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Matrix inequalities and the complex Monge–Ampère operator

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Abstract. We study two known theorems regarding Hermitian matrices: Bellman's principle and Hadamard's theorem. Then we apply them to problems for the complex Monge–Ampère operator. We use Bellman's principle and the theory for plurisubharmonic functions of finite energy to prove a version of subadditivity for the complex Monge–Ampère operator. Then we show how Hadamard's theorem can be extended to polyradial plurisubharmonic functions.

1. Introduction. We begin our study with a matrix equality that is reasonably well known in pluripotential theory, nevertheless the proof is not as simple as it may appear, so we give a careful proof of all steps.

Lemma 1.1 (Bellman's principle [Gav77]). *Let $\mathcal{A}$ denote the family of all positive definite $n \times n$ Hermitian matrices with determinant* 1. *For any positive definite Hermitian matrix B we have*

$$(\det B)^{1/n} = \frac{1}{n} \inf_{A \in \mathcal{A}} \operatorname{tr}(AB).$$

To prove the equality in Lemma 1.1 one first shows that for an arbitrary matrix $A \in \mathcal{A}$ the left-hand side is less than the right-hand side, which would surely be a simple matter if the product of two symmetric matrices were symmetric. Since it is symmetric only if the matrices commute, some preliminary work is necessary.

The proof of the lemma above is based on conjunctive reduction of two matrices. The principle of conjunctive reduction can be found in the literature (see e.g. [MM64].) Furthermore a careful proof of Lemma 1.1 gives explicit algorithms for calculating the matrix A for a given matrix B.

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THEOREM 1.2. *Let A and B be two Hermitian matrices, and let A be positive definite. Then there is a nonsingular matrix Q such that*

$$Q^* A Q = I, \tag{1.1}$$

$$Q^* B Q = \operatorname{diag}(k_1, \ldots, k_n) = D_k, \tag{1.2}$$

where D_k is the diagonal matrix with the eigenvalues of $A^{-1}B$ as entries.

Proof. Note that $A^{1/2}$ and $A^{-1/2}$ are both positive semidefinite Hermitian matrices. Thus $A^{-1/2}BA^{-1/2}$ is Hermitian. Since

$$\begin{aligned}\det(A^{-1}B - \lambda I) \\ &= \det(A^{1/2})\det(A^{-1}B-\lambda I)\det(A^{-1/2}) = \det(A^{1/2}(A^{-1}B-\lambda I)A^{-1/2}) \\ &= \det(A^{-1/2}BA^{-1/2} - \lambda I),\end{aligned}$$

it follows that $A^{-1}B$ has real eigenvalues $(k_1, \ldots, k_n)$. Let P be a unitary matrix such that $P^* A^{-1/2} B A^{-1/2} P = \operatorname{diag}(k_1, \ldots, k_n)$. Finally, let $Q = A^{-1/2}P$. ∎

1.1. *Proof of Lemma 1.1.* Let A and B be as in Theorem 1.2, with $\det A = 1$. With the above notation,

$$\begin{aligned}\det B = \det(A^{-1}B) = \prod k_i \\ &\le \left(\frac{1}{n}\sum k_i\right)^n = \left(\frac{1}{n}\operatorname{tr}(D_k)\right)^n \\ &= \left(\frac{1}{n}\operatorname{tr}(Q^*BQ)\right)^n = \left(\frac{1}{n}\operatorname{tr}(QQ^*B)\right)^n = \left(\frac{1}{n}\operatorname{tr}(A^{-1}B)\right)^n,\end{aligned}$$

where the second to last equality follows from the fact that $\operatorname{tr}(FG) = \operatorname{tr}(GF)$ for any two matrices F and G. The last equality follows from the definition of Q.

