r,m)} &= \frac{1}{h_0} (z^{(r+1,m)} - z^{(r,m)}), \\ \delta_i z^{(r,m)} &= \frac{1}{2} (\delta_i^+ z^{(r,m)} + \delta_i^- z^{(r,m)}), \quad 1 \leq i \leq n. \end{aligned}$$

The difference operator $\delta^{(2)} = [\delta_{ij}]_{i,j=1,\dots,n}$ of the second order is defined in the following way:

$$\delta_{ii} z^{(r,m)} = \delta_i^+ \delta_i^- z^{(r,m)} \quad \text{for } 1 \leq i \leq n$$

and

$$\begin{aligned} \delta_{ij} z^{(r,m)} &= \frac{1}{2} (\delta_i^+ \delta_j^- z^{(r,m)} + \delta_i^- \delta_j^+ z^{(r,m)}) \quad \text{for } (i, j) \in J_-, \\ \delta_{ij} z^{(r,m)} &= \frac{1}{2} (\delta_i^+ \delta_j^+ z^{(r,m)} + \delta_i^- \delta_j^- z^{(r,m)}) \quad \text{for } (i, j) \in J_+. \end{aligned}$$

Suppose that the functions

$$f_h : \Sigma_h \rightarrow \mathbb{R}, \quad \varphi_h : [-b, b] \rightarrow \mathbb{R}, \quad \varphi_{h,j} : \partial_0 E_{h,j} \rightarrow \mathbb{R}, \quad 1 \leq j \leq n,$$

are given. We consider the difference equations

$$(4) \quad \delta_0 z^{(r,m)} = f_h(t^{(r)}, x^{(m)}, z, \delta z^{(r,m)}, \delta^{(2)} z^{(r,m)}), \quad -N \leq m \leq N,$$

$$(5) \quad z(t^{(r)}, x^{(m+\alpha)}) = z(t^{(r)}, x^{(m-\alpha)})$$

$$+ 2 \sum_{j=1}^n \alpha_j h_j \varphi_{h,j}(t^{(r)}, x^{(m)}) \quad \text{on } \partial_0 E_h, \quad \alpha \in A^{(m)},$$

with the initial condition

$$(6) \quad z^{(0,m)} = \varphi_h^{(m)} \quad \text{for } x^{(m)} \in [-b, b].$$

We assume that f_h satisfies the *Volterra condition*: for each $(t^{(r)}, x^{(m)}, q, s) \in \Sigma'_h \times \mathbb{R}^n \times M[n]$ and $z, \bar{z} \in F(E_h, \mathbb{R})$ such that $z(\tau, y) = \bar{z}(\tau, y)$ for $(\tau, y) \in E_{h,r}$ we have

$$f_h(t^{(r)}, x^{(m)}, z, q, s) = f_h(t^{(r)}, x^{(m)}, \bar{z}, q, s).$$

Then there exists exactly one solution $u_h : E_h^+ \rightarrow \mathbb{R}$ of problem (4)–(6).

3. Approximate solutions of difference functional problems. We will denote by F_h the Nemytskii operator corresponding to (4), i.e.

$$F_h[z]^{(r,m)} = f_h(t^{(r)}, x^{(m)}, z, \delta z^{(r,m)}, \delta^{(2)} z^{(r,m)}), \quad (t^{(r)}, x^{(m)}) \in E'_h.$$

Suppose that $u_h : E_h^+ \rightarrow \mathbb{R}$ is the solution of problem (4)–(6) and $v_h : E_h^+ \rightarrow \mathbb{R}$ satisfies the following conditions:

$$(7) \quad |\delta_0 v_h^{(r,m)} - F_h[v_h]^{(r,m)}| \leq \gamma(h) \quad \text{on } E'_h,$$

$$(8) \quad \left| v_h^{(r,m+\alpha)} - v_h^{(r,m-\alpha)} - 2 \sum_{j=1}^n \alpha_j h_j \varphi_{j,h}^{(r,m)} \right| \leq \gamma_1(h) \|h'\|^2 \quad \text{on } \partial_0 E_h, \quad \alpha \in A^{(m)},$$

$$(9) \quad |\varphi^{(m)} - \varphi_h^{(m)}| \leq \gamma_0(h), \quad x^{(m)} \in [-b, b],$$

where $\gamma, \gamma_0, \gamma_1 : H \rightarrow \mathbb{R}_+$ and

$$(10) \quad \lim_{h \rightarrow 0} \gamma(h) = 0, \quad \lim_{h \rightarrow 0} \gamma_0(h) = 0, \quad \lim_{h \rightarrow 0} \gamma_1(h) = 0.$$

The function v_h satisfying the above relations is considered to be an approximate solution of problem (4)–(6). We prove an estimate of the difference between the exact and approximate solutions of (4)–(6). Put

$$I_h = \{t^{(r)} : 0 \leq r \leq N_0\}, \quad I'_h = I_h \setminus \{t^{(N_0)}\}.$$

For a function $\eta : I_h \rightarrow \mathbb{R}$ we write $\eta^{(r)} = \eta(t^{(r)})$.

