ANNALES
POLONICI MATHEMATICI
83.3 (2004)

Stability of the Cauchy functional equation in quasi-Banach spaces

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Abstract. Let X be a quasi-Banach space. We prove that there exists $K > 0$ such that for every function $w : \mathbb{R} \to X$ satisfying

$$\|w(s+t) - w(s) - w(t)\| \le \varepsilon(|s| + |t|) \quad \text{for } s, t \in \mathbb{R},$$

there exists a unique additive function $a : \mathbb{R} \to X$ such that $a(1) = 0$ and

$$\|w(s) - a(s) - s\theta(\log_2 |s|)\| \le K\varepsilon|s| \quad \text{for } s \in \mathbb{R},$$

where $\theta : \mathbb{R} \to X$ is defined by $\theta(k) := w(2^k)/2^k$ for $k \in \mathbb{Z}$ and extended in a piecewise linear way over the rest of $\mathbb{R}$.

1. Introduction. In this paper we investigate the behavior of functions $w : X \to Y$ satisfying

$$\|w(x+y) - w(x) - w(y)\| \le \varepsilon(\|x\| + \|y\|) \quad \text{for } x, y \in X. \tag{1}$$

Such functions are called *ε-quasi-additive*. The function which is ε-quasi-additive for a certain $\varepsilon > 0$ is called simply *quasi-additive*. The reader which is not familiar with this notion is referred to [3] and also to [2] where the class of quasi-linear functions is studied. Let us briefly mention that quasi-additive and quasi-linear functions are useful in investigation of the geometric structure of Banach spaces.

One of the main results of F. Sánchez [5] is that for every ε-quasi-additive function $w : \mathbb{R} \to X$, where X is a Banach space, there exists an additive function $a : \mathbb{R} \to X$ and a $\theta : \mathbb{R} \to X$ Lipschitz with constant ε such that

$$\|w(s) - a(s) - s\theta(\log_2 |s|)\| \le 19\varepsilon|s| \quad \text{for } s \in \mathbb{R}. \tag{2}$$

Since the natural setting for quasi-additive functions are quasi-Banach spaces, F. Sánchez also considers the case when X is a quasi-Banach space. He proves that for every p-Banach space X and every ε-quasi-additive map

2000 *Mathematics Subject Classification*: 39B82, 46A16.
Key words and phrases: stability, Cauchy functional equation.

$w : \mathbb{R} \to X$ there exists an additive function $a : \mathbb{R} \to X$ such that

$$\|w(x) - a(x)\| \le \varepsilon |s| \sqrt[p]{K_p + L_p |\log_2 |s||} \quad \text{for } s \in \mathbb{R}, \tag{3}$$

where K_p, L_p are constants depending on p.

In this paper we answer the problem posed by F. Sánchez in the last section of [5] concerning classification of quasi-additive maps from the real line into quasi-Banach spaces, and simultaneously generalize both the above described results.

We show that a result similar to (2) holds for quasi-Banach spaces. As an easy consequence we obtain an improvement of (3). Moreover, in both cases we obtain better approximation constants.

The main difference between the cases when the target space is a Banach space and when it is a quasi-Banach space is that a locally Lipschitz function with constant one with values in a quasi-Banach space may not be globally Lipschitz.

2. Quasi-Banach spaces. In this section we recall some basic facts concerning quasi-Banach spaces (for a detailed study we refer to [4, 2]) and prove some preliminary results.

Let X be a linear space. A *quasi-norm* is a real-valued function on X satisfying

- $\|x\| \ge 0$ and $\|x\| = 0$ if and only if $x = 0$.
- $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$, $x \in X$.
- There is a constant $K > 0$ such that $\|x + y\| \le K(\|x\| + \|y\|)$ for all $x, y \in X$.

A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\| \|$ is called a *p-norm* $(0 < p \le 1)$ if

$$\|x + y\|^p \le \|x\|^p + \|y\|^p \quad \text{for } x, y \in X.$$

In this case a quasi-Banach space is called a *p-Banach space*. Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki–Rolewicz Theorem [4] (see also [2]) each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms then quasi-norms, henceforth we restrict our attention mainly to p-norms.

From now on we fix $p \in (0, 1]$, a p-Banach space X and a constant $\varepsilon \ge 0$.

If X is a Banach space then a function $f : \mathbb{R} \to X$ locally Lipschitz with constant ε is globally Lipschitz with the same constant ε. We need the following generalization of this result for p-Banach spaces.

Proposition 2.1. *Let $f : \mathbb{R} \to X$ be such that*

$$\|f(s) - f(t)\| \le \varepsilon |s - t| \quad \textit{for } k \in \mathbb{Z},\ s, t \in [k, k + 1].$$

Then

(4) $\|f(s)-f(t)\| \le \varepsilon 2^{1/p-1}|s-t|$ *for* $s,t\in\mathbb{R}$, $|s-t|\le 1$,

(5) $\|f(s)-f(t)\| \le \varepsilon(|s-t|+2(1-p))^{1/p}$ *for* $s,t\in\mathbb{R}$.

