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A decomposition of complex Monge–Ampère measures

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Abstract. We prove a decomposition theorem for complex Monge–Ampère measures of plurisubharmonic functions in connection with their pluripolar sets.

1. Introduction. The purpose of this paper is to give a decomposition of complex Monge–Ampère measures associated to pluripolar sets of plurisubharmonic functions in the class $\mathcal{F}(\Omega)$ defined in [C1]. We denote by $\mathrm{PSH}(\Omega)$ the class of plurisubharmonic functions in a hyperconvex domain Ω and by $\mathrm{PSH}^-(\Omega)$ the subclass of negative functions. Recall that a set $\Omega \subset \mathbb{C}^n$ is said to be a *hyperconvex domain* if it is open, bounded, connected and there exists $\varrho \in \mathrm{PSH}^-(\Omega)$ such that $\{z \in \Omega;\, \varrho(z) < -c\} \subset\subset \Omega$ for any $c > 0$. The class $\mathcal{F}(\Omega)$ consists of all plurisubharmonic functions u in Ω such that there exists a sequence $u_j \in \mathcal{E}_0(\Omega)$ with $u_j \searrow u$ as $j \to \infty$ and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$, where $\mathcal{E}_0(\Omega)$ is the class of bounded plurisubharmonic functions v with $\lim_{z\to\zeta} v(z) = 0$ for all $\zeta \in \partial\Omega$ and $\int_\Omega (dd^c v)^n < \infty$. We also need the subclass $\mathcal{F}^a(\Omega)$ of functions from $\mathcal{F}(\Omega)$ whose Monge–Ampère measures put no mass on pluripolar subsets of Ω. It is known that Monge–Ampère measures $(dd^c u)^n$ for $u \in \mathcal{F}(\Omega)$ are well-defined finite measures in Ω (see [C1] for details).

Our main result is the following: Restriction of the complex Monge–Ampère measure of a function $u \in \mathcal{F}(\Omega)$ onto its pluripolar set is still a Monge–Ampère measure of some function in $\mathcal{F}(\Omega)$. As an application we find that every Monge–Ampère measure of a function in $\mathcal{F}(\Omega)$ can be written as a sum of two Monge–Ampère measures, one of which has zero mass on any pluripolar set and the other is carried by the pluripolar set of the corresponding function.

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2. Theorems and proofs. We need an inequality.

LEMMA ([X2]). *Let* $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ *be such that*

$$\liminf_{z \to \partial\Omega} (u(z) - v(z)) \geq 0.$$

Then for any $-1 \leq w \in \mathrm{PSH}^-(\Omega)$ *we have*

$$(n!)^{-2} \int_{u<v} (v-u)^n (dd^c w)^n + \int_{u<v} (-w)(dd^c v)^n \leq \int_{u<v} (-w)(dd^c u)^n.$$

Recall [X2] that a sequence $\{u_j\}$ of functions in $\mathrm{PSH}(\Omega)$ is said to be *convergent in* C_n to a function u on a subset E of Ω if for any $\delta > 0$ we have $C_n\{z \in E;\ |u_j(z) - u(z)| > \delta\} \to 0$ as $j \to \infty$, where C_n denotes the inner capacity introduced by Bedford and Taylor in [BT].

We denote by χ_A the characteristic function of the set A.

THEOREM 1. *Let* $v \in \mathcal{F}(\Omega)$. *Then there exists* $u \in \mathcal{F}(\Omega)$ *with* $u \geq v$ *in* Ω *such that*

$$(dd^c u)^n = \chi_{\{v=-\infty\}} (dd^c v)^n \quad \text{in } \Omega.$$

Furthermore, let g *be the unique function in* $\mathcal{F}^a(\Omega)$ *with* $(dd^c g)^n = \chi_{\{v>-\infty\}}(dd^c v)^n$. *Then* $v \geq u + g$ *in* Ω.

Proof. By Theorem 2.1 in [C1] we can take a sequence $v_j \in \mathcal{E}_0(\Omega)$ such that $v_j \searrow v$ as $j \to \infty$. By [C2], [K] there exist $u_j^k \in \mathcal{E}_0(\Omega)$ such that $(dd^c u_j^k)^n = -\max(v/k, -1)(dd^c v_j)^n$. From the comparison theorem [BT] it follows that $u_j^{k+1} \geq u_j^k \geq v_j \geq v$. By passing to a subsequence if necessary, we assume that $u_j^k \to u^k \in \mathcal{F}(\Omega)$ weakly as $j \to \infty$, and $u^k \nearrow u \in \mathcal{F}(\Omega)$ as $k \to \infty$. Then Theorem 2 below shows that $(dd^c u^k)^n = -\max(v/k, -1)(dd^c v)^n$, which implies $(dd^c u)^n = \chi_{\{v=-\infty\}}(dd^c v)^n$. If furthermore $\chi_{\{v>-\infty\}}(dd^c v)^n = (dd^c g)^n$ for $g \in \mathcal{F}^a(\Omega)$, then we take $g_j^k \in \mathcal{E}_0(\Omega)$ such that

