

Holomorphic line bundles on a domain of a two-dimensional Stein manifold

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Abstract. Let D be an open subset of a two-dimensional Stein manifold S . Then D is Stein if and only if every holomorphic line bundle L on D is the line bundle associated to some (not necessarily effective) Cartier divisor \mathfrak{d} on D .

1. Introduction. It is well known that every holomorphic line bundle L on a projective algebraic manifold P is associated to some (not necessarily effective) Cartier divisor \mathfrak{d} on P (see Griffiths–Harris [5, p. 161]). A similar fact holds for reduced Stein spaces. For every holomorphic line bundle L on a reduced Stein space S there exists a holomorphic global section s of L such that s does not vanish identically on any irreducible component of S . Therefore L is the line bundle associated to the effective Cartier divisor $\text{div}(s)$ on S (see Gunning [6, pp. 120–122]).

On the other hand a holomorphic line bundle L on the punctured disk $X := \Delta^2 \setminus \{(0, 0)\}$, which is not Stein, is holomorphically trivial if and only if L is the line bundle associated to some Cartier divisor on X (see Remark 1.4 of Ballico [1]).

In this paper we prove that an open subset D of a two-dimensional Stein manifold S is Stein if and only if every holomorphic line bundle L on D is the line bundle associated to some (not necessarily effective) Cartier divisor \mathfrak{d} on D (Theorem 3). By using the method of Kajiwara–Kazama [10] the proof is deduced from the above-mentioned property of the punctured disk.

If S is a Stein manifold of dimension more than two, then there can exist an open subset D of S such that D is not Stein and every holomorphic line bundle L on D is the line bundle associated to some effective Cartier divisor \mathfrak{d} on D .

2. Lemmas. Let X be a reduced complex space. Let \mathfrak{d} be a Cartier divisor on X defined by a meromorphic Cousin-II distribution $\{(U_i, m_i)\}_{i \in I}$

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on X . We denote by $[\mathfrak{d}]$ the holomorphic line bundle on X defined by the cochain $\{m_i/m_j\} \in Z^1(\{U_i\}_{i \in I}, \mathcal{O}^*)$. We say that $[\mathfrak{d}]$ is the *holomorphic line bundle associated to \mathfrak{d}* .

Let $\Delta := \{t \in \mathbb{C} \mid |t| < 1\}$. A holomorphic line bundle L on the punctured didisk $X := \Delta^2 \setminus \{(0, 0)\}$ is holomorphically trivial if and only if there exists a Cartier divisor \mathfrak{d} on X such that $L = [\mathfrak{d}]$ (see Remark 1.4 of Ballico [1]). More generally we have the following lemma.

LEMMA 1. *Let S be a two-dimensional Stein manifold with $H^2(S, \mathbb{Z}) = 0$. Let A be a non-empty discrete subset of S . Let L be a holomorphic line bundle on the open set $S \setminus A$. Then L is holomorphically trivial if and only if there exists a Cartier divisor \mathfrak{d} on $S \setminus A$ such that $L = [\mathfrak{d}]$.*

Proof. Assume that there exists a Cartier divisor \mathfrak{d} on $D := S \setminus A$ such that $L = [\mathfrak{d}]$. Since D is Cousin-II (see Theorem 8.25 of Hitotumatu [7, p. 174]), $L = [\mathfrak{d}]$ is holomorphically trivial. The converse is clear. ■

Let S be a reduced complex space and D an open subset of S . Let p be a boundary point of D in S . Then we say that D is *locally Stein* at p if there exists a neighborhood U of p in S such that $U \cap D$ is Stein. By using the method of the proof of Lemma 11 of Kajiwara–Kazama [10] we prove the following lemma.