Proof. We prove (4). If $[s,t]\subset[k,k+1]$ for some $k\in\mathbb{Z}$, then $\|f(s)-f(t)\| \le \varepsilon|s-t|$. If there exists $k\in\mathbb{Z}$ such that $k\in(s,t)$, then

$$\begin{aligned}\|f(s)-f(t)\|^p &\le \|f(s)-f(k)\|^p+\|f(k)-f(t)\|^p\\ &\le \varepsilon^p((k-s)^p+(t-k)^p).\end{aligned}$$

Since the function $[0,1]\ni t\mapsto t^p$ is concave we obtain $(k-s)^p+(t-k)^p \le 2((k-s)/2+(t-k)/2)^p = 2^{1-p}(t-s)^p$, which proves (4).

So let us now deal with (5). If $[s,t]\subset[k,k+1]$, then since $p\in(0,1]$ we get

$$\|f(s)-f(t)\| \le \varepsilon|s-t| \le \varepsilon(|s-t|+2(1-p))^{1/p}.$$

In the other case there exist $k,l\in\mathbb{Z}$ such that $[k,l]\subset[s,t]\subset[k-1,l+1]$. Then

$$\begin{aligned}\|f(s)-f(t)\|^p &\le \|f(s)-f(k)\|^p+\|f(k+1)-f(k)\|^p+\ldots+\\ &\qquad +\|f(l)-f(l-1)\|^p+\|f(t)-f(l)\|^p\\ &\le \varepsilon^p(k-s)^p+\varepsilon^p(l-k)+\varepsilon^p(t-l)^p\\ &= \varepsilon^p(t-s)+\varepsilon^p((k-s)^p-(k-s))+\varepsilon^p((t-l)^p-(t-l)).\end{aligned}$$

As the maximal value of the function $t\mapsto t^p-t$ on $[0,1]$ is $p^{p/(1-p)}(1-p)\le 1-p$, we obtain the assertion of the lemma. ■

3. Hyers theorem in quasi-Banach spaces. To deal with quasi-additive functions we will need a version of the Hyers–Rassias–Gajda theorem (see [3]) for quasi-Banach spaces. We modify the idea of K. Baron and P. Volkmann [1].

By 0^R, where $R<0$, we understand ∞.

THEOREM 3.1. *Let G be a quasi-Banach space and let $f:G\to X$ and $R\in\mathbb{R}\setminus\{1\}$ be such that*

$$\|f(x+y)-f(x)-f(y)\| \le \varepsilon(\|x\|^R+\|y\|^R) \quad \textit{for } x,y\in G.$$

Then there exists a unique additive function $a:G\to X$ such that

$$\|f(x)-a(x)\| \le \frac{\varepsilon}{|1-2^{p(R-1)}|^{1/p}}\|x\|^R \quad \textit{for } x\in G.$$

Proof. For $n\in\mathbb{Z}$ we define

$$f_n(x) := f(2^n x)/2^n \quad \text{for } x\in G.$$

One can easily check that

$$\|f_n(x)-f_{n+1}(x)\|^p \le \varepsilon^p(2^{p(R-1)})^n\|x\|^{Rp} \quad \text{for } x\in G\setminus\{0\}.$$

We first consider the case $R \in (-\infty, 1)$. We show that for every $x \in G \setminus \{0\}$ the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $k, n \in \mathbb{N}$, $k > n$. We have

$$\begin{aligned}\|f_n(x) - f_k(x)\|^p &\le \sum_{l=n}^{k-1} \|f_l(x) - f_{l+1}(x)\|^p \le \varepsilon^p \sum_{l=n}^{k-1} (2^{p(R-1)})^l \|x\|^{Rp} \\ &\le \varepsilon^p \, \frac{2^{p(R-1)n}}{1 - 2^{p(R-1)}} \, \|x\|^{Rp}.\end{aligned}$$

This shows that the limit $a(x) := \lim_{k\to\infty} f_k(x)$ is well defined and that

$$\|f(x) - a(x)\|^p = \lim_{k\to\infty} \|f_0(x) - f_k(x)\|^p \le \frac{\varepsilon^p}{1 - 2^{p(R-1)}} \, \|x\|^{Rp}.$$

By the standard procedure one can show that a is a unique additive function which satisfies the assertion of the theorem.

In the case when $R \in (1, \infty)$ we apply a similar reasoning to the sequence $\{f_{-n}(x)\}_{n\in\mathbb{N}}$ and easily obtain the desired result. ∎

In the case when $R = 0$ we obtain the following direct corollary.

Corollary 3.1. *Let G be a quasi-Banach space and let $f : G \to X$ be such that*

$$\|f(x + y) - f(x) - f(y)\| \le \varepsilon \quad \textit{for } x, y \in G.$$

Then there exists a unique additive function $a : G \to X$ such that

$$\|f(x) - a(x)\| \le \frac{\varepsilon}{(2^p - 1)^{1/p}} \quad \textit{for } x \in G.$$

Thus the Cauchy functional equation is stable for $R \ne 1$. Surprisingly, in the case $R = 1$ one can construct examples which show that there is no stability (see [3]). This means that it is important to describe the approximate solutions.