ASSUMPTION H[σ_h]. The function $\sigma_h : I'_h \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is such that

- 1) σ_h is nondecreasing with respect to the second variable and $\sigma_h(t, 0) = 0$ for $t \in I'_h$,

2) the difference problem

$$(11) \quad \eta^{(r+1)} = \eta^{(r)} + h_0 \sigma_h(t^{(r)}, \eta^{(r)}), \quad 0 \leq r \leq N_0 - 1,$$

$$(12) \quad \eta^{(0)} = 0,$$

is stable in the following sense: if $\bar{\gamma}, \bar{\gamma}_0 : H \rightarrow \mathbb{R}_+$ are functions such that

$$\lim_{h \rightarrow 0} \bar{\gamma}(h) = 0, \quad \lim_{h \rightarrow 0} \bar{\gamma}_0(h) = 0$$

and $\eta_h : I_h \rightarrow \mathbb{R}_+$ is a solution of the difference problem

$$(13) \quad \eta^{(r+1)} = \eta^{(r)} + h_0 \sigma_h(t^{(r)}, \eta^{(r)}) + h_0 \bar{\gamma}(h), \quad 0 \leq r \leq N_0 - 1,$$

$$(14) \quad \eta^{(0)} = \bar{\gamma}_0(h),$$

then there is $\tilde{\alpha} : H \rightarrow \mathbb{R}_+$ such that $\eta_h^{(r)} \leq \tilde{\alpha}(h)$ for $t^{(r)} \in I_h$ and $\lim_{h \rightarrow 0} \tilde{\alpha}(h) = 0$.

ASSUMPTION H[f_h]. The function $f_h : \Sigma_h \rightarrow \mathbb{R}$ of variables (t, x, w, q, s) , where $q = (q_1, \dots, q_n)$, $s = [s_{ij}]_{i,j=1,\dots,n}$, satisfies the conditions:

1) $f_h(t, x, z, \cdot, \cdot) \in C(\mathbb{R}^n \times M[n], \mathbb{R})$ and the derivatives

$$\partial_q f_h = (\partial_{q_1} f_h, \dots, \partial_{q_n} f_h), \quad \partial_s f_h = [\partial_{s_{ij}} f_h]_{i,j=1,\dots,n},$$

exist on Σ_h and $\partial_q f_h(t, x, z, \cdot, \cdot) \in C(\mathbb{R}^n \times M[n], \mathbb{R}^n)$, $\partial_s f_h(t, x, z, \cdot, \cdot) \in C(\mathbb{R}^n \times M[n], M[n])$ for each $(t, x, z) \in E'_h \times F(E_h, \mathbb{R})$,

2) the matrix $\partial_s f_h$ is symmetric and

$$(15) \quad \partial_{s_{ij}} f_h(P) \geq 0 \text{ for } (i, j) \in J_+, \quad \partial_{s_{ij}} f_h(P) \leq 0 \text{ for } (i, j) \in J_-,$$

$$(16) \quad 1 - 2h_0 \sum_{j=1}^n \frac{1}{h_j^2} \partial_{s_{jj}} f_h(P) + h_0 \sum_{(i,j) \in J} \frac{1}{h_i h_j} |\partial_{s_{ij}} f_h(P)| \geq 0,$$

$$(17) \quad -\frac{1}{2} |\partial_{q_i} f_h(P)| + \frac{1}{h_i} \partial_{s_{ii}} f_h(P) - \sum_{j=1, j \neq i}^n \frac{1}{h_j} |\partial_{s_{ij}} f_h(P)| \geq 0, \quad 1 \leq i \leq n,$$

where $P = (x, y, z, q, s) \in \Sigma_h$,

3) there is a function σ_h satisfying Assumption H[σ_h] such that

$$\|f_h(t, x, z, q, s) - f_h(t, x, \bar{z}, q, s)\| \leq \sigma_h(t, \|z - \bar{z}\|_{h,r})$$

on Σ_h .

REMARK 1. It is assumed in condition 2) of H[f_h] that the functions

$$g_{h,ij} = \text{sign } \partial_{s_{ij}} f_h, \quad (i, j) \in J,$$

are constant on Σ_h . Relations (15) can be considered as definitions of the sets J_+ and J_- .

REMARK 2. Suppose that

- (i) conditions 1), 2) of Assumption H[f_h] are satisfied,
- (ii) there is $\tilde{p} > 0$ such that

$$\partial_{s_{ii}} f_h(P) - \sum_{\substack{j=1 \\ j \neq i}}^n |\partial_{s_{ij}} f_h(P)| \geq \tilde{p}, \quad i = 1, \dots, n,$$

where $P = (t, x, z, q, s) \in \Sigma_h$.

Then there is $\tilde{\varepsilon} > 0$ such that for $\|h'\| < \tilde{\varepsilon}$ assumption (17) is satisfied.

