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On scalar-valued nonlinear absolutely summing mappings

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Abstract. We investigate cases ("coincidence situations") in which every scalar-valued continuous n-homogeneous polynomial (or every continuous n-linear mapping) is absolutely $(p;q)$-summing. We extend some well known coincidence situations and obtain several non-coincidence results, inspired by a linear technique due to Lindenstrauss and Pełczyński.

1. Introduction. Throughout this note $X, X_1, \ldots, X_n, Y$ will stand for Banach spaces and the scalar field $\mathbb{K}$ can be either the real or the complex numbers.

An m-homogeneous polynomial P from X into Y is said to be *absolutely* $(p;q)$-*summing* $(p \geq q/m)$ if there is a constant L so that

$$\Big(\sum_{j=1}^{k} \|P(x_j)\|^p\Big)^{1/p} \leq L\|(x_j)_{j=1}^k\|_{w,q}^m \tag{1.1}$$

for every natural k, where $\|(x_j)_{j=1}^k\|_{w,q} = \sup_{\varphi \in B_{X'}} (\sum_{j=1}^k |\varphi(x_j)|^q)^{1/q}$. This is a natural generalization of the concept of $(p;q)$-summing operators and in the last years it has been studied by several authors. The infimum of the $L > 0$ for which the inequality holds defines a norm $\|\cdot\|_{\mathrm{as}(p;q)}$ for $p \geq 1$, or a p-norm for $p < 1$, on the space of $(p;q)$-summing homogeneous polynomials. The space of all m-homogeneous $(p;q)$-summing polynomials from X into Y is denoted by $\mathcal{P}_{\mathrm{as}(p;q)}(^mX;Y)$ ($\mathcal{P}_{\mathrm{as}(p;q)}(^mX)$ if $Y = \mathbb{K}$).

When $p = q/m$ we have an important particular case, since in this situation there is an analogue of the Grothendieck–Pietsch Domination Theorem. The $(q/m;q)$-summing m-homogeneous polynomials from X into Y are said to be q-*dominated* and this space is denoted by $\mathcal{P}_{\mathrm{d},q}(^mX;Y)$ ($\mathcal{P}_{\mathrm{d},q}(^mX)$ if $Y = \mathbb{K}$).

The Banach space of all continuous m-homogeneous polynomials P from X into Y with the sup norm is denoted by $\mathcal{P}(^mX,Y)$ ($\mathcal{P}(^mX)$ if Y is

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the scalar field). Analogously, the space of all continuous m-linear mappings from $X_1 \times \cdots \times X_m$ into Y (with the sup norm) is denoted by $\mathcal{L}(X_1, \ldots, X_m; Y)$ ($\mathcal{L}(X_1, \ldots, X_m)$ if $Y = \mathbb{K}$). The concept of absolutely summing multilinear mapping follows the same pattern (for details we refer to [5]). Henceforth every polynomial and multilinear mapping are supposed to be continuous and every $\mathcal{L}_p$-space is assumed to be infinite-dimensional.

A natural problem is to find situations in which the space of absolutely summing polynomials coincides with the space of continuous polynomials (*coincidence situations*). When Y is the scalar field, these situations are not rare as we can see in the next two well known results:

THEOREM 1. *Every scalar-valued n-linear mapping is absolutely* $(1;1)$-*summing. In particular, every scalar-valued n-homogeneous polynomial is absolutely* $(1;1)$-*summing* (*and, a fortiori*, $(q;1)$-*summing for every* $q \geq 1$).

THEOREM 2 (D. Pérez-García [6]). *If $n \geq 2$ and X is an $\mathcal{L}_\infty$-space, then every scalar-valued n-linear mapping on X is* $(1;2)$-*summing. In particular, every scalar-valued n-homogeneous polynomial on X is* $(1;2)$-*summing* (*and, a fortiori*, $(q;2)$-*summing for every* $q \geq 1$).