We will need a local version of Theorem 3.1 for $R = 0$ and functions defined on a subinterval of $\mathbb{R}$. The following theorem is a modification of the result of F. Skof [6] who proved it for Banach spaces.

Proposition 3.1. *Let $a > 0$ and $f : (0, a] \to X$ be such that*

$$\|f(s + t) - f(s) - f(t)\| \le \varepsilon \quad \textit{for } s, t, s + t \in (0, a].$$

Then there exists an additive function $A : \mathbb{R} \to X$ such that $A(a/2) = f(a/2)$ and

$$\|f(s) - A(s)\| \le \left(1 + \frac{2}{2^p - 1}\right)^{1/p} \varepsilon \quad \textit{for } s \in (0, a]. \tag{6}$$

Proof. We define the function $\widetilde{f} : \mathbb{R} \to X$ by $\widetilde{f}(s) := f(s - ka/2) + kf(a/2)$ for $s \in (ka/2, (k + 1)a/2]$, $k \in \mathbb{Z}$.

We now verify that $\widetilde{f}$ satisfies

$$\|\widetilde{f}(s+t)-\widetilde{f}(s)-\widetilde{f}(t)\| \le 2^{1/p}\varepsilon \quad \text{for } s,t\in(0,\infty). \tag{7}$$

So let $k,l\in\mathbb{Z}$ and $s\in(ka/2,(k+1)a/2]$, $t\in(la/2,(l+1)a/2]$. If $s+t\in((k+l)a/2,(k+l+1)a/2]$ then

$$\|\widetilde{f}(s+t)-\widetilde{f}(s)-\widetilde{f}(t)\| = \|f(s+t-(k+l)a/2)-f(s-ka/2)-f(t-la/2)\| \le \varepsilon.$$

If $s+t\in((k+l+1)a/2,(k+l+2)a/2]$ then

$$\begin{aligned}
&\|\widetilde{f}(s+t)-\widetilde{f}(s)-\widetilde{f}(t)\|^p \\
&\quad = \|f(s+t-(k+l+1)a/2)+f(a/2)-\widetilde{f}(s-ka/2)-\widetilde{f}(t-la/2)\|^p \\
&\quad \le \|f(s+t-(k+l+1)a/2)+f(a/2)-f(s+t-(k+l)a/2)\|^p \\
&\qquad + \|f(s+t-(k+l)a/2)-\widetilde{f}(s-ka/2)-\widetilde{f}(t-la/2)\|^p \\
&\quad \le \varepsilon^p+\varepsilon^p = 2\varepsilon^p.
\end{aligned}$$

Thus (7) is proved.

By Corollary 3.1 we obtain a unique additive function $A:\mathbb{R}\to X$ such that

$$\|\widetilde{f}(s)-A(s)\| \le \frac{2^{1/p}}{(2^p-1)^{1/p}}\,\varepsilon \quad \text{for } s\in(0,\infty). \tag{8}$$

By the definition of $\widetilde{f}$ we obtain

$$A(a/2)=\lim_{k\to\infty}A(ka/2)/k=\lim_{k\to\infty}\widetilde{f}(ka/2)/k=f(a/2).$$

We check (6). Let $s\in(0,a]$ be arbitrary. If $s\in(0,a/2]$ then $\widetilde{f}(s)=f(s)$, and (8) yields $\|f(s)-A(s)\|\le 2^{1/p}\varepsilon/(2^p-1)^{1/p}$. If $s\in(a/2,a]$ then $s=(s-a/2)+a/2$, and we get

$$\begin{aligned}
\|f(s)-A(s)\|^p &\le \|f(s)-f(s-a/2)-f(a/2)\|^p+\|f(a/2)+f(s-a/2)-A(s)\|^p \\
&\le \left(1+\frac{2}{2^p-1}\right)\varepsilon^p. \ \blacksquare
\end{aligned}$$

As an easy consequence we deduce that quasi-additive functions can be locally approximated by additive ones:

Corollary 3.2. *Let $f:(0,\infty)\to X$ be ε-quasi-additive. Then there exists a unique additive function $a:\mathbb{R}\to X$ such that $a(1)=0$ and*

$$\left|f(s)-a(s)-\frac{s}{q}\,f(q)\right| \le \left(1+\frac{2}{2^p-1}\right)^{1/p} 2q\varepsilon \quad \textit{for } s\in(0,2q],\ q\in\mathbb{Q}_+.$$