THEOREM 3.1. Suppose that Assumption H[f_h] is satisfied and

- 1) $u_h : E_h^+ \rightarrow \mathbb{R}$ is a solution of (4)–(6) and the function $v_h : E_h^+ \rightarrow \mathbb{R}$ satisfies (7)–(9),
- 2) there is $\tilde{c} > 0$ such that $\|h'\|^2 \leq \tilde{c}h_0$.

Then there is $\alpha : H \rightarrow \mathbb{R}_+$ such that

$$|(u_h - v_h)^{(r,m)}| \leq \alpha(h) \quad \text{on } E_h$$

and

$$\lim_{h \rightarrow 0} \alpha(h) = 0.$$

Proof. Let $\Gamma_h : E'_h \rightarrow \mathbb{R}$, $\Gamma_{0,h} : E_{0,h} \rightarrow \mathbb{R}$, and $\Gamma_{\partial,h} : \partial_0 E_h \rightarrow \mathbb{R}$ be defined by the relations

$$\begin{aligned} \delta_0 v_h^{(r,m)} &= F_h[v_h]^{(r,m)} + \Gamma_h^{(r,m)} \quad \text{on } E'_h, \\ v_h^{(r,m+\alpha)} - v_h^{(r,m-\alpha)} &= 2 \sum_{j=1}^n \alpha_j h_j \varphi_{j,h}^{(r,m)} + \Gamma_{\partial,h}^{(r,m)} \quad \text{on } \partial_0 E_h, \quad \alpha \in A^{(m)}, \\ v_h^{(0,m)} &= \varphi_h^{(m)} + \Gamma_{0,h}^{(m)}, \quad x^{(m)} \in [-b, b]. \end{aligned}$$

It follows from (7)–(10) that

$$\begin{aligned} |\Gamma_h^{(r,m)}| &\leq \gamma(h) \quad \text{on } E'_h, \quad |\Gamma_{\partial,h}^{(r,m)}| \leq \gamma_1(h) \|h'\|^2 \quad \text{on } \partial_0 E_h, \\ |\Gamma_{0,h}^{(m)}| &\leq \gamma_0(h) \quad \text{for } x^{(m)} \in [-b, b] \end{aligned}$$

and

$$\lim_{h \rightarrow 0} \gamma(h) = 0, \quad \lim_{h \rightarrow 0} \gamma_0(h) = 0, \quad \lim_{h \rightarrow 0} \gamma_1(h) = 0.$$

Write $z_h = u_h - v_h$ and

$$\begin{aligned} \Xi_h^{(r,m)} &= z_h^{(r,m)} + h_0 [f_h(t^{(r)}, x^{(m)}, v_h, \delta u_h^{(r,m)}, \delta^{(2)} u_h^{(r,m)}) \\ &\quad - f_h(t^{(r)}, x^{(m)}, v_h, \delta v_h^{(r,m)}, \delta^{(2)} v_h^{(r,m)})], \\ \Lambda_h^{(r,m)} &= (t^{(r)}, x^{(m)}, u_h, \delta u_h^{(r,m)}, \delta^{(2)} u_h^{(r,m)}) \\ &\quad - (t^{(r)}, x^{(m)}, v_h, \delta v_h^{(r,m)}, \delta^{(2)} v_h^{(r,m)}). \end{aligned}$$

Thus we have

$$(18) \quad z_h^{(r+1,m)} = \Xi_h^{(r,m)} + h_0 \Lambda_h^{(r,m)} - h_0 \Gamma_h^{(r,m)} \quad \text{on } E'_h$$

and

$$z_h^{(r,m+\alpha)} = z_h^{(r,m-\alpha)} + \Gamma_{\partial.h}^{(r,m)} \quad \text{on } \partial_0 E_h, \quad \alpha \in A^{(m)}.$$

We first estimate $\Xi_h^{(r,m)}$. Write

$$\begin{aligned} A_{i,+}(Q) &= \frac{1}{2h_i} \partial_{q_i} f_h(Q) + \frac{1}{h_i^2} \partial_{s_{ii}} f_h(Q) - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{h_i h_j} |\partial_{s_{ij}} f_h(Q)|, \\ A_{i,-}(Q) &= -\frac{1}{2h_i} \partial_{q_i} f_h(Q) + \frac{1}{h_i^2} \partial_{s_{ii}} f_h(Q) - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{h_i h_j} |\partial_{s_{ij}} f_h(Q)|, \\ B(Q) &= 1 - 2 \sum_{i=1}^n \frac{h_0}{h_i^2} \partial_{s_{ii}} f_h(Q) + \sum_{(i,j) \in J} \frac{h_0}{h_i h_j} |\partial_{s_{ij}} f_h(Q)|, \\ C_i(Q) &= \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{h_i h_j} |\partial_{s_{ij}} f_h(Q)|, \end{aligned}$$

