On scalar-valued nonlinear absolutely summing mappings

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Abstract. We investigate cases ("coincidence situations") in which every scalar-valued continuous $n$-homogeneous polynomial (or every continuous $n$-linear mapping) is absolutely $(p; q)$-summing. We extend some well known coincidence situations and obtain several non-coincidence results, inspired by a linear technique due to Lindenstrauss and Pełczyński.

1. Introduction. Throughout this note $X, X_1, \ldots, X_n, Y$ will stand for Banach spaces and the scalar field $\mathbb{K}$ can be either the real or the complex numbers.

An $m$-homogeneous polynomial $P$ from $X$ into $Y$ is said to be absolutely $(p; q)$-summing ($p \geq q/m$) if there is a constant $L$ so that

$$
\left( \sum_{j=1}^{k} \| P(x_j) \|^p \right)^{1/p} \leq L \| (x_j)_{j=1}^{k} \|_{w; q}^m
$$

for every natural $k$, where $\| (x_j)_{j=1}^{k} \|_{w; q} = \sup_{\varphi \in B_{X'}} (\sum_{j=1}^{k} |\varphi(x_j)|^q)^{1/q}$. This is a natural generalization of the concept of $(p; q)$-summing operators and in the last years it has been studied by several authors. The infimum of the $L > 0$ for which the inequality holds defines a norm $\| \cdot \|_{as(p; q)}$ for $p \geq 1$, or a $p$-norm for $p < 1$, on the space of $(p; q)$-summing homogeneous polynomials. The space of all $m$-homogeneous $(p; q)$-summing polynomials from $X$ into $Y$ is denoted by $P_{as(p; q)}(mX; Y)$ ($P_{as(p; q)}(mX)$ if $Y = \mathbb{K}$).

When $p = q/m$ we have an important particular case, since in this situation there is an analogue of the Grothendieck–Pietsch Domination Theorem. The $(q/m; q)$-summing $m$-homogeneous polynomials from $X$ into $Y$ are said to be $q$-dominated and this space is denoted by $P_{d,q}(mX; Y)$ ($P_{d,q}(mX)$ if $Y = \mathbb{K}$).

The Banach space of all continuous $m$-homogeneous polynomials $P$ from $X$ into $Y$ with the sup norm is denoted by $P(mX; Y)$ ($P(mX)$ if $Y$ is...
the scalar field). Analogously, the space of all continuous $m$-linear mappings from $X_1 \times \cdots \times X_m$ into $Y$ (with the sup norm) is denoted by $\mathcal{L}(X_1, \ldots, X_m; Y)$ ($\mathcal{L}(X_1, \ldots, X_m)$ if $Y = \mathbb{K}$). The concept of absolutely summing multilinear mapping follows the same pattern (for details we refer to [5]). Henceforth every polynomial and multilinear mapping are supposed to be continuous and every $\mathcal{L}_p$-space is assumed to be infinite-dimensional.

A natural problem is to find situations in which the space of absolutely summing polynomials coincides with the space of continuous polynomials (coincidence situations). When $Y$ is the scalar field, these situations are not rare as we can see in the next two well known results:

**Theorem 1.** Every scalar-valued $n$-linear mapping is absolutely $(1; 1)$-summing. In particular, every scalar-valued $n$-homogeneous polynomial is absolutely $(1; 1)$-summing (and, a fortiori, $(q; 1)$-summing for every $q \geq 1$).

**Theorem 2** (D. Pérez-García [6]). If $n \geq 2$ and $X$ is an $\mathcal{L}_\infty$-space, then every scalar-valued $n$-linear mapping on $X$ is $(1; 2)$-summing. In particular, every scalar-valued $n$-homogeneous polynomial on $X$ is $(1; 2)$-summing (and, a fortiori, $(q; 2)$-summing for every $q \geq 1$).

The proof of Theorem 1 can be found in [1] and is credited to A. Defant and J. Voigt. The case $n = 2$ of Theorem 2 was previously proved by Botelho [2] and is the unique known coincidence result for dominated polynomials.

In Section 2 we obtain new coincidence situations, generalizing Theorem 1 and extending the results of Theorem 2. Section 3 has a different purpose: to obtain a technical estimate (inspired by a linear result due to Lindenstrauss and Pełczyński [3]) and to explore its consequences. In particular, it is shown that Theorems 1 and 2 cannot be generalized in some other directions, and converses for the aforementioned theorems are obtained.