Proof. Fix $q\in\mathbb{Q}_+$. Since f is ε-quasi-additive we obtain

$$\|f(s+t)-f(s)-f(t)\| \le \varepsilon 2q \quad \text{for } s,t,s+t\in(0,2q].$$

By Proposition 3.1 we obtain an additive function $A_q : \mathbb{R} \to X$ such that

$$A_q(q) = f(q), \quad \|f(s) - A_q(s)\| \le S\varepsilon 2q \quad \text{for } s \in (0, 2q], \tag{9}$$

where $S = \left(1 + \frac{2}{2^p - 1}\right)^{1/p}$. Let $a_q(s) := A_q(s) - f(q)s/q$. Then a_q is additive and since q is rational, $a_q(1) = a_q(q)/q = 0$. We show that a_q does not depend on the choice of $q \in \mathbb{Q}_+$. So let $\widetilde{q} \in \mathbb{Q}_+$. Applying (9) for q and $\widetilde{q}$ we easily get $C > 0$ such that

$$\|a_q(s) - a_{\widetilde{q}}(s)\| \le C \quad \text{for } s \in (0, \min\{2q, 2\widetilde{q}\}].$$

Since $a_q(1) = a_{\widetilde{q}}(1)$ we obtain $a_q = a_{\widetilde{q}} =: a$.

The assertion of the corollary follows directly from (9). ∎

4. Quasi-additive functions on $(0, \infty)$**.** In this section we describe quasi-additive functions defined on $(0, \infty)$ and with values in a quasi-Banach space. In Theorems 4.1 and 4.2 we show that every quasi-additive function is approximately the sum of an additive function and a function of the type $s \mapsto s\theta(\log_2 s)$, where θ is a certain piecewise linear function.

To prove Theorem 4.1 we will need the following simple technical lemma (we use a different reasoning from that on page 507 of [5] as the latter contains a small mistake: the maximum of the function $|t \log_2 t|$ on the interval $[0, 1]$ is attained at $t = 1/e$ and not at $t = 1/2$ and is equal to $(\log_2 e)/e \approx 0.53$ and not to 0.5).

Lemma 4.1. *Let $K \in \mathbb{R}_+$ and let*

$$v_p(a) := \frac{K + \log_2(1 + a)}{(1 + a)^p} \quad \textit{for } a \in (0, \infty).$$

Let $g_p(a) := v_p(a) + v_p(1/a)$ for $a \in \mathbb{R}_+$. Denote by $\max(g_p)$ the maximal value of the function g_p. Then

$$\max(g_p) \le \frac{2}{e \ln 2} \frac{2^{Kp}}{p} \quad \textit{for } p \in (0, 1],\ Kp \le \log_2 e, \tag{10}$$

$$\max(g_p) \le 2K \quad \textit{for } p \in (0, 1],\ Kp > \log_2 e. \tag{11}$$

If $p = 1$ and $K = 0$ we additionally get

$$\max(g_1) = 1. \tag{12}$$

Proof. We first show (10) and (11). Since $g_p(a) = v_p(a) + v_p(1/a)$ we obtain $\max(g_p) \le 2\max(v_p)$. So let us now compute the maximal value of the function v_p. We have

$$v_p'(a) = \frac{-p}{(1 + a)^{p+1}} \left(K + \log_2(1 + a) - \frac{\log_2 e}{p}\right). \tag{13}$$

Let a_p be such that $K + \log_2(1 + a_p) = (\log_2 e)/p$. Then $(1 + a_p)^p = 2^{-Kp} e$. By (13), v_p is increasing for $a \le a_p$ and decreasing for $a \ge a_p$. We need to discuss two cases. If $2^{-Kp}e > 1$, then $a_p > 0$, and therefore by the above reasoning the function $(0,\infty) \ni s \mapsto v_p(s)$ attains its maximal value at a_p and therefore $\max(v_p) = 2^{Kp}/(e\ln(2)p)$. If $2^{-Kp}e \le 1$, then $a_p \le 0$ and therefore v_p is decreasing on $\mathbb{R}_+$, which yields $\max(v_p) = K$.

Now let us deal with (12). One can easily check that $\lim_{a\to 0+} g_1(a) = 0$. Since $g_1(a) = g_1(1/a)$ we get $\lim_{a\to\infty} g_1(a) = 0$. Notice that

$$g_1(a) = \frac{\log_2(1+a) + a(\log_2(1+a) - \log_2 a)}{1+a} = \log_2(1+a) - \frac{a}{1+a}\log_2 a.$$

Hence

$$g_1'(a) = \frac{\log_2 e}{1+a} - \frac{(1+a)-a}{(1+a)^2}\log_2 a - \frac{a}{1+a}\,\frac{\log_2 e}{a} = -\frac{\log_2 a}{(1+a)^2}.$$

This means that g_1 is increasing on $(0,1)$ and decreasing on $(1,\infty)$. Since $g_1(0) = g_1(\infty) = 0$, it follows that g_1 is a nonnegative function which attains its maximum at $a = 1$. Since $g_1(1) = 1$, the proof is complete. ∎

Now we can proceed to prove our main results. We need the following definition.