To show that the infimum is attained, suppose B is diagonal. If $\det B \neq 0$, we take A as the diagonal matrix with diagonal elements $a_{ii} = b_{ii}^{-1}(\det B)^{1/n}$. On the other hand, if $\det B = 0$, then—since B is diagonal—we have $b_{ii} = 0$ if $i \in \Lambda$, for some index set $\Lambda \subset \{1, \ldots, n\}$. Let k be the number of elements in Λ. Take A to be a diagonal matrix. With this notation we have

$$\operatorname{tr}(AB) = \sum_{i \notin \Lambda} a_{ii} b_{ii}.$$

So if we choose $a_{ii} = 1/k$ if $i \in \Lambda$, we have $\operatorname{tr}(AB) = k^{-1}\operatorname{tr}(B)$. Take $a_{ii} = k^{(n-p)/p}$ for $i \notin \Lambda$; then $\det A = 1$.

If B is not diagonal we simply diagonalize B. Let T be the matrix that diagonalizes B, i.e. $B = T^* D_B T$, where D_B has the eigenvalues of B on the diagonal. Now

$$\det B = \det D_B = \left(\frac{1}{n}\inf_{A\in\mathcal{A}}\{\operatorname{tr}(AD_B)\}\right)^n = \left(\frac{1}{n}\inf_{A\in\mathcal{A}}\{\operatorname{tr}(AT^*BT)\}\right)^n$$
$$= \left(\frac{1}{n}\inf_{A\in\mathcal{A}}\{\operatorname{tr}(TAT^*B)\}\right)^n = \left(\frac{1}{n}\inf_{A'\in\mathcal{A}}\{\operatorname{tr}(A'B)\}\right)^n.$$

The last equality is just change of coordinates. ∎

Remark. The proof of Lemma 1.1 gives an *algorithm for calculating the matrix* A:

(1) Suppose B is a Hermitian matrix. Calculate the diagonalization matrix T such that $T^*BT = \operatorname{diag}(\lambda_1,\ldots,\lambda_n)$, where $\lambda_k \le \lambda_{k+1}$.

(2) (a) If $\det B \neq 0$ let $A' = (\det B)^{1/n}\operatorname{diag}(1/\lambda_1,\ldots,1/\lambda_n)$.

(b) If $\det B = 0$, and B has rank $n-p$, let

$$A'_k = k^{n-p}\operatorname{diag}(k,\ldots,k,1/k,\ldots,1/k),$$

with p k's.

(3) (a) If $\det B \neq 0$, let $A = TAT^*$, and we have

$$\det B = \left(\frac{1}{n}\operatorname{tr}(AB)\right)^n.$$

(b) If $\det B = 0$, let $A_k = TA_kT^*$, and we have

$$\det B = \lim_{k\to\infty}\left(\frac{1}{n}\operatorname{tr}(A_kB)\right)^n.$$

The algorithm above must of course be improved if used in a numerical calculation.

2. Some applications of Bellman's principle to the complex Monge–Ampère operator. The complex Monge–Ampère operator, $(\mathrm{dd^c}u)^n$, can be defined on quite general sets of plurisubharmonic functions (see e.g. [BT76], [Ceg98]). If $u \in \mathcal{C}^2(\Omega)$ then

$$(\mathrm{dd^c}u)^n = 4^n \det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) dV, \tag{2.1}$$

where dV is the volume measure in $\mathbb{C}^n$. Let us denote the above determinant by $M(u)$.

If u is a $\mathcal{C}^{1,1}$ function, then u is twice differentiable almost everywhere, and the second partial derivatives of u defined pointwise as locally bounded functions coincide with distributional derivatives. Furthermore (2.1) holds for $u \in \mathcal{C}^{1,1}$ (see [Bło96]).

For every complex $n\times n$ Hermitian positive matrix A, consider the Kähler form

$$\Delta_A = \frac{1}{n}\sum_{j,k}\frac{a_{ij}\partial^2}{\partial z_j \partial \bar{z}_k}. \tag{2.2}$$

Then

$$M(u(z)) = (\inf_{A\in\mathcal{A}} \Delta_A u(z))^n \quad \text{if } u \in \mathcal{PSH}\cap\mathcal{C}^{1,1}. \tag{2.3}$$

As it is written, (2.3) does not hold on less smooth plurisubharmonic functions since the right hand side is a distribution, and it is generally hard to multiply distributions.