where $Q \in \Sigma_h$ and $1 \leq i \leq n$. It follows from the definition of difference operators and from condition 1) of Assumption $H[f_h]$ that there is $Q \in \Sigma_h$ such that

$$(19) \quad \begin{aligned} \Xi_h^{(r,m)} &= B(Q) z_h^{(r,m)} \\ &\quad + h_0 \sum_{i=1}^n A_{i,+}(Q) z_h^{(r,m+e_i)} + h_0 \sum_{i=1}^n A_{i,-}(Q) z_h^{(r,m-e_i)} \\ &\quad + h_0 \sum_{i=1}^n C_i(Q) [z_h^{(r,m+e_i+e_j)} + z_h^{(r,m-e_i-e_j)} + z_h^{(r,m+e_i-e_j)} + z_h^{(r,m-e_i+e_j)}] \end{aligned}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. We conclude from (15)–(17) that

$$A_{i,+}(Q) \geq 0, \quad A_{i,-}(Q) \geq 0, \quad B(Q) \geq 0, \quad C_i(Q) \geq 0.$$

It is important in our considerations that

$$(20) \quad h_0 \sum_{i=1}^n A_{i,+}(Q) + h_0 \sum_{i=1}^n A_{i,-}(Q) + B^{(r,m)} + 2h_0 \sum_{i=1}^n C_i(Q) = 1.$$

Write

$$\begin{aligned} \varepsilon_h^{(r)} &= \max\{|z_h^{(r,m)}| : (t^{(r)}, x^{(m)}) \in E_{h,r}\}, \\ \tilde{\varepsilon}_h^{(r)} &= \max\{|z_h^{(r,m)}| : (t^{(r)}, x^{(m)}) \in E_h^+ \cap ([0, t^{(r)}] \times \mathbb{R}^n)\}, \end{aligned}$$

where $0 \leq r \leq N_0$. It follows from (19)–(20) that

$$|\Xi_h^{(r,m)}| \leq \tilde{\varepsilon}_h^{(r)} \quad \text{on } E'_h.$$

Next, according to condition 3) of Assumption H[f_h] we have

$$|\Lambda_h^{(r,m)}| \leq \sigma_h(t^{(r)}, \|z\|_{h,r}) \quad \text{on } E'_h.$$

The above estimates and (18) imply

$$\varepsilon_h^{(r+1)} \leq \tilde{\varepsilon}_h^{(r)} + h_0\sigma_h(t^{(r)}, \varepsilon_h^{(r)}) + h_0\gamma(h),$$

where $0 \leq r \leq N_0 - 1$. It is easily seen that

$$\tilde{\varepsilon}_h^{(r)} \leq \varepsilon_h^{(r)} + h_0\gamma_1(h)\tilde{c}, \quad 0 \leq r \leq N_0 - 1.$$

Thus we see that the function ε_h satisfies the recurrent inequality

$$\varepsilon_h^{(r+1)} \leq \varepsilon_h^{(r)} + h_0\sigma_h(t^{(r)}, \varepsilon_h^{(r)}) + h_0(\gamma(h) + \tilde{c}\gamma_1(h)), \quad 0 \leq r \leq N_0 - 1,$$

and $\varepsilon_h^{(0)} \leq \gamma_0(h)$. Let us denote by $\bar{\eta}_h : I_h \rightarrow \mathbb{R}_+$ the solution of

$$\begin{aligned} \eta_h^{(r+1)} &= \eta_h^{(r)} + h_0\sigma_h(t^{(r)}, \eta_h^{(r)}) + h_0(\gamma(h) + \tilde{c}\gamma_1(h)), \quad 0 \leq r \leq N_0 - 1, \\ \eta_h^{(0)} &= \gamma_0(h). \end{aligned}$$

It follows easily that $\varepsilon_h^{(r)} \leq \eta_h^{(r)}$ for $0 \leq r \leq N_0$. Then the assertion of the theorem follows from the stability of problem (11), (12).

4. Convergence of implicit difference methods. For any $z \in C(E, \mathbb{R})$ we put

$$\|z\|_t = \max\{|z(\tau, x)| : (\tau, x) \in E_t\}, \quad 0 \leq t \leq a.$$

Now we give an example of an operator f_h associated with (1)–(3), and we prove that the corresponding difference method is convergent.