Definition 4.1. Let $\mathbf{x} := \{x_k\}_{k\in\mathbb{Z}}$ be a doubly infinite sequence of elements of X. We define $\theta_{\mathbf{x}} : \mathbb{R} \to X$ by

$$\theta_{\mathbf{x}}(s) := (x_{k+1} - x_k)(s-k) + x_k \quad \text{for } s \in [k,k+1],\ k \in \mathbb{Z}.$$

Then $\theta_{\mathbf{x}}(k) = x_k$ for $k \in \mathbb{Z}$, and $\theta_{\mathbf{x}}$ is affine on each interval $[k,k+1]$.

We will be interested in sequences $\mathbf{x} = \{x_k\}_{k\in\mathbb{Z}}$ which satisfy

$$\sup_{k\in\mathbb{Z}} \|x_k - x_{k+1}\| \le \varepsilon. \tag{14}$$

One can easily see that if $\mathbf{x}$ is a sequence of elements of a Banach space which satisfies (14) then $\theta_{\mathbf{x}}$ is Lipschitz with constant ε. However, this is no longer true for p-Banach spaces.

We now prove that the function $s \mapsto s\theta_{\mathbf{x}}(\log_2 s)$ is quasi-additive.

Theorem 4.1. *Let $\mathbf{x}$ be a sequence of elements of X satisfying* (14). *Let $w_{\mathbf{x}} : (0,\infty) \to X$ be defined by*

$$w_{\mathbf{x}}(s) := s\theta_{\mathbf{x}}(\log_2 s) \quad \textit{for } s \in (0,\infty).$$

Then

$$\|w_{\mathbf{x}}(s+t) - w_{\mathbf{x}}(s) - w_{\mathbf{x}}(t)\| \le \frac{4}{4^p}\left(\frac{C}{p}\right)^{1/p}\varepsilon(s+t) \quad \textit{for } s,t \in (0,\infty),$$

where $C = 2/(e\ln 2) \approx 1.0615$. Moreover, if $p = 1$ we have

$$\|w_{\mathbf{x}}(s+t) - w_{\mathbf{x}}(s) - w_{\mathbf{x}}(t)\| \le \varepsilon(s+t) \quad \textit{for } s,t \in (0,\infty).$$

Proof. Applying Proposition 2.1 we get

$$\begin{aligned}
&\|w_{\mathbf{x}}(s+t)-w_{\mathbf{x}}(s)-w_{\mathbf{x}}(t)\|^p\\
&\quad=\|(s+t)\theta_{\mathbf{x}}(\log_2(s+t))-s\theta_{\mathbf{x}}(\log_2 s)-t\theta_{\mathbf{x}}(\log_2 t)\|^p\\
&\quad=\|s(\theta_{\mathbf{x}}(\log_2(s+t))-\theta_{\mathbf{x}}(\log_2 s))+t(\theta_{\mathbf{x}}(\log_2(s+t))-\theta_{\mathbf{x}}(\log_2 t))\|^p\\
&\quad\le s^p\|\theta_{\mathbf{x}}(\log_2(s+t))-\theta_{\mathbf{x}}(\log_2 s)\|^p+t^p\|\theta_{\mathbf{x}}(\log_2(s+t))-\theta_{\mathbf{x}}(\log_2 t)\|^p\\
&\quad\le \varepsilon^p s^p\Big(\log_2\frac{s+t}{s}+2(1-p)\Big)+\varepsilon^p t^p\Big(\log_2\frac{s+t}{t}+2(1-p)\Big)\\
&\quad=\varepsilon^p(s+t)^p\bigg(\frac{1}{(1+t/s)^p}\Big(\log_2\Big(1+\frac{t}{s}\Big)+2(1-p)\Big)\\
&\qquad+\frac{1}{(1+s/t)^p}\Big(\log_2\Big(1+\frac{s}{t}\Big)+2(1-p)\Big)\bigg).
\end{aligned}$$

Since $2(1-p)p\le\log_2 e$, by Lemma 4.1 we obtain the assertion of the theorem. ∎

We show that every solution to the inequality (1) can be approximated by a function of the type introduced in the previous theorem.

THEOREM 4.2. *Let $w:(0,\infty)\to X$ be such that*

$$\|w(s+t)-w(s)-w(t)\|\le\varepsilon(s+t)\quad \textit{for } s,t\in(0,\infty).$$

Define the sequence $\mathbf{x}$ of elements of X by $x_k:=w(2^k)/2^k$ for $k\in\mathbb{Z}$. Let $a:(0,\infty)\to X$ be the unique additive function such that $a(1)=0$ and $w-a$ is locally bounded. Then $\mathbf{x}$ satisfies (14) *and*

$$\|w(s)-a(s)-s\theta_{\mathbf{x}}(\log_2 s)\|\le K_p^{1/p}\varepsilon s\quad \textit{for } s\in(0,\infty),$$

where $K_p=2^p\big(1+\frac{2}{2^p-1}\big)+1$ for $p\in(0,1)$, $K_1=4$.