2.1. *Examples*

EXAMPLE 2.1. Let $u(z) = |z_1 z_2|^2$. Then the Hessian H_u is given by

$$H_u(z) = \begin{bmatrix} |z_1|^2 & z_1\bar{z}_2 \\ z_2\bar{z}_1 & |z_2|^2 \end{bmatrix}, \tag{2.4}$$

and the Monge–Ampère operator of u is zero everywhere.

H_u has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = |z_1|^2 + |z_2|^2$, and away from both coordinate axes, is diagonalized by

$$T_u = \begin{bmatrix} -\dfrac{\bar{z}_2}{\bar{z}_1\sqrt{1+|z_2/z_1|^2}} & \dfrac{1}{\sqrt{1+|z_2/z_1|^2}} \\ \dfrac{z_1}{z_2\sqrt{1+|z_1/z_2|^2}} & \dfrac{1}{\sqrt{1+|z_1/z_2|^2}}. \end{bmatrix}. \tag{2.5}$$

Since

$$D = T_u H_u T_u^* = \begin{bmatrix} 0 & 0 \\ 0 & |z_1|^2+|z_2|^2 \end{bmatrix}$$

we choose, according to the proof of Lemma 1.1,

$$A = \begin{bmatrix} k & 0 \\ 0 & 1/k \end{bmatrix}.$$

Straightforward calculations give

$$\operatorname{tr}(T_u^* A T_u H_u) = \frac{|z_1|^2+|z_2|^2}{k}.$$

The Monge–Ampère operator of u is zero and there is an eigenvalue $\lambda_1 = 0$ everywhere. Our choice of T_u is a change of coordinates in $\mathbb{C}^2$ such that in the new coordinate $z' = T_u(z)$ we have $\partial^2_{z_1'\bar{z}_1'} u = 0$. That is, u is harmonic on the line $\{z_2' = 0\}$.

EXAMPLE 2.2. Let $u = |z_1 + z_3|^4 + |z_2|^4$. Then the Hessian H_u is given by

$$H_u(z) = \begin{pmatrix} 4|z_1+z_3|^2 & 0 & 4|z_1+z_3|^2 \\ 0 & 4|z_2|^2 & 0 \\ 4|z_1+z_3|^2 & 0 & 4|z_1+z_3|^2 \end{pmatrix}, \tag{2.6}$$

and the Monge–Ampère operator of u is zero everywhere. Suppose we stay away from $z_2 = 0$. Then H_u has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 4|z_2|^2$, and $\lambda_3 = 8|z_1 + z_3|^2$, and is diagonalized by

$$T_u = \begin{pmatrix} -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}. \tag{2.7}$$

We choose, according to the proof of Lemma 1.1,

$$A = \mathrm{diag}(k^2, 1/k, 1/k).$$

Straightforward calculations give

$$\mathrm{tr}(T_u^* A T_u H_u) = \frac{4|z_2|^2 + 4|z_1 + z_3|^2}{k}.$$

If $z_2 = 0$ we get the eigenvalues $\lambda_{1,2} = 0$ and $\lambda_3 = 8|z_1 + z_3|^2$, and H_u is diagonalized by exactly the same matrix T_u as above. Choosing $A = \mathrm{diag}(k, k, 1/k^2)$, we get

$$\mathrm{tr}(T_u^* A T_u H_u) = \frac{8|z_1 + z_3|^2}{k}.$$

If $z_3 + z_1 = 0$, we get the eigenvalues 0 and $4|z_2|^2$, and

$$\mathrm{tr}(T_u^* A T_u H_u) = \frac{4|z_2|^2}{k^2}.$$

In my opinion the examples above show quite clearly why the following theorem is true:

THEOREM 2.3 ([BK77]). *Let $p \in \mathbb{C}^2$ and assume $u \in \mathcal{PSH} \cap \mathcal{C}^3$, and $(dd^c u)^2 = 0$ in a neighbourhood of p. If $dd^c u(p) \neq 0$, then there is a complex manifold M through p such that $u|_M$ is harmonic on M.*