Equation (1) contains the function variable z which is an element of the space $C(E, \mathbb{R})$. So we need an interpolating operator $T_h : F(E_h, \mathbb{R}) \rightarrow C(E, \mathbb{R})$. We give an example of such an operator as follows. Put

$$\mathfrak{S} = \{\lambda = (\lambda_1, \dots, \lambda_n) : \lambda_i \in \{0, 1\} \text{ for } 0 \leq i \leq n\}.$$

Let $z \in F(E_h, \mathbb{R})$ and $(t, x) \in E$, $t \leq N_0h_0$. There exists $(r, m) \in \mathbb{Z}^{1+n}$ such that $t^{(r)} \leq t \leq t^{(r+1)}$, $x^{(m)} \leq x \leq x^{(m+1)}$ and $(t^{(r)}, x^{(m)})$, $(t^{(r+1)}, x^{(m+1)}) \in E_h$ where $m + 1 = (m_1 + 1, \dots, m_n + 1)$. We define

$$\begin{aligned} T_h[z](t, x) &= \frac{t - t^{(r)}}{h_0} \sum_{\lambda \in \mathfrak{S}} z^{(r+1, m+\lambda)} \left(\frac{x - x^{(m)}}{h'} \right)^\lambda \left(1 - \frac{x - x^{(m)}}{h'} \right)^{1-\lambda} \\ &+ \left(1 - \frac{t - t^{(r)}}{h_0} \right) \sum_{\lambda \in \mathfrak{S}} z^{(r, m+\lambda)} \left(\frac{x - x^{(m)}}{h'} \right)^\lambda \left(1 - \frac{x - x^{(m)}}{h'} \right)^{1-\lambda} \end{aligned}$$

where

$$\left(\frac{x - x^{(m)}}{h'}\right)^\lambda = \prod_{i=1}^n \left(\frac{x_i - x_i^{(m_i)}}{h_i}\right)^{\lambda_i},$$

$$\left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda} = \prod_{i=1}^n \left(1 - \frac{x_i - x_i^{(m_i)}}{h_i}\right)^{1-\lambda_i},$$

and we take $0^0 = 1$ in the above formulas. If $(t, x) \in E$ and $N_0 h_0 < t \leq a$ then we put $T_h[z](t, x) = T_h[z](N_0 h_0, x)$. Thus we have defined $T_h z$ on E . It follows easily that $T_h z \in C(E, \mathbb{R})$, and that $\|T_h[z]\|_{t^{(r)}} = \|z\|_{h,r}$, $0 \leq r \leq N_0$.

The above operator T_h was first applied in [2] to construct difference methods for first order partial differential functional equations.

We approximate solutions of (1)–(3) with solutions of the difference equation

$$(21) \quad \delta_0 z^{(r,m)} = f(t^{(r)}, x^{(m)}, T_h[z], \delta z^{(r,m)}, \delta^{(2)} z^{(r,m)})$$

with initial boundary condition (5), (6).

LEMMA 1. *Suppose that $z : E \rightarrow \mathbb{R}$ and*

- 1) $z(t, \cdot) : [-b, b] \rightarrow \mathbb{R}$ is of class C^2 for $t \in [0, a]$ and $z_h = z|_{E_h}$,
- 2) $\tilde{d} \in \mathbb{R}_+$ is a constant such that

$$|\partial_{x_j x_k} z(t, x)| \leq \tilde{d}, \quad (t, x) \in E, \quad j, k = 1, \dots, n,$$

- 3) there is $L \in \mathbb{R}_+$ such that

$$(22) \quad |z(t, x) - z(\bar{t}, x)| \leq L|t - \bar{t}|.$$

Then

$$\|T_h[z_h] - z\|_t \leq Lh_0 + \tilde{d}\|h'\|^2, \quad t \in [0, N_0 h_0].$$

Proof. Let $(t, x) \in E$ and $t^{(r)} \leq t \leq t^{(r+1)}$, $x^{(m)} \leq x \leq x^{(m+1)}$ where $(t^{(r)}, x^{(m)}), (t^{(r+1)}, x^{(m+1)}) \in E_h$. Write

$$U(t, x) = \frac{t - t^{(r)}}{h_0} \times \left\{ \sum_{\lambda \in \mathfrak{S}} z^{(r+1, m+\lambda)} \left(\frac{x - x^{(m)}}{h'}\right)^\lambda \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda} - z(t^{(r+1)}, x) \right\},$$

$$V(t, x) = \left(1 - \frac{t - t^{(r)}}{h_0}\right) \times \left\{ \sum_{\lambda \in \mathfrak{S}} z^{(r, m+\lambda)} \left(\frac{x - x^{(m)}}{h'}\right)^\lambda \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda} - z(t^{(r)}, x) \right\},$$

$$W(t, x) = \frac{t - t^{(r)}}{h_0} [z(t^{(r+1)}, x) - z(t, x)] + \left(1 - \frac{t - t^{(r)}}{h_0}\right) [z(t^{(r)}, x) - z(t, x)].$$

Thus we have

$$T_h[z](t, x) - z(t, x) = U(t, x) + V(t, x) + W(t, x).$$

It follows from ([2, Theorem 5.27]) that

$$|U(t, x)| + |V(t, x)| \leq \tilde{d} \|h'\|^2.$$

According to condition (22) we have $|W(t, x)| \leq Lh_0$. Hence, the proof is complete.