Proof. We first consider the case $p\in(0,1)$. By Corollary 3.2 there exists a unique additive function $a:\mathbb{R}\to X$ such that for every $k\in\mathbb{Z}$ we have

$$\left\|w(s)-a(s)-\frac{w(2^k)}{2^k}\,s\right\|\le S\varepsilon 2^{k+1}\quad\text{for } s\in(0,2^{k+1}],\tag{15}$$

where $S=\big(1+\frac{2}{2^p-1}\big)^{1/p}$.

Fix $k\in\mathbb{Z}$ and $s\in[2^k,2^{k+1}]$. Then $\log_2 s\in[k,k+1]$, and by the definition of $\theta_{\mathbf{x}}$ we obtain

$$\theta_{\mathbf{x}}(\log_2 s)=\bigg(\frac{w(2^{k+1})}{2^{k+1}}-\frac{w(2^k)}{2^k}\bigg)(\log_2 s-k)+\frac{w(2^k)}{2^k}.$$

Then for $s\in[2^k,2^{k+1}]$, by (15) we get

$$
\begin{aligned}
&\|w(s) - a(s) - s\theta_{\mathbf{x}}(\log_2 s)\|^p \\
&\leq \left\| w(s) - a(s) - \frac{w(2^k)}{2^k}\, s \right\|^p + \left\| \frac{w(2^k)}{2^k}\, s - s\theta_{\mathbf{x}}(\log_2 s) \right\|^p \\
&\leq S^p \varepsilon^p 2^{p(k+1)} + \left\| \frac{w(2^k)}{2^k}\, s - \left[\left(\frac{w(2^{k+1})}{2^{k+1}} - \frac{w(2^k)}{2^k} \right) \log_2(s/2^k) + w(2^k)/2^k \right] s \right\|^p \\
&\leq S^p \varepsilon^p 2^{p(k+1)} + \left\| \frac{w(2^{k+1})}{2^{k+1}} - \frac{w(2^k)}{2^k} \right\|^p \log_2^p(s/2^k) \cdot s^p \\
&\leq S^p \varepsilon^p 2^p s^p + \varepsilon^p s^p = (2^p S^p + 1)\varepsilon^p s^p.
\end{aligned}
$$

Now we consider the case $p = 1$. Let $M \geq 0$ be a minimal constant such that

$$\|w(s) - a(s) - s\theta_{\mathbf{x}}(\log_2 s)\| \leq M\varepsilon s \quad \text{for } s \in (0, \infty).$$

We have proved that $M < \infty$. We are going to show that $M \leq 4$. Let $w_{\mathbf{x}}(s) := \log_2(s) \cdot \theta_{\mathbf{x}}(s)$. Fix $k \in \mathbb{Z}$ and $s \in [2^k, 2^{k+1})$. Let $t = s - 2^k$. Then $t \leq s/2$. Making use of Theorem 4.1 and the equalities $w(2^k) = w_{\mathbf{x}}(2^k)$, $a(2^k) = 0$ we get

$$
\begin{aligned}
&\|w(s) - a(s) - w_{\mathbf{x}}(s)\| \\
&\qquad \leq \|w(s) - w(t) - w(2^k)\| + \|w(t) - a(t) - w_{\mathbf{x}}(t)\| \\
&\qquad\quad + \|w(2^k) - a(2^k) - w_{\mathbf{x}}(2^k)\| + \|w_{\mathbf{x}}(s) - w_{\mathbf{x}}(t) - w_{\mathbf{x}}(2^k)\| \\
&\qquad \leq \varepsilon s + \varepsilon M t + \varepsilon s \leq (2 + M/2)\varepsilon s.
\end{aligned}
$$

Since s was arbitrary, by the definition of M we obtain $M \leq (2 + M/2)$, and consequently $M \leq 4$. ■

As a direct consequence of Theorem 5.1 and Proposition 4 we get an improvement of Proposition 1 of [5].

Remark 4.1. Proposition 1 of [5] is formulated for the dyadic group and our corollary is formulated for the real line and therefore is not a generalization of the result of Sánchez. However, by inspecting the proofs one can easily notice that Corollary 4.1 is valid for all dense subgroups of $\mathbb{R}$ which allow division by 2.

Corollary 4.1. *Let $w : (0, \infty) \to X$ with $w(1) = 0$ be such that*

$$\|w(s + t) - w(s) - w(t)\| \leq \varepsilon(s + t) \quad \text{for } s, t \in (0, \infty).$$

Then there exists an additive function $a : \mathbb{R} \to X$ with $a(1) = 0$ such that

$$\|w(s) - a(s)\| \leq \varepsilon(K_p + 2(1 - p) + |\log_2 s|)^{1/p} s \quad \text{for } s \in (0, \infty),$$

where K_p is given in Theorem 4.2.