Proof. Let H_u be the complex Hessian of u. The null space of H_u is a complex subspace of $\mathbb{C}^2$. Thus the integral curves of the vector field of eigenvectors corresponding to the eigenvalue 0 over every point form the required complex manifold M. Since $0 = M_u = \lim_{k\to 0} \mathrm{tr}(A_k' H_u)$, where $A_k' = T_u^* \,\mathrm{diag}(k, 1/k) T_u$, and T_u is the matrix that diagonalizes H_u, it is clear that u is harmonic along M. ∎

Using estimates for the *energy class* $\mathcal{F}$ of plurisubharmonic functions with bounded (classical) energy, developed in the papers [Ceg98], [Ceg01], and [Ceg02] we can generalize two theorems by Błocki [Bło96].

THEOREM 2.4. *Let Ω be a hyperconvex set and suppose $u \in \mathcal{F}(\Omega)$. Furthermore assume that ν is a positive measure absolutely continuous with respect to the Lebesgue measure on Ω. Then the following are equivalent*:

(1) $(\mathrm{dd^c} u)^n \geq \nu$,
(2) $\Delta_A u \geq \nu^{1/n}$ *for all* $A \in \mathcal{A}$,
(3) $(\mathrm{dd^c} u_\delta)^n \geq (\nu^{1/n} * \varrho_\delta)^n$, *where* $u = \varrho_\delta$ *is the usual regularization with an approximative identity.*

Proof. Part of this proof is just a line by line copy of the proof of Theorem 3.10 in [Bło96].

We begin by showing that (3)⇒(1). Since $(\mathrm{dd^c} u_\delta)^n \to (\mathrm{dd^c} u)^n$ weakly as $\delta \to 0$, the implication follows from the convergence of $\nu^{1/n} * \varrho_\delta$.

Assume $\Delta_A u \geq \nu^{1/n}$ for all $A \in \mathcal{A}$. From (2.3) it follows that $(\mathrm{dd^c} u_\delta)^n = (\inf \Delta_A u_\delta)^n = (\inf(\Delta_A u) * \varrho_\delta)^n \geq \nu^{1/n}$. Note that since $u \in \mathcal{F}$, u has finite classical energy so the second to last equality holds. Hence (2)⇒(3).

To show that (1)⇒(2), fix $A \in \mathcal{A}$ and take any plurisubharmonic $\psi \in \mathcal{E}_0 \cap \mathcal{C}^\infty$. Set $G := (\mathrm{dd^c}\psi)^n$. Then $G^{1/n}$ is smooth. Let B be any ball in Ω and solve the equation $\Delta_A \varphi = G^{1/n}$ with boundary data $\varphi = \psi$ on ∂B. It is clear that $\varphi \in \mathcal{C}^\infty(B)$, and from (2.3) it follows that $(\mathrm{dd^c}\varphi)^n \leq G$ wherever φ is plurisubharmonic. According to [Bło96, Theorem 3.9, p. 735] it follows that $\psi \leq \varphi$, and so $\int_B \Delta_A \psi \geq \int_B \Delta_A \varphi \geq \int_B G^{1/n}\, dV$.

Let $u \in \mathcal{F}(\Omega)$. Then there is a sequence $\{\psi_j\}_j \subset \mathcal{E}_0 \cap \mathcal{C}^\infty(\Omega)$ such that $\psi_j \to u$ in capacity (see [Ceg02]). Let $G_j = (\mathrm{dd^c}\psi_j)^n$. Then $\Delta_A \psi_j \geq G_j^{1/n}$ and since $(\mathrm{dd^c}\psi_j)^n$ converges weak* to $(\mathrm{dd^c} u)^n$ as $j \to \infty$, we get $\Delta_A u \geq \nu^n$. ∎

Using the fact that the mapping $(\det B)^{1/n}$ is concave on the set of positive definite Hermitian matrices, together with Theorem 2.4 one gets the following subadditivity theorem:

THEOREM 2.5. *Let Ω be a hyperconvex set and suppose $u, v \in \mathcal{F}(\Omega)$. Furthermore assume that ν and μ are positive measures absolutely continuous with respect to the Lebesgue measure on Ω, and that $(\mathrm{dd^c} u)^n \geq \mu$ and $(\mathrm{dd^c} v)^n \geq \nu$. Then*

$$(\mathrm{dd^c}(u+v))^n \geq (\mu^{1/n} + \nu^{1/n})^n. \tag{2.8}$$

Note how this contrasts with the following lemma:

LEMMA 2.6. *Suppose Ω is a hyperconvex domain in $\mathbb{C}^n$ and that $u, v \in \mathcal{F}(\Omega)$. Then*

$$\int_\Omega (dd^c(u+v))^n \le \Big[\Big(\int_\Omega (dd^c u)^n\Big)^{1/n} + \Big(\int_\Omega (dd^c v)^n\Big)^{1/n}\Big]^n. \tag{2.9}$$

For a proof see [CW03].

3. Lelong numbers. The *Lelong number* of a function u at $x \in \mathbb{C}^n$ is defined as

$$\nu(u,x) = \lim_{r\to 0} \frac{1}{(2\pi)^n} \int_{\|z-x\|\le r} dd^c u \wedge (dd^c \log\|z-x\|)^{n-1}. \tag{3.1}$$

Let ω_n be the $(2n-1)$-volume of the unit sphere in $\mathbb{C}^n$. Let $B(r;x)$ be the ball of radius r with center x, and write $\partial B(r;x) = S(r;x)$. Furthermore let $B(r) = B_r = B(r;0)$, and the same for the sphere. If we define

$$M(u,r,x) = \sup_{z\in B(r;x)} u(z),$$
$$\lambda(u,r,x) = \frac{1}{\omega_n} \int_{S(1;0)} u(x+rz)\, dS_1(z),$$

we have

$$\nu(u,x) = \lim_{r\to 0} \frac{M(u,r,x)}{\log r} = \lim_{r\to 0} \frac{\lambda(u,r,x)}{\log r} \tag{3.2}$$

(see e.g. [Kis79]).

Given a function $u : \mathbb{C}^n \supset \Omega \to \mathbb{R} \cup \{-\infty\}$ we define the *slice* of u through 0 and $p \in \mathbb{C}^n$ by $u_p(\zeta) := u(\zeta p)$, $\zeta \in \mathbb{C}$, wherever this expression makes sense.

LEMMA 3.1. *If $u : \mathbb{C}^n \supset \Omega \to \mathbb{R} \cup \{-\infty\}$ and $0 \in \Omega$, then $\nu(u_p, 0) \ge \nu(u,0)$.*

Proof. Since $\log r < 0$ if $r < 1$, we have

$$\lim_{r\to 0} \frac{\sup_{\|z\|\le r} u(z)}{\log r} \le \lim_{r\to 0} \frac{\sup_{\|\zeta\|\le r} u(\zeta p)}{\log r}. \ ∎$$

Using this lemma we can prove that the reverse inequality holds almost everywhere.

LEMMA 3.2. *If $u \in \mathcal{PSH}(B)$, then $\nu(u_q,0) = \nu(u,0)$ for all $q \in B \setminus A$, where A is a set of Lebesgue measure zero.*

Proof. This follows from

$$\nu(u,0) = \frac{1}{\omega_n} \int_{S(1)} \nu(u(\zeta y), 0)\, dS_1(y),$$

and from Lemma 3.1 above. ∎

For functions radial in at least one variable Lemma 3.2 above can be considerably strengthened.

Lemma 3.3. *Assume* $u : \mathbb{C}^2 \supset B \to \mathbb{R} \cup \{-\infty\}$, $u \in \mathcal{PSH}(B)$, *and* $u(|z|, w) = u(z, w)$. *Suppose that* $\nu(u, 0) = 0$. *Then for all* $y = (y_1, y_2) \in B$ *such that* $y_1 y_2 \neq 0$, $\nu(u_y, 0) = 0$.