ASSUMPTION H[σ]. The function $\sigma : [0, a] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is such that

- 1) if $(t, p), (\bar{t}, \bar{p}) \in [0, a] \times \mathbb{R}_+$ and $t \leq \bar{t}, p \leq \bar{p}$ then $\sigma(t, p) \leq \sigma(\bar{t}, \bar{p})$,
- 2) $\sigma(t, 0) = 0$ for $t \in [0, a]$, and the maximal solution of the Cauchy problem

$$\zeta'(t) = \sigma(t, \zeta(t)), \quad \zeta(0) = 0,$$

is $\zeta(t) = 0$ for $t \in [0, a]$.

ASSUMPTION H[f]. The function $f : \Sigma \rightarrow \mathbb{R}$ of variables (t, x, z, q, s) satisfies the conditions:

- 1) $f(t, x, z, \cdot, \cdot) \in C(\mathbb{R}^n \times M[n], \mathbb{R})$, the derivatives

$$\partial_q f = (\partial_{q_1} f, \dots, \partial_{q_n} f), \quad \partial_s f = [\partial_{s_{ij}} f]_{i,j=1,\dots,n},$$

exist on Σ and

$$\partial_q f(t, x, z, \cdot, \cdot) \in C(\mathbb{R}^n \times M[n], \mathbb{R}^n),$$

$$\partial_s f(t, x, z, \cdot, \cdot) \in C(\mathbb{R}^n \times M[n], M[n])$$

for each $(t, x, z) \in E' \times F(E_h, \mathbb{R})$,

- 2) the matrix $\partial_s f$ is symmetric and

$$\partial_{s_{ij}} f(P) \geq 0 \quad \text{for } (i, j) \in J_+, \quad \partial_{s_{ij}} f(P) \leq 0 \quad \text{for } (i, j) \in J_-$$

$$1 - 2h_0 \sum_{j=1}^n \frac{1}{h_j^2} \partial_{s_{jj}} f(P) + h_0 \sum_{(i,j) \in J} \frac{1}{h_i h_j} |\partial_{s_{ij}} f(P)| \geq 0,$$

$$-\frac{1}{2} |\partial_{q_i} f(P)| + \frac{1}{h_i} \partial_{s_{ii}} f(P) - \sum_{j=1, j \neq i} \frac{1}{h_j} |\partial_{s_{ij}} f(P)| \geq 0, \quad 1 \leq i \leq n,$$

where $P = (t, x, z, q, s) \in \Sigma$,

- 3) there is a function σ satisfying Assumption H[σ] such that

$$\|f(t, x, z, q, s) - f(t, x, \bar{z}, q, s)\| \leq \sigma(t, \|z - \bar{z}\|_t)$$

on Σ_h .

We can now formulate our main results.

THEOREM 4.1. *Suppose that Assumption H[f] is satisfied and $\Omega \subset \mathbb{R}^{1+n}$ is an open and bounded set such that $E \subset \Omega$ and*

- 1) *the function $v : \Omega \rightarrow \mathbb{R}$ is the solution of (1)–(3),*

- 2) $v(\cdot, x)$ is of class C^1 and $v(t, \cdot)$ is of class C^2 ,
- 3) the function $u_h : E_h^+ \rightarrow \mathbb{R}$ is a solution of (5), (6), (21), where $\varphi_{h,j} = \varphi_j|_{\partial_0 E_h}$ for $1 \leq j \leq n$ and there is $\gamma_0 : H \rightarrow \mathbb{R}_+$ such that

$$|\varphi_0^{(m)} - \varphi_h^{(m)}| \leq \gamma_0(h) \quad \text{for } x^{(m)} \in [-b, b] \quad \text{and} \quad \lim_{h \rightarrow 0} \gamma_0(h) = 0,$$

- 4) there exist $c, \tilde{c} > 0$ such that $h_k \leq ch_j$ for $1 \leq k, j \leq n$, and $\|h'\|^2 \leq \tilde{c}h_0$.

Then there exist $\varepsilon_0 > 0$ and a function $\alpha : H \rightarrow \mathbb{R}_+$ such that for $\|h\| < \varepsilon_0$, $h \in H$ we have

$$(23) \quad |(u_h - v_h)^{(r,m)}| \leq \alpha(h) \quad \text{on } E_h \quad \text{and} \quad \lim_{h \rightarrow 0} \alpha(h) = 0,$$

where v_h is the restriction of v to E_h .

Proof. We will use Theorem 3.1 on the error estimation. Write

$$f_h(t, x, z, q, s) = f(t, x, T_h[z], q, s) \quad \text{on } \Sigma_h, \quad \sigma_h(t, p) = \sigma(t, p) \quad \text{on } I'_h \times \mathbb{R}_+.$$

It follows that conditions (7)–(9) are satisfied. Now we prove that problem (11), (12) is stable. Let $\eta_h : I_h \rightarrow \mathbb{R}_+$ be a solution of (13), (14) where $\bar{\gamma}_0, \bar{\gamma} : H \rightarrow \mathbb{R}_+$ and $\lim_{h \rightarrow 0} \bar{\gamma}_0(h) = \lim_{h \rightarrow 0} \bar{\gamma}(h) = 0$. Let $\tilde{\eta}_h : [0, a] \rightarrow \mathbb{R}_+$ be the maximal solution of the Cauchy problem

$$(24) \quad \zeta'(t) = \sigma(t, \zeta(t)) + \bar{\gamma}(h), \quad \zeta(0) = \alpha_0(h).$$