LEMMA 2. *Let S be a purely two-dimensional reduced Stein space and D an open subset of S . Assume that for every holomorphic line bundle L on D there exists a Cartier divisor \mathfrak{d} on D such that $L = [\mathfrak{d}]$. Then D is locally Stein at every point $p \in \partial D \setminus \text{Sing}(S)$.*

Proof. Assume that there exists a point $p \in \partial D \setminus \text{Sing}(S)$ such that D is not locally Stein at p . Since S is Stein, there exist a holomorphic map $\psi : S \rightarrow \mathbb{C}^2$ and a neighborhood W of p such that W is non-singular, $\psi(W)$ is an open subset of \mathbb{C}^2 and $\psi|_W : W \rightarrow \psi(W)$ is biholomorphic (see Grauert–Remmert [4, p. 151]). Take a Stein open subset V of \mathbb{C}^2 such that $\psi(p) \in V \Subset \psi(W)$. Then $U := \psi^{-1}(V) \cap W$ is a Stein neighborhood of p and $\psi(U) = V$. Since D is not locally Stein at p , the open set $\psi(D \cap U)$ is not Stein. By Lemma 1 of Kajiwara–Kazama [10] (see also the proof of Lemma 11 of [10]) there exist $H, P, \varepsilon, \varphi$ and (b_1, b_2) with the following properties:

$$\begin{aligned}
 H &= \{(w_1, w_2) \in \mathbb{C}^2 \mid |w_1| < 1, |w_2| < 1\} \\
 &\quad \cup \{(w_1, w_2) \in \mathbb{C}^2 \mid 1 - 2\varepsilon < |w_1| < 1 + 2\varepsilon, |w_2| < 1 + 2\varepsilon\}, \\
 P &= \{(w_1, w_2) \in \mathbb{C}^2 \mid |w_1| < 1 + 2\varepsilon, |w_2| < 1 + 2\varepsilon\}, \quad 0 < \varepsilon < 1/2, \\
 \varphi : \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \text{ is a biholomorphic map, } \quad \varphi(H) \subset \psi(D \cap U), \\
 |b_1| &\leq 1 - 2\varepsilon, \quad |b_2| = 1, \quad \varphi(b_1, b_2) \in \partial(\psi(D \cap U)).
 \end{aligned}$$