The proof of Theorem 1 can be found in [1] and is credited to A. Defant and J. Voigt. The case $n = 2$ of Theorem 2 was previously proved by Botelho [2] and is the unique known coincidence result for dominated polynomials.

In Section 2 we obtain new coincidence situations, generalizing Theorem 1 and extending the results of Theorem 2. Section 3 has a different purpose: to obtain a technical estimate (inspired by a linear result due to Lindenstrauss and Pełczyński [3]) and to explore its consequences. In particular, it is shown that Theorems 1 and 2 cannot be generalized in some other directions, and converses for the aforementioned theorems are obtained.

2. Coincidence situations. The next theorem, inspired by a result of C. A. Soares, leads us to extensions of the two theorems stated in the first section:

THEOREM 3. *Let $A \in \mathcal{L}(X_1, \ldots, X_n; Y)$ and suppose that there exists $C > 0$ so that for any $x_1 \in X_1, \ldots, x_r \in X_r$, the s-linear ($s = n-r$) mapping $A_{x_1 \ldots x_r}(x_{r+1}, \ldots, x_n) = A(x_1, \ldots, x_n)$ is absolutely* $(1; q_1, \ldots, q_s)$-*summing and*

$$\|A_{x_1 \ldots x_r}\|_{\mathrm{as}(1;q_1,\ldots,q_s)} \leq C\|A\| \, \|x_1\| \ldots \|x_r\|.$$

Then A is absolutely $(1; 1, \ldots, 1, q_1, \ldots, q_s)$-*summing.*

Proof. For $x_1^{(1)},\ldots,x_1^{(m)}\in X_1,\ldots,x_n^{(1)},\ldots,x_n^{(m)}\in X_n$, consider $\varphi_j\in B_{Y'}$ such that

$$\|A(x_1^{(j)},\ldots,x_n^{(j)})\|=\varphi_j(A(x_1^{(j)},\ldots,x_n^{(j)}))$$

for every $j=1,\ldots,m$. Then, denoting by $r_j(t)$ the Rademacher functions on $[0,1]$ and by λ the Lebesgue measure on $I=[0,1]^r$, we have

$$\begin{aligned}
&\int_I\sum_{j=1}^m\Big(\prod_{l=1}^r r_j(t_l)\Big)\\
&\qquad\times\varphi_j A\Big(\sum_{j_1=1}^m r_{j_1}(t_1)x_1^{(j_1)},\ldots,\sum_{j_r=1}^m r_{j_r}(t_r)x_r^{(j_r)},x_{r+1}^{(j)},\ldots,x_n^{(j)}\Big)\,d\lambda\\
&\quad=\sum_{j,j_1,\ldots,j_r=1}^m\varphi_j A(x_1^{(j_1)},\ldots,x_r^{(j_r)},x_{r+1}^{(j)},\ldots,x_n^{(j)})\\
&\qquad\times\int_0^1 r_j(t_1)r_{j_1}(t_1)\,dt_1\ldots\int_0^1 r_j(t_r)r_{j_r}(t_r)\,dt_r\\
&\quad=\sum_{j=1}^m\sum_{j_1=1}^m\cdots\sum_{j_r=1}^m\varphi_j A(x_1^{(j_1)},\ldots,x_r^{(j_r)},x_{r+1}^{(j)},\ldots,x_n^{(j)})\delta_{jj_1}\ldots\delta_{jj_r}\\
&\quad=\sum_{j=1}^m\varphi_j A(x_1^{(j)},\ldots,x_n^{(j)})=\sum_{j=1}^m\|A(x_1^{(j)},\ldots,x_n^{(j)})\|=(*).
\end{aligned}$$