Proof. Take $p = (z, w)$, $q = (z', w') \in \mathbb{C}^2$ with $|z| < |z'|$ and $|w| = |w'| = R$. Since $u(r, w)$ is an increasing function in the radial variable we have

$$\sup_{|w|=R} u(|z|, w) \le \sup_{|w'|=R} u(|z'|, w').$$

That is, $\nu(u_p, 0) \ge c\nu(u_q, 0)$ for some constant c. Take $y = (y_1, y_2)$ such that neither y_1 nor y_2 is zero. According to Lemma 3.2 there is a point $y' = (y_1', y_2)$ with $|y_1'| < |y_1|$ such that $\nu(u_{y'}, 0) = 0$, but then $0 \ge \nu(u_y, 0)$. ■

4. Bellman's principle for polyradial functions. In $\mathbb{C}^n$ it is well known that the Lelong number is dominated by the Monge–Ampère operator in the following way:

$$(2\pi\nu(u, x))^n \le (\mathrm{dd^c} u)^n(\{x\}). \tag{4.1}$$

Note that if $u(z_1, z_2) = \max\{1/k \log|z_1|, k\log|z_2|\}$, then one can show that $(\mathrm{dd^c} u)^2(0) = 4\pi^2\delta_0$, and $\nu(u, 0) = 1/k$, so inequality (4.1) cannot be reversed.

For polyradial functions at least, it seems that the "correct" version of (4.1) should be in the flavor of Bellman's principle.

We begin with a simple lemma, concerning subharmonic functions in the complex plane.

Lemma 4.1. *Let* D *denote the unit disc in* $\mathbb{C}^1$, *and suppose* $u \in \mathcal{SH}(D)$, $u \not\equiv -\infty$, *is radial* (*i.e.* $u(|z|) = u(z)$). *Take* $\varepsilon > 0$, *and let* $D_\varepsilon = \{z : u(z) > (k + \varepsilon)\log|z|\} \cup \{0\}$. *Then* $\nu(u, 0) = k = \mathrm{const} \cdot \partial\overline{\partial} u(\{0\})$ *if and only if* D_ε *is a disc of positive radius centered at the origin.*

Proof. Since u is radial and $u \not\equiv -\infty$, it is a convex function in $\log r$, continuous except for a possible pole at the origin. The result follows. ■

The following theorem, for polyradial functions, is an analogue to Bellman's principle for plurisubharmonic twice differentiable functions.

Theorem 4.2. *Let* Ω *be a domain in* $\mathbb{C}^n$, *containing the origin. Assume* $u : \mathbb{C}^n \to \mathbb{R}$ *is plurisubharmonic and polyradial on* Ω. *Take any point* p_j *on the* z_j-*axis*, *and set*

$$v_j = \nu_{p_j}(u, 0).$$

Then $(\mathrm{dd^c} u)^n \le (2\pi)^n v_1 \cdots v_n$.

Proof. For simplicity, assume that the unit polydisc $D_1 \times \cdots \times D_1$ is contained in Ω.

Fix $\varepsilon > 0$. According to Lemma 4.1 there is a disc $D^1_\varepsilon \subset \mathbb{C}^1$ such that $u(z) \geq (v_1 + \varepsilon) \log |z_1|$ on $D^1_\varepsilon \times \{0\} \times \cdots \times \{0\}$. Using the maximum principle in one complex variable, and the fact that u is polyradial, we clearly obtain $u(z) \geq (v_1 + \varepsilon) \log |z_1|$ on $D^1_\varepsilon \times D_1 \times \cdots \times D_1$. As a result, $\max\{u(z), (v_1 + \varepsilon) \log |z_1|\} = u(z)$ on $D^1_\varepsilon \times D_1 \times \cdots \times D_1$.

By repeating this reasoning one gets

$$\max\{u(z), (v_1 + \varepsilon) \log |z_1|, \ldots, (v_n + \varepsilon) \log |z_n|\} = u(z)$$

on $D^1_\varepsilon \times \cdots \times D^1_\varepsilon$. On the other hand

$$\max\{u(z), (v_1 + \varepsilon) \log |z_1|, \ldots, (v_n + \varepsilon) \log |z_n|\} \\ \geq \max\{(v_1 + \varepsilon) \log |z_1|, \ldots, (v_n + \varepsilon) \log |z_n|\}.$$