### 2. Coincidence situations.

The next theorem, inspired by a result of C. A. Soares, leads us to extensions of the two theorems stated in the first section:

**Theorem 3.** Let $A \in \mathcal{L}(X_1, \ldots, X_n; Y)$ and suppose that there exists $C > 0$ so that for any $x_1 \in X_1, \ldots, x_r \in X_r$, the $s$-linear ($s = n - r$) mapping $A_{x_1 \ldots x_r}(x_{r+1}, \ldots, x_n) = A(x_1, \ldots, x_n)$ is absolutely $(1; q_1, \ldots, q_s)$-summing and

$$\|A_{x_1 \ldots x_r}\|_{\text{as}(1; q_1, \ldots, q_s)} \leq C\|A\| \|x_1\| \cdots \|x_r\|.$$  

Then $A$ is absolutely $(1; 1, \ldots, 1, q_1, \ldots, q_s)$-summing.
Proof. For \( x_1^{(1)}, \ldots, x_1^{(m)} \in X_1, \ldots, x_n^{(1)}, \ldots, x_n^{(m)} \in X_n \), consider \( \varphi_j \in B_Y \) such that
\[
\|A(x_1^{(j)}, \ldots, x_n^{(j)})\| = \varphi_j(A(x_1^{(j)}, \ldots, x_n^{(j)}))
\]
for every \( j = 1, \ldots, m \). Then, denoting by \( r_j(t) \) the Rademacher functions on \([0,1]\) and by \( \lambda \) the Lebesgue measure on \( I = [0,1]^r \), we have
\[
\sum_{I} \left( \prod_{l=1}^{r} r_j(t_l) \right) \times \varphi_j A \left( \sum_{j_1=1}^{m} r_{j_1}(t_1) x_1^{(j_1)}, \ldots, \sum_{j_r=1}^{m} r_{j_r}(t_r) x_r^{(j_r)}, x_{r+1}^{(j)}, \ldots, x_n^{(j)} \right) d\lambda
\]
\[
= \sum_{j_1, \ldots, j_r=1}^{m} \varphi_j A(x_1^{(j_1)}, \ldots, x_r^{(j_r)}, x_{r+1}^{(j)}, \ldots, x_n^{(j)})
\]
\[
\times \int_{0}^{1} r_j(t_1) r_{j_1}(t_1) dt_1 \ldots \int_{0}^{1} r_j(t_r) r_{j_r}(t_r) dt_r
\]
\[
= \sum_{j=1}^{m} \sum_{j_1=1}^{m} \ldots \sum_{j_r=1}^{m} \varphi_j A(x_1^{(j_1)}, \ldots, x_r^{(j_r)}, x_{r+1}^{(j)}, \ldots, x_n^{(j)}) \delta_{jj_1} \ldots \delta_{jj_r}
\]
\[
= \sum_{j=1}^{m} \varphi_j A(x_1^{(j)}, \ldots, x_n^{(j)}) = \sum_{j=1}^{m} \|A(x_1^{(j)}, \ldots, x_n^{(j)})\| = (*).
\]
So, for each \( l = 1, \ldots, r \), assuming \( z_l = \sum_{j=1}^{m} r_j(t_l) x_l^{(j)} \) we obtain
\[
(*) = \sum_{l=1}^{m} \left( \prod_{l=1}^{r} r_j(t_l) \right) \times \varphi_j A \left( \sum_{j_1=1}^{m} r_{j_1}(t_1) x_1^{(j_1)}, \ldots, \sum_{j_r=1}^{m} r_{j_r}(t_r) x_r^{(j_r)}, x_{r+1}^{(j)}, \ldots, x_n^{(j)} \right) d\lambda
\]
\[
\leq \sum_{l=1}^{m} \left( \prod_{l=1}^{r} r_j(t_l) \right) \times \varphi_j A \left( \sum_{j_1=1}^{m} r_{j_1}(t_1) x_1^{(j_1)}, \ldots, \sum_{j_r=1}^{m} r_{j_r}(t_r) x_r^{(j_r)}, x_{r+1}^{(j)}, \ldots, x_n^{(j)} \right) d\lambda
\]
\[
\leq \sum_{l=1}^{m} \left\| A \left( \sum_{j_1=1}^{m} r_{j_1}(t_1) x_1^{(j_1)}, \ldots, \sum_{j_r=1}^{m} r_{j_r}(t_r) x_r^{(j_r)}, x_{r+1}^{(j)}, \ldots, x_n^{(j)} \right) \right\| d\lambda
\]
We have the following straightforward consequence, generalizing Theorem 1:

**Corollary 1.** If
\[ \mathcal{L}(X_1, \ldots, X_m; Y) = \mathcal{L}_{as(1; q_1, \ldots, q_m)}(X_1, \ldots, X_m; Y) \]
then, for any Banach spaces \( X_{m+1}, \ldots, X_n \), we have
\[ \mathcal{L}(X_1, \ldots, X_n; Y) = \mathcal{L}_{as(1; q_1, \ldots, q_m, 1, \ldots, 1)}(X_1, \ldots, X_n; Y). \]

The following corollary (whose proof is simple and we omit it) is a consequence of Theorems 2 and 3.

**Corollary 2.** If \( X_1, \ldots, X_s \) are \( \mathcal{L}_\infty \)-spaces then, for any Banach spaces \( X_{s+1}, \ldots, X_n \), we have
\[ \mathcal{L}(X_1, \ldots, X_n) = \mathcal{L}_{as(1; q_1, \ldots, q_n)}(X_1, \ldots, X_n), \]
where \( q_1 = \cdots = q_s = 2 \) and \( q_{s+1} = \cdots = q_n = 1 \).

It is obvious that Corollary 2 is still true if we replace the scalar field by any finite-dimensional Banach space. A natural question is whether Corollary 2 can be stated for some infinite-dimensional Banach space in place of \( \mathbb{K} \). Precisely, the question is:

- If \( X_1, \ldots, X_k \) are \( \mathcal{L}_\infty \)-spaces, is there some infinite-dimensional Banach space \( Y \) such that
\[ \mathcal{L}(X_1, \ldots, X_k, \ldots, X_n; Y) = \mathcal{L}_{as(1; q_1, \ldots, q_n)}(X_1, \ldots, X_k, \ldots, X_n; Y), \]
where \( q_1 = \cdots = q_k = 2 \) and \( q_{k+1} = \cdots = q_n = 1 \), regardless of the choice of the Banach spaces \( X_{k+1}, \ldots, X_n \)?

The answer to this question is no, as shown by the following proposition:

**Proposition 1.** Suppose that \( X_1, \ldots, X_k \) are \( \mathcal{L}_\infty \)-spaces. If \( q_1 = \cdots = q_k = 2 \), \( q_{k+1} = \cdots = q_n = 1 \) and
\[ \mathcal{L}(X_1, \ldots, X_k, \ldots, X_n; Y) = \mathcal{L}_{as(1; q_1, \ldots, q_n)}(X_1, \ldots, X_k, \ldots, X_n; Y), \]
regardless of the choice of the Banach spaces \( X_{k+1}, \ldots, X_n \), then \( \dim Y < \infty \).
Proof. By a standard localization argument, it suffices to prove that if \( \dim Y = \infty \), then
\[
\mathcal{L}(^{n}c_{0};Y) \neq \mathcal{L}_{\text{as}(q_{1},\ldots,q_{n})}(^{n}c_{0};Y),
\]
where \( q_{1} = \ldots = q_{k} = 2 \) and \( q_{k+1} = \ldots = q_{n} = 1 \). But from [5, Theorem 8] we even have
\[
\mathcal{L}(^{n}c_{0};Y) \neq \mathcal{L}_{\text{as}(q_{1},\ldots,q_{n})}(^{n}c_{0};Y)
\]
for any \( q < 2 \) and \( q_{1},\ldots,q_{n} \geq 1 \).

3. Non-coincidence situations. Assume that \( X \) is an infinite-dimensional Banach space and suppose that \( X \) has a normalized unconditional Schauder basis \( (x_{n}) \) with coefficient functionals \( (x_{n}^{*}) \). If \( \mathcal{P}_{\text{as}(q;1)}(mX;Y) = \mathcal{P}(^{m}X;Y) \), it is natural to ask:

What is the infimum of the \( t \) such that in this situation \( (x_{n}^{*}(x)) \in l_{t} \) for each \( x \in X \)? This infimum will be denoted by \( \mu = \mu(X,Y,q,m) \).