Proof. By Theorem 4.2 there exists an additive function $a : \mathbb{R} \to X$ with $a(1) = 0$ such that

$$\|w(s) - a(s) - s\theta_{\mathbf{x}}(\log_2 s)\|^p \le K_p \varepsilon^p s^p \quad \text{for } s \in (0, \infty).$$

On the other hand, by Proposition 2.1 we get

$$\|\theta_{\mathbf{x}}(s) - \theta_{\mathbf{x}}(0)\|^p \le \varepsilon^p (s + 2(1-p)) \quad \text{for } s \in (0, \infty).$$

Since $w(1) = 0$ and $\theta_{\mathbf{x}}(0) = 0$, combining the above two inequalities we get

$$\|w(s) - a(s)\|^p \le K_p \varepsilon^p s^p + \|s\theta_{\mathbf{x}}(\log_2 s)\|^p \le \varepsilon^p s^p (K_p + 2(1-p) + |\log_2 s|)$$

for $s \in (0, \infty)$. ∎

5. Stability on $\mathbb{R}$. In this section we discuss the case when the domain space is $\mathbb{R}$.

THEOREM 5.1. *Let $w : \mathbb{R} \to X$ be such that*

$$\|w(s+t) - w(s) - w(t)\| \le \varepsilon(|s| + |t|) \quad \textit{for } s, t \in \mathbb{R}.$$

Then there exists an additive function $a : \mathbb{R} \to X$ and a sequence $\mathbf{x}$ satisfying (14) *such that*

$$\|w(s) - a(s) - s\theta_{\mathbf{x}}(\log_2 |s|)\| \le L_p^{1/p} \varepsilon |s| \quad \textit{for } s \in \mathbb{R},$$

where $L_p = 2^p\big(2 + \frac{2}{2^p - 1}\big) + 1$ for $p \in (0,1)$ and $L_1 = 5$ (here $0 \cdot \theta_{\mathbf{x}}(\log_2 0)$ is understood to be 0).

Proof. We first discuss the case $p \in (0,1)$. Let $\mathbf{x} = \{w(2^k)/2^k\}_{k \in \mathbb{Z}}$. Then by Theorem 4.2 there exists an additive function $a : \mathbb{R} \to X$ such that

$$\|w(t) - a(t) - t\theta_{\mathbf{x}}(\log_2 t)\|^p \le K_p \varepsilon^p t^p \quad \text{for } t \in (0, \infty),$$

where K_p is given in Theorem 4.2. Since $\|w(t) + w(-t)\|^p \le 2^p \varepsilon^p |t|^p$, the above implies

$$\|w(t) - a(t) - t\theta_{\mathbf{x}}(\log_2 |t|)\|^p \le (2^p + K_p)\varepsilon^p |t| \quad \text{for } t \in (-\infty, 0),$$

which yields the assertion of the theorem.

Now we consider the case $p = 1$. Let $\mathbf{x}_-$, $\mathbf{x}_+$ be the sequences $\{w(2^k)/2^k\}_{k \in \mathbb{Z}}$, $\{-w(2^{-k})/2^k\}_{k \in \mathbb{Z}}$, respectively. By Theorem 4.2 there exist additive functions $a_- : \mathbb{R} \to X$ and $a_+ : \mathbb{R} \to X$ such that

$$\left.\begin{aligned} \|w(s) - a_-(s) - s\theta_{\mathbf{x}_-}(\log_2 |s|)\| &\le 4\varepsilon |s| \\ \|w(t) - a_+(t) - t\theta_{\mathbf{x}_+}(\log_2 |t|)\| &\le 4\varepsilon |t| \end{aligned}\right. \quad \text{for } s \in (-\infty, 0),\ t \in (0, \infty).$$

Since $\|w(s) + w(-s)\| \le 2\varepsilon |s|$, the above inequalities yield

$$\left.\begin{aligned} \|w(s) - a_-(s) - s\theta_{\mathbf{x}_-}(\log_2 |s|)\| &\le 6\varepsilon |s| \\ \|w(t) - a_+(t) - t\theta_{\mathbf{x}_+}(\log_2 |t|)\| &\le 6\varepsilon |t| \end{aligned}\right. \quad \text{for } s \in (0, \infty),\ t \in (-\infty, 0).$$

Let $\mathbf{x} = (\mathbf{x}_- + \mathbf{x}_+)/2$, $a = (a_- + a_+)/2$. Let $s \in (0, \infty)$ (the case $s \in (-\infty, 0)$ is symmetric). Then

$$\begin{aligned}\|w(s) - a(s) - s\theta_-(\log_2 |s|)\| &\le \tfrac{1}{2}\|w(s) - a_+(s) - s\theta_{\mathbf{x}_+}(\log_2 |s|)\| \\ &\quad + \tfrac{1}{2}\|w(s) - a_-(s) - s\theta_{\mathbf{x}_-}(\log_2 |s|)\| \\ &\le \left(\tfrac{1}{2} \cdot 4 + \tfrac{1}{2} \cdot 6\right)\varepsilon|s| = 5\varepsilon|s|. \ \blacksquare\end{aligned}$$

By proceeding analogously to the proof of Corollary 4.1 one can prove the following generalization of Corollary 1 from [5].