$$\begin{aligned}(dd^c g_j^k)^n &= \max((v+k)/k, 0)(dd^c v_j)^n \\ &= \max((v+k)/k, 0)(dd^c \max(v_j, -k-1))^n.\end{aligned}$$

By the comparison theorem [BT] we have $0 > g_j^k \geq \max(v_j, -k-1) \geq v$. By Theorem 2 again, we assume that g_j^k converges to a bounded psh function g^k in C_n on each $E \subset\subset \Omega$. Letting $j \to \infty$ we get

$$(dd^c g^k)^n = \max((v+k)/k, 0)(dd^c v)^n = \max((v+k)/k, 0)(dd^c g)^n \leq (dd^c g)^n,$$

which implies $0 > g^k \geq g$. Hence g^k decreases to some $g_1 \in \mathcal{F}^a(\Omega)$. By Theorem 5.15 in [C1] we have $g_1 = g$. Since $(dd^c(g_j^k + u_j^k))^n \geq (dd^c g_j^k)^n + (dd^c u_j^k)^n = (dd^c v_j)^n$ we get $v_j \geq g_j^k + u_j^k$ and hence $v \geq g + u$. The proof of Theorem 1 is complete.

THEOREM 2. *Suppose that* $v \in \mathcal{F}(\Omega)$, $v_j \in \mathcal{E}_0(\Omega)$ *and* $-1 \le \psi \in \mathrm{PSH}^-(\Omega)$ *are such that* $v_j \searrow v$ *as* $j \to \infty$ *and* v *is bounded on* $\{z \in \Omega;\ \psi(z) \ne -1\}$. *If* $u_j \in \mathcal{E}_0(\Omega)$ *are such that* $(dd^c u_j)^n = -\psi(dd^c v_j)^n$ *and* $u_j \to u \in \mathrm{PSH}(\Omega)$ *weakly in* Ω, *then* $(dd^c u)^n = -\psi(dd^c v)^n$, $u \ge v$ *and hence* $u \in \mathcal{F}(\Omega)$.

Proof. By the comparison theorem [BT] we get $0 \ge u_j \ge v_j \ge v$. Hence $u \ge v$ and $u \in \mathcal{F}(\Omega)$. To prove $(dd^c u)^n = -\psi(dd^c v)^n$, by Theorem 7 in [X1] or [C1] we have $-\psi(dd^c v_j)^n \to -\psi(dd^c v)^n$ weakly as $j \to \infty$, and hence it is enough to show that $u_j \to u$ in C_n on each $E \subset\subset \Omega$ as $j \to \infty$. Take $t < \inf_{\{\psi \ne -1\}} v$. Since

$$\begin{aligned}(dd^c v_j)^n &= \chi_{\{v_j > t\}}(dd^c v_j)^n + \chi_{\{v_j \le t\}}(dd^c v_j)^n \\ &\le (dd^c \max(v_j, t))^n + (dd^c u_j)^n \le (dd^c(\max(v_j, t) + u_j))^n,\end{aligned}$$

we have $v_j \ge u_j + \max(v_j, t)$ and thus $v \ge u + t$. Given $E \subset\subset \Omega$ and $0 < \varepsilon < -t$, Theorem 6.10 of [BT] shows that there exists $0 < \delta < 1$ such that $C_n\{z \in E;\ (1-\delta)v \le -\varepsilon\} < \varepsilon$. By quasicontinuity of psh functions and Hartogs' lemma, we only need to show that

$$C_n\{z \in E;\ u(z) > u_j(z) + 3\varepsilon\} \to 0 \quad \text{as } j \to \infty.$$