So, for each $l=1,\ldots,r$, assuming $z_l=\sum_{j=1}^m r_j(t_l)x_l^{(j)}$ we obtain

$$\begin{aligned}
(*)&=\int_I\sum_{j=1}^m\Big(\prod_{l=1}^r r_j(t_l)\Big)\\
&\qquad\times\varphi_j A\Big(\sum_{j_1=1}^m r_{j_1}(t_1)x_1^{(j_1)},\ldots,\sum_{j_r=1}^m r_{j_r}(t_r)x_r^{(j_r)},x_{r+1}^{(j)},\ldots,x_n^{(j)}\Big)\,d\lambda\\
&\le\int_I\Big|\sum_{j=1}^m\Big(\prod_{l=1}^r r_j(t_l)\Big)\\
&\qquad\times\varphi_j A\Big(\sum_{j_1=1}^m r_{j_1}(t_1)x_1^{(j_1)},\ldots,\sum_{j_r=1}^m r_{j_r}(t_r)x_r^{(j_r)},x_{r+1}^{(j)},\ldots,x_n^{(j)}\Big)\Big|\,d\lambda\\
&\le\int_I\sum_{j=1}^m\Big\|A\Big(\sum_{j_1=1}^m r_{j_1}(t_1)x_1^{(j_1)},\ldots,\sum_{j_r=1}^m r_{j_r}(t_r)x_r^{(j_r)},x_{r+1}^{(j)},\ldots,x_n^{(j)}\Big)\Big\|\,d\lambda
\end{aligned}$$

$$\leq \sup_{t_l\in[0,1],\, l=1,\dots,r} \sum_{j=1}^{m} \Big\| A\Big(\sum_{j_1=1}^{m} r_{j_1}(t_1)x_1^{(j_1)}, \dots, \sum_{j_r=1}^{m} r_{j_r}(t_r)x_r^{(j_r)}, x_{r+1}^{(j)}, \dots, x_n^{(j)}\Big)\Big\|$$

$$\leq \sup_{t_l\in[0,1],\, l=1,\dots,r} \|A_{z_1\dots z_r}\|_{\mathrm{as}(1;q_1,\dots,q_s)} \|(x_{r+1}^{(j)})_{j=1}^m\|_{w,q_1} \cdots \|(x_n^{(j)})_{j=1}^m\|_{w,q_s}$$

$$\leq \sup_{t_l\in[0,1],\, l=1,\dots,r} C\|A\|\,\|z_1\|\cdots\|z_r\|\,\|(x_{r+1}^{(j)})_{j=1}^m\|_{w,q_1} \cdots \|(x_n^{(j)})_{j=1}^m\|_{w,q_s}$$

$$\leq C\|A\| \Big(\prod_{l=1}^{r} \|(x_l^{(j)})_{j=1}^m\|_{w,1}\Big)\Big(\prod_{l=r+1}^{n} \|(x_l^{(j)})_{j=1}^m\|_{w,q_l}\Big).$$

We have the following straightforward consequence, generalizing Theorem 1:

Corollary 1. *If*

$$\mathcal{L}(X_1,\dots,X_m;Y) = \mathcal{L}_{\mathrm{as}(1;q_1,\dots,q_m)}(X_1,\dots,X_m;Y)$$

then, for any Banach spaces $X_{m+1},\dots,X_n$, *we have*

$$\mathcal{L}(X_1,\dots,X_n;Y) = \mathcal{L}_{\mathrm{as}(1;q_1,\dots,q_m,1,\dots,1)}(X_1,\dots,X_n;Y).$$

The following corollary (whose proof is simple and we omit it) is a consequence of Theorems 2 and 3.

Corollary 2. *If* $X_1,\dots,X_s$ *are* $\mathcal{L}_\infty$*-spaces then, for any Banach spaces* $X_{s+1},\dots,X_n$, *we have*

$$\mathcal{L}(X_1,\dots,X_n) = \mathcal{L}_{\mathrm{as}(1;q_1,\dots,q_n)}(X_1,\dots,X_n),$$

where $q_1=\dots=q_s=2$ *and* $q_{s+1}=\dots=q_n=1$.