In [5], inspired by an important linear result due to Lindenstrauss and Pelczyński, we have proved:

**Theorem 4 (Pellegrino [5, Theorem 5]).** Let \( X \) and \( Y \) be infinite-dimensional Banach spaces. Suppose that \( X \) has an unconditional Schauder basis \( (x_{n}) \). If \( Y \) finitely factors the formal inclusion \( l_{p} \to l_{\infty} \) and \( \mathcal{P}_{\text{as}(q;1)}(mX;Y) = \mathcal{P}(^{m}X;Y) \) with \( 1/m \leq q \), then

(a) \( \mu \leq mpq/(p-q) \) if \( q < p \),
(b) \( \mu \leq mq \) if \( q \geq p/2 \).

However, by inspecting the proof of this theorem in [5], one can see that it is by no means necessary to assume that \( \dim Y = \infty \). Only in Corollary of [5] (when the Dvoretzky–Rogers Theorem is invoked) is it indeed necessary to assume \( \dim Y = \infty \). A slight change in the proof of [5, Theorem 5] yields the following result:

**Theorem 5.** Let \( X \) be an infinite-dimensional Banach space with a normalized unconditional Schauder basis \( (x_{n}) \). If \( \mathcal{P}_{\text{as}(q;1)}(^{m}X) = \mathcal{P}(^{m}X) \), then

(a) \( \mu \leq mq/(1-q) \) if \( q < 1 \),
(b) \( \mu \leq mq \) if \( q \leq 1/2 \).

**Proof.** If \( x = \sum_{j=1}^{\infty} a_{j}x_{j} \) and \( \{\mu_{i}\}_{i=1}^{n} \) is such that \( \sum_{j=1}^{n} |\mu_{j}|^{1/q} = 1 \), define \( P : X \to \mathbb{K} \) by \( Px = \sum_{j=1}^{n} |\mu_{j}|^{1/q}a_{j}^{m} \).

Since \( (x_{n}) \) is an unconditional basis, there exists a \( \theta > 0 \) satisfying
\[
\left\| \sum_{j=1}^{n} \varepsilon_{j}a_{j}x_{j} \right\| \leq \theta \|x\| \quad \text{for every } n \text{ and any } \varepsilon_{j} = \pm 1.
\]
Hence
\[ |Px| \leq \sum_{j=1}^{n} |\mu_j|^{1/q}a_j^m \leq g^m|x|^m \sum_{j=1}^{n} |\mu_j|^{1/q}, \]
and thus \( \|P\| \leq g^m \) and \( \|P\|_{as(q;1)} \leq Cg^m \). Therefore
\[
\left( \sum_{j=1}^{n} |a_j|^m |\mu_j|^{1/q} \right)^{1/q} \leq \left( \sum_{j=1}^{n} |Pa_jx_j|^q \right)^{1/q} \\
\leq \|P\|_{as(q;1)} \max_{\varepsilon_j \in \{1,-1\}} \left\| \sum_{j=1}^{n} \varepsilon_j a_jx_j \right\|^m \\
\leq \|P\|_{as(q;1)} (g \|x\|)^m \leq Cg^{2m} \|x\|^m. 
\]
Defining \( s = 1/q \), we have \( \frac{1}{s} + \frac{1}{s-1} = 1 \) and
\[
\left( \sum_{j=1}^{n} |a_j|^{s-1}mq \right)^{1/s} \leq \sup \left\{ \sum_{j=1}^{n} |\mu_j| |a_j|^{mq} : \sum_{j=1}^{n} |\mu_j|^s = 1 \right\}. 
\]
Since (3.1) is true whenever \( \sum_{j=1}^{n} |\mu_j|^s = 1 \), by (3.1) and (3.2) we obtain
\[
\left( \sum_{j=1}^{n} |a_j|^{s-1}mq \right)^{1/s} \leq [Cg^{2m} \|x\|^m]^{1/m}. 
\]
But \( \frac{s-1}{s-1}mq = \frac{mq}{1-q} \) and \( n \) is arbitrary, and hence part (a) is proved. Now, if \( 1/m \leq q \leq 1/2 \), define \( S : X \to \mathbb{K} \) by \( Sx = \sum_{j=1}^{n} a_j^m \). Since \( m \geq \frac{s}{s-1}mq \), we obtain
\[
|Sx| \leq \sum_{j=1}^{n} |a_j|^m \leq \left[ \left( \sum_{j=1}^{n} |a_j|^{s-1}mq \right)^{1/s} \right]^m \leq Cg^{2m} \|x\|^m. 
\]
Thus \( \|S\| \leq Cg^{2m} \) and \( \|S\|_{as(q;1)} \leq C^2g^{2m} \). Therefore
\[
\sum_{j=1}^{n} |a_j|^q = \sum_{j=1}^{n} |Sa_jx_j|^q \leq \|S\|_{as(q;1)}^q \max_{\varepsilon_j \in \{1,-1\}} \left\| \sum_{j=1}^{n} \varepsilon_j a_jx_j \right\|^{mq} \\
\leq (C^2g^{2m})^q (g \|x\|)^{mq}. 
\]
Consequently, since \( n \) is arbitrary, we have \( \sum_{j=1}^{\infty} |a_j|^mq < \infty \) whenever \( x = \sum_{j=1}^{\infty} a_jx_j \in X \).