Let $l_j := \min_\Omega(\delta u_j + \varepsilon)$. Since $C_n\{z \in E;\ u_j(z) \le \delta u_j(z) - \varepsilon\} \le C_n\{z \in E;\ (1-\delta)\,v \le -\varepsilon\} < \varepsilon$, we have

$$C_n\{z \in E;\ u(z) > u_j(z) + 3\varepsilon\} \le C_n\{z \in \Omega;\ u(z) > \delta u_j(z) + 2\varepsilon\} + \varepsilon,$$

which, by the definition of C_n, does not exceed

$$\begin{aligned}&\sup\bigg\{\frac{1}{\varepsilon^n}\int\limits_{u > \delta u_j + \varepsilon} (u - \delta u_j - \varepsilon)^n (dd^c w)^n;\ w \in \mathrm{PSH}(\Omega),\ 0 < w < 1\bigg\} + \varepsilon \\ &\qquad = \sup\bigg\{\frac{1}{\varepsilon^n}\int\limits_{\max(u, l_j) > \delta u_j + \varepsilon} (\max(u, l_j) - \delta u_j - \varepsilon)^n (dd^c w)^n; \\ &\qquad\qquad\qquad w \in \mathrm{PSH}(\Omega),\ 0 < w < 1\bigg\} + \varepsilon,\end{aligned}$$

which by the Lemma is less than

$$\begin{aligned}\frac{(n!)^2\delta^n}{\varepsilon^n}\int\limits_{\max(u, l_j) > \delta u_j + \varepsilon} (dd^c u_j)^n + \varepsilon &\le \frac{(n!)^2\delta^n}{\varepsilon^n}\int\limits_{u > \delta u_j + \varepsilon} (dd^c v_j)^n + \varepsilon \\ &\le \frac{(n!)^2\delta^n}{\varepsilon^n}\int\limits_{u > \delta u_j + \varepsilon} \phi(dd^c v_j)^n + 2\varepsilon\end{aligned}$$

for some $\phi \in C_0^\infty(\Omega)$ with $0 \le \phi \le 1$, where we have used the fact that there exists $E_1 \subset\subset \Omega$ such that $\int_{\Omega \setminus E_1} (dd^c v_j)^n \le \varepsilon^{n+1}/(n!)^2\delta^n$ for all j, which follows from $(dd^c v_j)^n \to (dd^c v)^n$ weakly and $\lim_{j\to\infty}\int_\Omega (dd^c v_j)^n = \int_\Omega (dd^c v)^n < \infty$. Since $v - t \ge u \ge v$ and $u_j \ge v$, we have $\{u > \delta u_j + \varepsilon\} \subset$

$\{v > a\}$ for $a := (\varepsilon + t)/(1-\delta) < 0$. So the last integral equals

$$\int\limits_{\max(u,a)>\delta\max(u_j,a)+\varepsilon} \phi(dd^c v_j)^n \le \frac{1}{\varepsilon} \int\limits_{\max(u,a)>\delta\max(u_j,a)+\varepsilon} \phi(\max(u,a)-\max(u_j,a))(dd^c v_j)^n.$$

Since $v_j \ge u_j + t$ and $v_j \ge v \ge u + t$ we have $\max(u,a) - \max(u_j,a) = 0$ if $v_j \le a + t$. By the quasicontinuity of u there exists an open subset $O_\varepsilon \subset \Omega$ such that $C_n(O_\varepsilon) < \varepsilon^{n+2}$ and $u \in C(\Omega \setminus O_\varepsilon)$. It then follows from Hartogs' lemma that $\varepsilon^{n+2} + \max(u,a) \ge \max(u_j,a)$ on $\operatorname{supp}\phi \setminus O_\varepsilon$ for all j large enough. Hence by the definition of C_n, for all j large enough we have

$$\begin{aligned} C_n\{z \in E;\, u(z) > u_j(z) + 3\varepsilon\} &\le \frac{(n!)^2\delta^n}{\varepsilon^{n+1}} \int\limits_\Omega \phi(\varepsilon^{n+2} + \max(u,a) - \max(u_j,a))(dd^c v_j)^n \\ &\quad + 2\varepsilon + \varepsilon(n!)^2(\varepsilon^{n+2} - a)(-a-t)^n \sup_\Omega |\phi| \\ &= \frac{(n!)^2\delta^n}{\varepsilon^{n+1}} \int\limits_\Omega \phi(\max(u,a) - \max(u_j,a))((dd^c \max(v_j, a+t))^n \\ &\qquad\qquad - (dd^c \max(v, a+t))^n) \\ &\quad + \frac{(n!)^2\delta^n}{\varepsilon^{n+1}} \int\limits_\Omega \phi(\max(u,a) - \max(u_j,a))(dd^c \max(v,a+t))^n + \mathrm{O}(\varepsilon) \\ &= \mathrm{O}(\varepsilon) \quad \text{as } j \to \infty, \end{aligned}$$

where the last estimate follows from Theorem 1 and Corollary 1 in [X1] or [C2]. By the arbitrariness of $\varepsilon > 0$ we see that $u_j \to u$ in C_n on E as $j \to \infty$, which concludes the proof of Theorem 2.