It is obvious that Corollary 2 is still true if we replace the scalar field by any finite-dimensional Banach space. A natural question is whether Corollary 2 can be stated for some infinite-dimensional Banach space in place of $\mathbb{K}$. Precisely, the question is:

- If $X_1,\dots,X_k$ are $\mathcal{L}_\infty$-spaces, is there some infinite-dimensional Banach space Y such that

  $$\mathcal{L}(X_1,\dots,X_k,\dots,X_n;Y) = \mathcal{L}_{\mathrm{as}(1;q_1,\dots,q_n)}(X_1,\dots,X_k,\dots,X_n;Y),$$

  where $q_1=\dots=q_k=2$ and $q_{k+1}=\dots=q_n=1$, regardless of the chioce of the Banach spaces $X_{k+1},\dots,X_n$?

The answer to this question is no, as shown by the following proposition:

Proposition 1. *Suppose that* $X_1,\dots,X_k$ *are* $\mathcal{L}_\infty$*-spaces. If* $q_1=\dots=q_k=2$, $q_{k+1}=\dots=q_n=1$ *and*

$$\mathcal{L}(X_1,\dots,X_k,\dots,X_n;Y) = \mathcal{L}_{\mathrm{as}(1;q_1,\dots,q_n)}(X_1,\dots,X_k,\dots,X_n;Y),$$

regardless of the choice of the Banach spaces $X_{k+1},\dots,X_n$, *then* $\dim Y<\infty$.

Proof. By a standard localization argument, it suffices to prove that if $\dim Y = \infty$, then

$$\mathcal{L}(^n c_0; Y) \neq \mathcal{L}_{\mathrm{as}(1;q_1,\ldots,q_n)}(^n c_0; Y),$$

where $q_1 = \ldots = q_k = 2$ and $q_{k+1} = \cdots = q_n = 1$. But from [5, Theorem 8] we even have

$$\mathcal{L}(^n c_0; Y) \neq \mathcal{L}_{\mathrm{as}(q;q_1,\ldots,q_n)}(^n c_0; Y)$$

for any $q < 2$ and $q_1, \ldots, q_n \geq 1$.

3. Non-coincidence situations. Assume that X is an infinite-dimensional Banach space and suppose that X has a normalized unconditional Schauder basis (x_n) with coefficient functionals (x_n^*). If $\mathcal{P}_{\mathrm{as}(q;1)}(^m X; Y) = \mathcal{P}(^m X; Y)$, it is natural to ask:

What is the infimum of the t such that in this situation $(x_n^*(x)) \in l_t$ for each $x \in X$? This infimum will be denoted by $\mu = \mu(X, Y, q, m)$.

In [5], inspired by an important linear result due to Lindenstrauss and Pełczyński, we have proved:

Theorem 4 (Pellegrino [5, Theorem 5]). *Let X and Y be infinite-dimensional Banach spaces. Suppose that X has an unconditional Schauder basis (x_n). If Y finitely factors the formal inclusion $l_p \to l_\infty$ and $\mathcal{P}_{\mathrm{as}(q;1)}(^m X; Y) = \mathcal{P}(^m X; Y)$ with $1/m \leq q$, then*

(a) $\mu \leq mpq/(p-q)$ *if* $q < p$,
(b) $\mu \leq mq$ *if* $q \leq p/2$.

However, by inspecting the proof of this theorem in [5], one can see that it is by no means necessary to assume that $\dim Y = \infty$. Only in Corollary of [5] (when the Dvoretzky–Rogers Theorem is invoked) is it indeed necessary to assume $\dim Y = \infty$. A slight change in the proof of [5, Theorem 5] yields the following result:

Theorem 5. *Let X be an infinite-dimensional Banach space with a normalized unconditional Schauder basis (x_n). If $\mathcal{P}_{\mathrm{as}(q;1)}(^m X) = \mathcal{P}(^m X)$, then*

(a) $\mu \leq mq/(1-q)$ *if* $q < 1$,
(b) $\mu \leq mq$ *if* $q \leq 1/2$.