Then $\lim_{h \rightarrow 0} \tilde{\eta}_h(t) = 0$ uniformly on $[0, a]$. The function $\tilde{\eta}_h$ is convex on $[0, a]$, therefore we have

$$\tilde{\eta}_h^{(r+1)} \geq \tilde{\eta}_h^{(r)} + h_0\sigma(t^{(r)}, \tilde{\eta}_h^{(r)}) + h_0\bar{\gamma}(h), \quad 0 \leq r \leq N_0 - 1.$$

Since η_h satisfies (13) we have $\eta_h^{(r)} \leq \tilde{\eta}_h^{(r)} \leq \tilde{\eta}_h(a)$ for $0 \leq i \leq N_0$, which completes the proof of the stability of problem (11), (12). It follows from Assumption H[f] that

$$\begin{aligned} |f_h(t, x, z, q, s) - f_h(t, x, \bar{z}, q, s)| &= |f(t, x, T_h[z], q, s) - f(t, x, T_h[\bar{z}], q, s)| \\ &\leq \sigma(t, \|T_h[z] - T_h[\bar{z}]\|_t) \leq \sigma(t, \|z - \bar{z}\|_{h,r}) = \sigma_h(t, \|z - \bar{z}\|_{h,r}). \end{aligned}$$

Thus all the assumptions of Theorem 3.1 are satisfied and the proof of (23) is complete.

REMARK 3. Suppose that Assumption H[f] is satisfied with

$$\sigma(t, p) = Lp, \quad (t, p) \in [0, a] \times \mathbb{R}_+, \quad \text{where } L \in \mathbb{R}_+$$

and f satisfies the Lipschitz condition with respect to the function variable. Then we obtain the following error estimates:

$$\|u_h^{(i,m)} - v_h^{(i,m)}\| \leq \alpha_0(h)e^{La} + \bar{\gamma}(h) \frac{e^{La} - 1}{L} \quad \text{on } E_h \text{ if } L > 0$$

and

$$\|u_h^{(i,m)} - v_h^{(i,m)}\| \leq \alpha_0(h) + a\bar{\gamma}(h) \quad \text{on } E_h \text{ if } L = 0.$$

The above inequality follows from (23) with $\alpha(h) = \tilde{\eta}_h(a)$ where $\tilde{\eta}_h : [0, a] \rightarrow \mathbb{R}_+$ is a solution of (24).

5. Numerical examples

EXAMPLE 1. Write

$$E = [0, 0.2] \times [-1, 1] \times [-1, 1],$$

$$\partial_0 E = [0, 0.2] \times [([-1, 1] \times [-1, 1]) \setminus ((-1, 1) \times (-1, 1))].$$

Consider the differential equation with deviated variables

$$(25) \quad \partial_t z(t, x, y) = \partial_{xx} z(t, x, y) + \partial_{yy} z(t, x, y) + \frac{1}{4} \partial_{xy} z(t, x, y) \\ + z(t, x, y) + z\left(t, \frac{x+y}{2}, \frac{x-y}{2}\right) + f(t, x, y)$$

and the initial boundary conditions

$$(26) \quad z(0, x, y) = 0 \quad \text{for } (x, y) \in [-1, 1] \times [-1, 1],$$

$$(27) \quad \partial_x z(t, x, y) = 0 \quad \text{for } t \in [0, 0.2], y \in [-1, 1], x = 1 \text{ or } x = -1,$$

$$(28) \quad \partial_y z(t, x, y) = 0 \quad \text{for } t \in [0, 0.2], x \in [-1, 1], y = 1 \text{ or } y = -1,$$

where

$$f(t, x, y) \\ = 4(1 - e^t)((3x^2 - 1)(y^2 - 1)^2 + (3y^2 - 1)(x^2 - 1)^2 + xy(x^2 - 1)(y^2 - 1)) \\ + (x^2 - 1)^2(y^2 - 1)^2 + (1 - e^t)((x^2 - y^2)^2 - 8(x^2 + y^2 - 2))^2 : 16^2$$

The function

$$v(t, x, y) = (1 - e^t)(x^2 - 1)^2(y^2 - 1)^2$$

is a solution of problem (25)–(28).

We found approximate solutions of (25)–(28) using our numerical method, and taking the following steps of the mesh: $h_0 = 0.00005$, $h_1 = 0.02$, $h_2 = 0.02$.

Let u_h denote the approximate solution of (25)–(28) which is obtained by the difference scheme.