COROLLARY 1. *A positive measure μ in Ω can be written as $\mu = (dd^c v)^n$ for $v \in \mathcal{F}(\Omega)$ if and only if*

$$\mu = (dd^c u_1)^n + \chi_{\{u_2=-\infty\}}(dd^c u_2)^n$$

for some $u_1 \in \mathcal{F}^a(\Omega)$ and $u_2 \in \mathcal{F}(\Omega)$.

Proof. The "only if" part. By [C2], [K] there exists a decreasing sequence $g_k \in \mathcal{E}_0(\Omega)$ such that $g_k \ge v$ in Ω and $(dd^c g_k)^n = \chi_{\{v>-k\}}(dd^c v)^n$. Then $u_1 := \lim_{k\to\infty} g_k \in \mathcal{F}^a(\Omega)$ and $(dd^c u_1)^n = \chi_{\{v\ne-\infty\}}(dd^c v)^n$. Hence we have $\mu = (dd^c u_1)^n + \chi_{\{v=-\infty\}}(dd^c v)^n$.

The "if" part. From Theorem 1 it turns out that there exists $h \in \mathcal{F}(\Omega)$ such that $\mu = (dd^c u_1)^n + (dd^c h)^n$. By Theorem 5.11 in [C1] there exist $\psi \in \mathcal{E}_0(\Omega)$ and $f \in L_{\mathrm{loc}}((dd^c\psi)^n)$ such that $(dd^c u_1)^n = f(dd^c\psi)^n$. Take a sequence $h_j \in \mathcal{E}_0(\Omega)$ such that $h_j \searrow h$ as $j \to \infty$. Since $\min(f, k^n)(dd^c\psi)^n +$

$(dd^c h_j)^n \le (dd^c(k\psi + h_j))^n$, by [C2], [K] there exist $v_j^k \in \mathcal{E}_0(\Omega)$ such that $(dd^c v_j^k)^n = \min(f, k^n)(dd^c\psi)^n + (dd^c h_j)^n$ and hence the comparison theorems in [BT], [C1] imply that $0 > v_j^k \ge k\,\psi + h \ge u_1 + h$. Repeating the proof of Theorem 2 we obtain an increasing sequence v^k in $\mathcal{F}(\Omega)$ such that $(dd^c v^k)^n = \min(f, k^n)(dd^c\psi)^n + (dd^c h)^n$ and $0 > v^k \ge u_1 + h$. Therefore, $v := (\lim_{k\to\infty} v^k)^* \in \mathcal{F}(\Omega)$ and $\mu = (dd^c v)^n$. The proof of Corollary 1 is complete.

COROLLARY 2. *For any set $B = \{z_1, \ldots, z_m\}$ of points in Ω and nonnegative constants $c_1, \ldots, c_m$ there exists a function $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus B)$ such that $u = 0$ on $\partial\Omega$ and $(dd^c u)^n = \sum_{j=1}^m c_j \delta_{z_j}$ in Ω, where δ_{z_j} denotes the Dirac measure at z_j.*

Proof. Take the pluricomplex Green function g_{z_j} of Ω with logarithmic pole at z_j and set $v = \sum_{j=1}^m c_j^{1/n} g_{z_j}$. Then $v \in \mathcal{F}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus B)$ and $v = 0$ on $\partial\Omega$. By Lemma 5 in [X3], $(dd^c v)^n$ has zero mass at any point $z \notin B$ and has mass c_j at z_j. Therefore, by Theorem 1 we get the required function u and the proof is complete.

References

[BT] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.

[C1] U. Cegrell, *The general definition of the complex Monge–Ampère operator*, Ann. Inst. Fourier (Grenoble) 54 (2004), 159–197.

[C2] —, *Pluricomplex energy*, Acta Math. 180 (1998), 187–217.

[K] S. Kołodziej, *The range of the complex Monge–Ampère operator, II*, Indiana Univ. Math. J. 44 (1995), 765–782.

[X1] Y. Xing, *Convergence in capacity*, Umeå Univ., Research Reports No 1, 2007.

[X2] —, *Continuity of the complex Monge–Ampère operator*, Proc. Amer. Math. Soc. 124 (1996), 457–467.

[X3] —, *The complex Monge–Ampère equations with a countable number of singular points*, Indiana Univ. Math. J. 48 (1999), 749–765.

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