Proof. If $x = \sum_{j=1}^{\infty} a_j x_j$ and $\{\mu_i\}_{i=1}^n$ is such that $\sum_{j=1}^n |\mu_j|^{1/q} = 1$, define $P : X \to \mathbb{K}$ by $Px = \sum_{j=1}^n |\mu_j|^{1/q} a_j^m$.

Since (x_n) is an unconditional basis, there exists a $\varrho > 0$ satisfying

$$\Big\| \sum_{j=1}^n \varepsilon_j a_j x_j \Big\| \leq \varrho \|x\| \quad \text{for every } n \text{ and any } \varepsilon_j = \pm 1.$$

Hence

$$|Px| \le \sum_{j=1}^{n} \big| |\mu_j|^{1/q} a_j^m \big| \le \varrho^m \|x\|^m \sum_{j=1}^{n} |\mu_j|^{1/q},$$

and thus $\|P\| \le \varrho^m$ and $\|P\|_{\mathrm{as}(q;1)} \le C\varrho^m$. Therefore

$$\begin{aligned}(3.1)\qquad \Big(\sum_{j=1}^{n} \big|a_j^m |\mu_j|^{1/q}\big|^q\Big)^{1/q} &\le \Big(\sum_{j=1}^{n} |Pa_jx_j|^q\Big)^{1/q} \\ &\le \|P\|_{\mathrm{as}(q;1)} \max_{\varepsilon_j \in \{1,-1\}} \Big\| \sum_{j=1}^{n} \varepsilon_j a_j x_j \Big\|^m \\ &\le \|P\|_{\mathrm{as}(q;1)} (\varrho\|x\|)^m \le C\varrho^{2m}\|x\|^m.\end{aligned}$$

Defining $s = 1/q$, we have $\frac{1}{s} + \frac{1}{\frac{s}{s-1}} = 1$ and

$$(3.2)\qquad \Big(\sum_{j=1}^{n} |a_j|^{\frac{s}{s-1}mq}\Big)^{1/\frac{s}{s-1}} \le \sup\Big\{ \sum_{j=1}^{n} |\mu_j|\, |a_j|^{mq} : \sum_{j=1}^{n} |\mu_j|^s = 1 \Big\}.$$

Since (3.1) is true whenever $\sum_{j=1}^{n} |\mu_j|^s = 1$, by (3.1) and (3.2) we obtain

$$\Big(\sum_{j=1}^{n} |a_j|^{\frac{s}{s-1}mq}\Big)^{1/\frac{s}{s-1}mq} \le [C\varrho^{2m}\|x\|^m]^{1/m}.$$

But $\frac{s}{s-1}mq = \frac{mq}{1-q}$ and n is arbitrary, and hence part (a) is proved. Now, if $1/m \le q \le 1/2$, define $S : X \to \mathbb{K}$ by $Sx = \sum_{j=1}^{n} a_j^m$. Since $m \ge \frac{s}{s-1}mq$, we obtain

$$|Sx| \le \sum_{j=1}^{n} |a_j^m| \le \Big[\Big(\sum_{j=1}^{n} |a_j|^{\frac{s}{s-1}mq}\Big)^{1/\frac{s}{s-1}mq}\Big]^m \le C\varrho^{2m}\|x\|^m.$$

Thus $\|S\| \le C\varrho^{2m}$ and $\|S\|_{\mathrm{as}(q;1)} \le C^2\varrho^{2m}$. Therefore

$$\begin{aligned}\sum_{j=1}^{n} |a_j^m|^q = \sum_{j=1}^{n} |Sa_jx_j|^q &\le \|S\|_{\mathrm{as}(q;1)}^q \max_{\varepsilon_j \in \{1,-1\}} \Big\| \sum_{j=1}^{n} \varepsilon_j a_j x_j \Big\|^{mq} \\ &\le (C^2\varrho^{2m})^q (\varrho\|x\|)^{mq}.\end{aligned}$$

Consequently, since n is arbitrary, we have $\sum_{j=1}^{\infty} |a_j|^{mq} < \infty$ whenever $x = \sum_{j=1}^{\infty} a_j x_j \in X$. ■

