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A note on LaSalle's problems

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Abstract. In LaSalle's book "The Stability of Dynamical Systems", the author gives four conditions which imply that the origin of a discrete dynamical system defined on $\mathbb{R}$ is a global attractor, and proposes to study the natural extensions of these conditions in $\mathbb{R}^n$. Although some partial results are obtained in previous papers, as far as we know, the problem is not completely settled. In this work we first study the four conditions and prove that just one of them implies that the origin is a global attractor in $\mathbb{R}^n$ for polynomial maps. Then we note that two of these conditions have a natural extension to ordinary differential equations. One of them gives rise to the well known Markus–Yamabe assumptions. We study the other condition and we prove that it does not imply that the origin is a global attractor.

1. Introduction and statement of the results. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ map and consider the dynamics of iterations of T:

$$x_{k+1} = T(x_k). \tag{1}$$

Assume that 0 is a fixed point of T. We say that it is a *global attractor* for (1) if the sequence x_k tends to 0 as k tends to infinity for any $x_0 \in \mathbb{R}^n$.

Let $A = (a_{ij})$ be a real $n \times n$ matrix. We denote by $\sigma(A)$ the spectrum of A, i.e., the set of eigenvalues of A, and we define $|A| = (|a_{ij}|)$. We also denote by $T'(x) = (\partial T_i(x)/\partial x_j)$ the Jacobian matrix of T at $x \in \mathbb{R}^n$. When $T(0) = 0$, we can write $T(x)$ in the form $T(x) = A(x)x$, where $A(x)$ is an $n \times n$ matrix function. Note that this $A(x)$ is not unique.

LaSalle [11] gives some possible generalizations of the sufficient conditions to have a global attractor for $n = 1$. Concretely, the conditions are the following:

(A_1) $\quad |\lambda| < 1$ for each $\lambda \in \sigma(A(x))$ and for all $x \in \mathbb{R}^n$,

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(A_2) $|\lambda| < 1$ for each $\lambda \in \sigma(|A(x)|)$ and for all $x \in \mathbb{R}^n$,
(B_1) $|\lambda| < 1$ for each $\lambda \in \sigma(T'(x))$ and for all $x \in \mathbb{R}^n$,
(B_2) $|\lambda| < 1$ for each $\lambda \in \sigma(|T'(x)|)$ and for all $x \in \mathbb{R}^n$.

Condition B_1 was treated by the authors in [5]. This condition only implies that the origin is a global attractor for planar polynomial maps. In the same paper there is an example of a planar rational map which has a periodic orbit. In [3] there are examples of polynomial maps defined in $\mathbb{R}^n$, $n \geq 3$, satisfying the condition and having unbounded orbits. We also refer the reader to [4].

On the other hand Mau-Hsiang Shih and Jinn-Wen Wu [13] present a planar map which satisfies condition A_1 and has an unbounded orbit. Their example is not smooth. See also [14].

Now for each $T : \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{C}^1$ we consider the dynamical system

$$\dot{x} = T(x). \tag{2}$$

Assume that 0 is a zero of T. We say that it is a *global attractor* for (2) if $\phi(t,x)$ tends to 0 as t tends to infinity for each $x \in \mathbb{R}^n$, where $\phi(t,x)$ is the solution of (2) with $\phi(0,x) = x$.

Conditions A_1, B_1 have natural extensions to ordinary differential equations of the form (2):

(C) $\mathrm{Re}(\lambda) < 0$ for each $\lambda \in \sigma(A(x))$ and for all $x \in \mathbb{R}^n$,
(D) $\mathrm{Re}(\lambda) < 0$ for each $\lambda \in \sigma(T'(x))$ and for all $x \in \mathbb{R}^n$.

The problem determined by condition D was also posed by Markus and Yamabe in 1960. It is known in the literature as the Markus–Yamabe Conjecture. It was proved for planar polynomial maps in 1988 (see [12]) and for planar $\mathcal{C}^1$ maps in 1993 (see [7, 9, 10]). In [1] and [2] there are examples of smooth maps defined in $\mathbb{R}^n$, $n \geq 4$, satisfying the condition and having a periodic orbit and in [3] there is an example of a polynomial map defined in $\mathbb{R}^n$, $n \geq 3$, satisfying the same condition and having some orbits that escape to infinity.

In this note we complete the study of LaSalle's problems by considering conditions A_1, A_2, B_2 and C. Concretely, for $n \geq 2$ we exhibit polynomial maps satisfying either A_1 or A_2 or C such that the origin is not a global attractor. In addition for $n \geq 2$ we show a rational map satisfying condition B_2 such that the origin is not a global attractor either. Lastly we prove the following theorem:

THEOREM. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map satisfying condition B_2 and $T(0) = 0$. Then the origin is a global attractor.*

The following table summarizes all the results about these conditions.