Now we list several important consequences of Theorem 5. For example, Corollaries 3 and 4 below give converses for Theorems 1 and 2, respectively. The proofs of Corollaries 3–6 are simple (using Theorem 5 and standard localization techniques in order to extend the results from \( c_0 \) to \( \mathcal{L}_\infty \)-spaces):

**Corollary 3.** Let \( m \) be a fixed natural number. Then \( P_{as(q;1)}^m(X) = P^m(X) \) for every \( X \) if and only if \( q \geq 1 \).
COROLLARY 4. If $m \geq 2$ and $X$ is an $\mathcal{L}_\infty$-space, then $P_{\text{as}(q;2)}(mX) = \mathcal{P}(mX)$ if and only if $q \geq 1$.

COROLLARY 5. If $m \geq 2$ and $X$ is an $\mathcal{L}_\infty$-space, then $P_{d,q}(mX) \neq \mathcal{P}(mX)$ for every $q < m$.

In particular, if $X$ is an $\mathcal{L}_\infty$-space and $m = 2$, then $P_{d,2}(2X) = \mathcal{P}(2X)$ and thus we have:

COROLLARY 6. If $X$ is an $\mathcal{L}_\infty$-space, then $P_{d,q}(2X) = \mathcal{P}(2X)$ if and only if $q \geq 2$.

We also have:

COROLLARY 7. If $q \leq 1/2$ and $X$ is an $\mathcal{L}_p$-space ($p \geq 2$), then $P_{\text{as}(q;1)}(mX) = \mathcal{P}(mX)$ if and only if $p \leq mq$.

Proof. A localization argument allows us to assume that $X = l_p$. If $P_{\text{as}(q;1)}(mX) = \mathcal{P}(mX)$, Theorem 5 ensures that $p \leq mq$. On the other hand, if $p \leq mq$ and $P \in \mathcal{P}(mX)$, then

$$\left(\sum_{j=1}^k \|P(x_j)\|^q\right)^{1/q} \leq \|P\|\left(\sum_{j=1}^k \|x_j\|^{mq}\right)^{1/q} \leq \|P\|\left(\sum_{j=1}^k \|x_j\|^p\right)^{m/p} \leq C_p(X)\|P\|\|(x_j)_{j=1}^k\|_{w,1},$$

where $C_p(X)$ is the cotype constant of $l_p$ and the last inequality holds since $l_p$ has cotype $p$ (for $p \geq 2$) and thus $\text{id} : l_p \to l_p$ is absolutely $(p;1)$-summing.

All these results can be adapted (including Theorem 5), mutatis mutandis, to the multilinear case. Furthermore, one can extend Corollary 2:

COROLLARY 8. Let $X_1, \ldots, X_s$ be $\mathcal{L}_\infty$-spaces, $q_1 = \cdots = q_s = 2$ and $q_{s+1} = \cdots = q_n = 1$. Then

$$\mathcal{L}(X_1, \ldots, X_n) = \mathcal{L}_{\text{as}(q_1, \ldots, q_n)}(X_1, \ldots, X_n),$$

for any choice of Banach spaces $X_{s+1}, \ldots, X_n$, if and only if $q \geq 1$.

REMARK 1. For the bilinear case it is not hard to prove that when $X$ is an $\mathcal{L}_\infty$-space, $\mathcal{L}_{d,q}(2X) \neq \mathcal{L}(2X)$ if $q < 2$. However, this result cannot be straightforwardly adapted for polynomials and thus Corollary 6 is in fact non-trivial. Non-coincidence results for absolutely summing multilinear mappings, in general, do not imply non-coincidence results for absolutely summing polynomials.
References


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