Now we list several important consequences of Theorem 5. For example, Corollaries 3 and 4 below give converses for Theorems 1 and 2, respectively. The proofs of Corollaries 3–6 are simple (using Theorem 5 and standard localization techniques in order to extend the results from c_0 to $\mathcal{L}_\infty$-spaces):

COROLLARY 3. *Let m be a fixed natural number. Then $\mathcal{P}_{\mathrm{as}(q;1)}(^mX) = \mathcal{P}(^mX)$ for every X if and only if $q \ge 1$.*

Corollary 4. *If $m \geq 2$ and X is an $\mathcal{L}_\infty$-space, then $\mathcal{P}_{\mathrm{as}(q;2)}({}^m X) = \mathcal{P}({}^m X)$ if and only if $q \geq 1$.*

Corollary 5. *If $m \geq 2$ and X is an $\mathcal{L}_\infty$-space, then $\mathcal{P}_{\mathrm{d},q}({}^m X) \neq \mathcal{P}({}^m X)$ for every $q < m$.*

In particular, if X is an $\mathcal{L}_\infty$-space and $m = 2$, then $\mathcal{P}_{\mathrm{d},2}({}^2 X) = \mathcal{P}({}^2 X)$ and thus we have:

Corollary 6. *If X is an $\mathcal{L}_\infty$-space, then $\mathcal{P}_{\mathrm{d},q}({}^2 X) = \mathcal{P}({}^2 X)$ if and only if $q \geq 2$.*

We also have:

Corollary 7. *If $q \leq 1/2$ and X is an $\mathcal{L}_p$-space ($p \geq 2$), then $\mathcal{P}_{\mathrm{as}(q;1)}({}^m X) = \mathcal{P}({}^m X)$ if and only if $p \leq mq$.*

Proof. A localization argument allows us to assume that $X = l_p$. If $\mathcal{P}_{\mathrm{as}(q;1)}({}^m X) = \mathcal{P}({}^m X)$, Theorem 5 ensures that $p \leq mq$. On the other hand, if $p \leq mq$ and $P \in \mathcal{P}({}^m X)$, then

$$\begin{aligned}\Big(\sum_{j=1}^{k} \|P(x_j)\|^q\Big)^{1/q} &\leq \|P\| \Big(\sum_{j=1}^{k} \|x_j\|^{mq}\Big)^{1/q} \\ &\leq \|P\| \Big(\sum_{j=1}^{k} \|x_j\|^{p}\Big)^{m/p} \leq C_p(X)\|P\|\, \|(x_j)_{j=1}^k\|_{w,1}^m,\end{aligned}$$

where $C_p(X)$ is the cotype constant of l_p and the last inequality holds since l_p has cotype p (for $p \geq 2$) and thus $\mathrm{id} : l_p \to l_p$ is absolutely $(p;1)$-summing.

All these results can be adapted (including Theorem 5), mutatis mutandis, to the multilinear case. Furthermore, one can extend Corollary 2:

Corollary 8. *Let $X_1, \ldots, X_s$ be $\mathcal{L}_\infty$-spaces, $q_1 = \cdots = q_s = 2$ and $q_{s+1} = \cdots = q_n = 1$. Then*

$$\mathcal{L}(X_1, \ldots, X_n) = \mathcal{L}_{\mathrm{as}(q;q_1,\ldots,q_n)}(X_1, \ldots, X_n),$$

for any choice of Banach spaces $X_{s+1}, \ldots, X_n$, if and only if $q \geq 1$.

Remark 1. For the bilinear case it is not hard to prove that when X is an $\mathcal{L}_\infty$-space, $\mathcal{L}_{\mathrm{d},q}({}^2 X) \neq \mathcal{L}({}^2 X)$ if $q < 2$. However, this result cannot be straightforwardly adapted for polynomials and thus Corollary 6 is in fact non-trivial. Non-coincidence results for absolutely summing multilinear mappings, in general, do not imply non-coincidence results for absolutely summing polynomials.

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