Let $\theta = (\theta_1, \theta_2) := \varphi^{-1} \circ \psi : S \rightarrow \mathbb{C}^2$. Let $T := \{|\theta_1| < 1 + 2\varepsilon\}$, $T_0 := \{|\theta_2| < 1 + 2\varepsilon\} \cap T \cap U$ and $T_1 := \{|\theta_2| > 1 + \varepsilon\} \cup (T \setminus \overline{U})$. Then T is a Stein open subset of S and $\{T_0, T_1\}$ is an open covering of T . The function $1/(\theta_2 - b_2)$ is holomorphic on $T_0 \cap T_1$. Since $H^1(\{T_0, T_1\}, \mathcal{O}) = 0$, there exist $v_i \in \mathcal{O}(T_i)$, $i = 1, 2$, such that $1/(\theta_2 - b_2) = v_1 - v_0$ on $T_0 \cap T_1$. We define a meromorphic function v on T by the equalities $v = v_0 + 1/(\theta_2 - b_2)$ on T_0 and $v = v_1$ on T_1 . Then v is holomorphic on $(\{\theta_2 \neq b_2\} \cap T_0) \cup T_1$. Let $D_1 := \{\theta_1 \neq b_1\} \cap D$ and $D_2 := ((\{\theta_2 \neq b_2\} \cap T) \cup (T \setminus \overline{U})) \cap D$. Then $\{D_1, D_2\}$ is an open covering of D and the function $v/(\theta_1 - b_1)$ is holomorphic on $D_1 \cap D_2$. By assumption there exist invertible meromorphic functions g_i on D_i , $i = 1, 2$, such that $\exp(v/(\theta_1 - b_1)) = g_1/g_2$ on $D_1 \cap D_2$. The function $g'_1 := \exp(-v_0/(\theta_1 - b_1)) g_1$ is meromorphic on $T_0 \cap D_1$ and $\exp(1/((\theta_1 - b_1)(\theta_2 - b_2))) = g'_1/g_2$ on $T_0 \cap D_1 \cap D_2$. Let $H_i := \{(w_1, w_2) \in H \mid w_i \neq b_i\}$ and $P_i := \{(w_1, w_2) \in P \mid w_i \neq b_i\}$ for $i = 1, 2$. Since P is the envelope of holomorphy of H , the open set P_i is the envelope of holomorphy of H_i for each $i = 1, 2$ by Satz 7 of Grauert–Remmert [3] (see Theorem 2.5.9 of Jarnicki–Pflug [8, p. 182]). Since $H_i \subset \theta(U)$ and $\theta^{-1}(H_i) \cap U \subset T_0 \cap D_i$ for $i = 1, 2$, the function $f_1 := g'_1 \circ (\theta|_U)^{-1}$ is meromorphic on H_1 and $f_2 := g_2 \circ (\theta|_U)^{-1}$ is meromorphic on H_2 . By Proposition 3 of Kajiwara–Sakai [11] there exists a meromorphic function \tilde{f}_i on P_i such that $\tilde{f}_i = f_i$ on H_i for each $i = 1, 2$. Since f_i is invertible on H_i , \tilde{f}_i is also invertible on P_i by the theorem of identity. We have $\exp(1/((w_1 - b_1)(w_2 - b_2))) = \tilde{f}_1/\tilde{f}_2$ on $P_1 \cap P_2$. This contradicts Lemma 1 because the function $\exp(1/((w_1 - b_1)(w_2 - b_2))) \in \mathcal{O}^*(P_1 \cap P_2)$ defines a non-trivial holomorphic line bundle on $P_1 \cup P_2 = P \setminus \{(b_1, b_2)\}$ (see Lemma 1 of Kajiwara [9] or Serre [12, p. 372]). ■

3. Theorem. We have the following theorem which characterizes a Stein open subset of a two-dimensional Stein manifold.

THEOREM 3. *Let S be a two-dimensional Stein manifold and D an open subset of S . Then the following four conditions are equivalent.*

- (1) D is Stein.
- (2) For every holomorphic line bundle L on D there exists an effective Cartier divisor \mathfrak{d} on D such that $L = [\mathfrak{d}]$.
- (3) For every holomorphic line bundle L on D there exists a Cartier divisor \mathfrak{d} on D such that $L = [\mathfrak{d}]$.
- (4) The image of the natural homomorphism $H^1(D, \mathcal{O}^*) \rightarrow H^1(D, \mathcal{M}^*)$ vanishes.

Proof. (3) \Rightarrow (1). By Lemma 2 the open set D is locally Stein at every boundary point p of D in S . It follows that D is Stein by the theorem of Docquier–Grauert [2].

(1) \Rightarrow (2). Every holomorphic line bundle L on an arbitrary reduced Stein space is associated to some Cartier divisor $[\mathfrak{d}]$. For the proof of this fact we refer to Gunning [6, pp. 120–122].

(2) \Rightarrow (3) \Leftrightarrow (4). Clear. ■

For an open subset D of a Stein manifold S such that $\dim S \geq 3$ the theorem above does not hold. As an example, we take a non-empty analytic subset A of S such that $\text{codim } A \geq 3$. The open subset $D := S \setminus A$ of S is not Stein. Let L be an arbitrary holomorphic line bundle on D . There exists a holomorphic line bundle \tilde{L} on S such that $\tilde{L}|_D = L$ (see Shiffman [13, p. 340]). Since S is Stein, there exists an effective divisor $\tilde{\mathfrak{d}}$ on S such that $\tilde{L} = [\tilde{\mathfrak{d}}]$. Then we have $L = [\tilde{\mathfrak{d}}|_D]$ and condition (2) of Theorem 3 is satisfied.

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