The average errors of the method are

$$(29) \quad \varepsilon_h^{(r)} = \frac{1}{(2N_1 + 1)(2N_2 + 1)} \sum_{i=-N_1}^{N_1} \sum_{j=-N_2}^{N_2} |u_h^{(r,i,j)} - v_h^{(r,i,j)}|$$

where $N_1 h_1 = 1$, $N_2 h_2 = 1$ and v_h is the restriction of v to the mesh.

A list of the average errors ε_h for fixed $t^{(r)}$ is given in Table 1.

Table 1. Errors ε_h
 ($h_0 = 0.00005$, $h_1 = 0.02$, $h_2 = 0.02$)

| $t^{(r)}$ | $\varepsilon_h^{(r)}$ |
|------------|-----------------------|
| $t = 0.01$ | 0.00002 |
| $t = 0.05$ | 0.00028 |
| $t = 0.10$ | 0.00105 |
| $t = 0.15$ | 0.00222 |
| $t = 0.20$ | 0.00375 |

The differential equation (25) contains the deviated variables $(t, \frac{x+y}{2}, \frac{x-y}{2})$ and the example has the following property: if $(t^{(r)}, x^{(m_1)}, y^{(m_2)})$ is a grid point then

$$(t^{(r)}, 0.5(x^{(m_1)} + y^{(m_2)}), 0.5(x^{(m_1)} - y^{(m_2)}))$$

is not in general a grid point. We approximate the value $z(t^{(r)}, 0.5(x^{(m_1)} + y^{(m_2)}), 0.5(x^{(m_1)} - y^{(m_2)}))$ using the interpolating operator T_h with $n = 2$.

EXAMPLE 2. Consider the integral differential equation

$$(30) \quad \partial_t z(t, x, y) = \partial_{xx} z(t, x, y) + \partial_{yy} z(t, x, y) + \partial_{xy} z(t, x, y) + \frac{\pi^4}{4} \int_{-x-y}^x \int_{-y}^y z(t, \tau, s) ds d\tau + \int_0^t z(\tau, x, y) d\tau + 2\pi^2 z(t, x, y) + \cos \pi x \cos \pi y$$

with the initial boundary conditions

$$(31) \quad z(0, x, y) = 0 \quad \text{for } (x, y) \in [-1, 1] \times [-1, 1],$$

$$(32) \quad \partial_x z(t, x, y) = 0 \quad \text{for } t \in [0, 0.2], y \in [-1, 1], x = 1 \text{ or } x = -1,$$

$$(33) \quad \partial_y z(t, x, y) = 0 \quad \text{for } t \in [0, 0.2], x \in [-1, 1], y = 1 \text{ or } y = -1,$$

The function $v(t, x, y) = \sin t \cos \pi x \cos \pi y$ is a solution of (30)–(33).

Let u_h denote the approximate solution of (30)–(33) which is obtained by the difference scheme. The average errors of the method $\varepsilon_h^{(r)}$ are defined by (29), where v_h is the restriction of v to the mesh.

We found approximate solutions of (30)–(33) using the numerical method, and taking the following steps of the mesh: $h_0 = 0.00005$, $h_1 = 0.02$, $h_2 = 0.02$. The method is also convergent and gives the average errors as in Table 2.

Table 2. Errors ε_h
 ($h_0 = 0.00005$, $h_1 = 0.02$, $h_2 = 0.02$)

| $t^{(r)}$ | $\varepsilon_h^{(r)}$ |
|------------|-----------------------|
| $t = 0.01$ | 0.0000004 |
| $t = 0.05$ | 0.0000098 |
| $t = 0.10$ | 0.0000396 |
| $t = 0.15$ | 0.0000972 |
| $t = 0.20$ | 0.0001695 |

The differential equation (30) contains integrals of the unknown function z . The corresponding difference equation includes the terms

$$\int_{-x^{(m_1)} - y^{(m_2)}}^{x^{(m_1)} - y^{(m_2)}} \int_{-x^{(m_1)} - y^{(m_2)}}^{y^{(m_2)}} z(t^{(r)}, \tau, s) ds d\tau \quad \text{and} \quad \int_0^{t^{(r)}} z(\tau, x^{(m_1)}, y^{(m_2)}) d\tau$$

where $z(t^{(r)}, x^{(m_1)}, y^{(m_2)})$ is a grid point. The above integrals are approximated by

$$\int_{-x^{(m_1)} - y^{(m_2)}}^{x^{(m_1)} - y^{(m_2)}} \int_{-x^{(m_1)} - y^{(m_2)}}^{y^{(m_2)}} T_h[z_h](t^{(r)}, \tau, s) ds d\tau \quad \text{and} \quad \int_0^{t^{(r)}} T_h[z_h](\tau, x^{(m_1)}, y^{(m_2)}) d\tau$$

where z_h is a solution of the difference equation. The above method is equivalent to the trapezoidal rule.

The computation was performed on a PC computer. Numerical results are consistent with our mathematical theory.

In difference schemes obtained by discretization of problem (1)–(3) a large number of previous values $z^{(r,m)}$ must be preserved, because they are needed to compute an approximate solution with $t = t^{(r+1)}$.

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