COROLLARY 5.1. *Let $w : \mathbb{R} \to X$ be such that*

$$\|w(s+t) - w(s) - w(t)\| \le \varepsilon(|s| + |t|) \quad \text{for } s, t \in \mathbb{R}.$$

Then there exists an additive function $a : \mathbb{R} \to X$ such that

$$\|w(s) - a(s)\| \le \varepsilon(L_p + 2(1-p) + |\log_2 |s|\,|)^{1/p}|s| \quad \text{for } s \in \mathbb{R},$$

where L_p is given in Theorem 5.1.

6. Stability on $\mathbb{R}^n$. To deal with quasi-additive functions on $\mathbb{R}^n$ we will need the following lemma:

LEMMA 6.1. *Let G be a p-Banach space and let $w : G \to X$ be such that*

$$\|w(a+b) - w(a) - w(b)\| \le \varepsilon(\|a\| + \|b\|) \quad \text{for } a, b \in G.$$

Let $n \in \mathbb{N}$. Then

$$\Big\|w\Big(\sum_{i=1}^{2^n} a_i\Big) - \sum_{i=1}^{2^n} w(a_i)\Big\|^p \le n\sum_{i=1}^{2^n} \|a_i\|^p \quad \text{for } a_1, \ldots, a_{2^n} \in G. \tag{16}$$

Proof. The proof is by induction on n. For $n = 1$ the equality is trivial. So, suppose that (16) holds for a given n. Then

$$\begin{aligned}&\Big\|w\Big(\sum_{i=1}^{2^{n+1}} a_i\Big) - \sum_{i=1}^{2^{n+1}} w(a_i)\Big\|^p \\ &\quad \le \Big\|w\Big(\sum_{i=1}^{2^n}(a_{2i-1} + a_{2i})\Big) - \sum_{i=1}^{2^n} w(a_{2i-1} + a_{2i})\Big\|^p \\ &\qquad + \sum_{i=1}^{2^n} \|w(a_{2i-1} + a_{2i}) - w(a_{2i-1}) - w(a_{2i})\|^p \\ &\quad \le n\sum_{i=1}^{2^n} \|a_{2i-1} + a_{2i}\|^p + \sum_{i=1}^{2^n}(\|a_{2i-1}\|^p + \|a_{2i}\|^p) \le (n+1)\sum_{i=1}^{2^{n+1}} \|a_i\|^p. \ \blacksquare\end{aligned}$$

By $E(s)$ we denote the smallest integer not less than s. Now we are ready to prove the main result of this section. In $\mathbb{R}^n$ we use the quasi-norm $\|(x_1, \ldots, x_n)\|^p = |x_1|^p + \cdots + |x_n|^p$.

THEOREM 6.1. *Let $w : \mathbb{R}^n \to X$ be such that*

$$\|w(x+y) - w(x) - w(y)\| \le \varepsilon(\|x\| + \|y\|) \quad \text{for } x, y \in \mathbb{R}^n.$$

Then there exist an additive function $A : \mathbb{R}^n \to X$ and sequences $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of elements of X satisfying (14) *such that*

$$\Big\| w(x) - A(x) - \sum_{i=1}^{n} x_i \theta_{\mathbf{x}_i}(\log_2 |x_i|) \Big\| \le M_p^{1/p} \varepsilon \|x\| \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where $M_p = E(\log_2 n) + L_p$, and L_p is given in Theorem 5.1.

Proof. Let $w_i : \mathbb{R} \to X$ be defined by $w_i(r) := w(0, \ldots, \overbrace{r}^{i}, \ldots, 0)$. Then

$$\|w_i(s+t) - w_i(s) - w_i(t)\| \le \varepsilon(|s| + |t|) \quad \text{for } s, t \in \mathbb{R},$$

and therefore there exists an additive function $a_i : \mathbb{R} \to X$ and a sequence $\mathbf{x}_i \subset X$ such that

$$\|w_i(s) - a_i(s) - s\theta_{\mathbf{x}_i}(\log_2 |s|)\|^p \le L_p \varepsilon^p |s|^p \quad \text{for } s \in \mathbb{R},$$

where L_p is given in Theorem 5.1. Let

$$A(x_1, \ldots, x_n) := a_1(x_1) + \cdots + a_n(x_n) \quad \text{for } (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Now by Lemma 6.1 we obtain

$$\Big\| w(x) - A(x) - \sum_{i=1}^{n} x_i \theta_{\mathbf{x}_i}(\log_2 |x_i|) \Big\|^p$$
$$\Big\| w(x_1, \ldots, x_n) - \sum_{i=1}^{n} w_i(x_i) \Big\|^p + \sum_{i=1}^{n} \|w_i(x_i) - a_i(x_i) - \theta_{\mathbf{x}_i}(\log_2 |x_i|)\|^p$$
$$\le E(\log_2 n)\varepsilon^p \sum_{i=1}^{n} |x_i|^p + \sum_{i=1}^{n} L_p \varepsilon^p |x_i|^p = (E(\log_2 n) + L_p)\varepsilon^p \|x\|^p. \ \blacksquare$$

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Reçu par la Rédaction le 20.10.2003
Révisé le 12.3.2004 (1477)