Let $u_\varepsilon(z) = \max\{u(z), (v_1 + \varepsilon) \log |z_1|, \ldots, (v_n + \varepsilon) \log |z_n|\}$ for $\varepsilon > 0$. Then $u_\varepsilon = u$ on $D^1_\varepsilon \times \cdots \times D^1_\varepsilon$, hence

$$\int\limits_{D^1_\varepsilon \times \cdots \times D^1_\varepsilon} (\mathrm{dd^c} u)^n = \int\limits_{D^1_\varepsilon \times \cdots \times D^1_\varepsilon} (\mathrm{dd^c} u_\varepsilon)^n \leq \int\limits_{D_1 \times \cdots \times D_1} (\mathrm{dd^c} u_\varepsilon)^n.$$

By the approximation theorem in [Ceg01] and monotone convergence, for every $\varepsilon > 0$ we have $(\mathrm{dd^c} u_\varepsilon)^n \to (2\pi)^n v_1 \cdots v_n$ as $\varepsilon \to 0$. ∎

To sum up, given a polyradial plurisubharmonic function u, we have

$$(2\pi\nu(u,0))^n \leq (\mathrm{dd^c} u)^n(\{0\}) \leq (2\pi)^n v_1 \cdots v_n, \tag{4.2}$$

where v_k is the Lelong number along the kth axis.

(4.2) is perhaps not so surprising if we consider the following special case of the Hadamard inequality:

PROPOSITION 4.3. *Let $B = (b_{ij})$ be a positive semidefinite Hermitian matrix. Then* $\det B \leq \prod b_{ii}$.

Proof. Since we might as well assume that $\det B \neq 0$, all diagonal elements b_{ii} are nonzero. Let $D = \mathrm{diag}(b_{11}^{-1/2}, \ldots, b_{nn}^{-1/2})$. Since $(DBDx, x) = (BDx, Dx)$, we see that DBD is positive semidefinite and thus

$$\frac{\det B}{\prod b_{ii}} = \det(BD^2) = \det(DBD) \leq \left(\frac{1}{n} \operatorname{tr}(DBD)\right)^n = 1,$$

where the inequality follows, as in the proof of Lemma 1.1, by comparison between arithmetic and geometric means. ∎

Note that Proposition 4.3 above is a nice analogue to Theorem 4.2 if we interpret $\det B$ as the residual Monge–Ampère mass at the origin of a polyradial function u, and b_{ii} as $\nu(u_i, 0)$, where $u_1 : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ is the slice of u along the z_1-axis, defined by $u_1(\zeta) = u(\zeta, 0, \ldots, 0)$ and so forth.

It is well known that for a large class of functions u, the determinant of the Hessian, $\det \partial^2_{j\bar{k}} u$, can be generalized to a measure. However it is not at all clear to me how to generalize $\prod \partial^2_{k\bar{k}} u$ for plurisubharmonic functions from some reasonable class.

We finish with an application of Proposition 4.3 to a very simple case. Let $u = \max(a_1 \log |z_1|, \dots, a_n \log |z_n|)$, and set $u_m = \max(u, -m)$. Let u_m^j be a family of smooth plurisubharmonic functions, decreasing to u_m. Then $(\mathrm{dd^c} u_m^j)^n \to (\mathrm{dd^c} u_m)^n$. Furthermore

$$\int M(u_m^j) \le \int \prod \partial^2_{k\bar{k}} u_m^j \tag{4.3}$$

by Proposition 4.3. Note that, in measure, $\partial^2_{1\bar{1}} u_m^j \to 2\pi a_1 \lambda_1$, where λ_1 is Lebesgue measure on the segment $|z_1| = e^{-m/a_1}$, and $|z_k| \le e^{-m/a_k}$. Thus it follows from (4.3) that

$$\int (\mathrm{dd^c} u_m)^n \le (2\pi)^n \prod a_k,$$

and using Bellman's principle on the sequence u_m^j above we obtain an inequality in the other direction. Hence $(\mathrm{dd^c} u)^n\{0\} = (2\pi)^n \prod a_k$. Note that with the notation of Theorem 4.2, $\nu_k = a_k$.

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