Table 1. Is the origin a global attractor for $x_{k+1} = T(x_k)$ or $\dot{x} = T(x)$? Here $A(x)$ is a matrix such that $T(x) = A(x)x$, $R_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{C}^- = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$. The results marked with $(*)$ are obtained in this paper.

Condition	$n = 2$	$n > 2$
$A_1 : \sigma(A(x)) \subset R_1$	FALSE: there are polynomial examples with periodic points $(*)$. See also [13].	FALSE $(*)$.
$A_2 : \sigma(\lvert A(x)\rvert) \subset R_1$	FALSE: there are polynomials examples with invariant algebraic hypersurfaces not passing through the origin $(*)$.	FALSE $(*)$.
$B_1 : \sigma(T'(x)) \subset R_1$ (Discrete Markus–Yamabe Problem)	TRUE for polynomials, FALSE for rational maps: there are examples with periodic points [5].	FALSE: there are polynomial examples with orbits going to infinity [3].
$B_2 : \sigma(\lvert T'(x)\rvert) \subset R_1$	TRUE for polynomials, FALSE for rational maps: there are examples with periodic points $(*)$.	TRUE for polynomials $(*)$.
$C : \sigma(A(x)) \subset \mathbb{C}^-$	FALSE: there are polynomial examples with invariant hyperplanes not passing through the origin $(*)$.	FALSE $(*)$.
$D : \sigma(T'(x)) \subset \mathbb{C}^-$ (Markus–Yamabe Problem)	TRUE in the C^1 case [7, 10].	FALSE: there are polynomial examples with orbits going to infinity [3].

The proof of the above theorem is given in Section 3. Section 2 is devoted to the examples which give negative answers to the other questions.

2. The examples. We begin by giving a lemma which will be the key to constructing polynomial examples.

LEMMA 2.1. *Let $A(x) = (p_{ij}(x))$ be an $n \times n$ matrix function with polynomial entries $p_{ij}(x) \in \mathbb{R}[x_1, \ldots, x_n]$ and such that $\bigcup_{x \in \mathbb{R}^n} \sigma(A(x))$ is a bounded set. Then the characteristic polynomial of $A(x)$ is independent of x.*

Proof. Let $P_x(\lambda)$ be the characteristic polynomial of $A(x)$ and let $\lambda_1, \ldots, \lambda_n$ be the roots of $P_x(\lambda)$. Then

$$P_x(\lambda) = \lambda^n - t_1\lambda^{n-1} + \ldots + (-1)^n t_n$$

where

$$t_j = \sum_{1 \le i_1 < \ldots < i_j \le n} \lambda_{i_1} \ldots \lambda_{i_j}, \qquad j = 1, \ldots, n.$$

Since there exists $k \in \mathbb{R}$ such that $|\lambda_i| < k$ for all $i = 1, \ldots, n$, there exist $k_j \in \mathbb{R}$ such that $|t_j| < k_j$ for all $j = 1, \ldots, n$. On the other hand since each t_j can be described as the sum of all minors of order j which have its diagonal on the principal diagonal of $A(x)$, we conclude that each t_j is a polynomial in x. Since the only bounded polynomials are the constants, the result follows. ■

In the simplest case $n = 2$ we can consider

$$A(x,y) = \begin{pmatrix} C + p(x,y) & q(x,y) \\ r(x,y) & C - p(x,y) \end{pmatrix}$$

with $r(x,y)q(x,y) = D^2 - p^2(x,y)$. Then $t_1 = 2C$, $t_2 = C^2 - D^2$ and the eigenvalues are $C \pm D$, which do not depend on $(x,y) \in \mathbb{R}^2$.

In order to have examples satisfying condition A_1 we take $A(x,y)$ in the above form with $|C \pm D| < 1$. For instance we can take $C = D = 0$, $q(x,y) \equiv 1$ and $r(x,y) = -p^2(x,y)$. Now we are going to impose that such maps have periodic points.

LEMMA 2.2. *Let $T(x,y) = A(x,y)(x,y)^T$ where $A(x,y)$ is of the form*

$$A(x,y) = \begin{pmatrix} p(x,y) & 1 \\ -p^2(x,y) & -p(x,y) \end{pmatrix}.$$

Let $(x_0,y_0) \in \mathbb{R}^2$ and set $p_n = p(x_n,y_n)$. Then (x_0,y_0) is a periodic point of period k if and only if

(i) $y_0 = -p_{k-1}x_0$,
(ii) $(p_0 - p_{k-1})(p_1 - p_0)(p_2 - p_1)\cdots(p_{k-1} - p_{k-2}) = 1$.

Proof. By iterating k times the map T, a simple computation gives

$$y_k = -p_{k-1}x_k,$$
$$x_k = (p_{k-1} - p_{k-2})(p_{k-2} - p_{k-3})\ldots(p_2 - p_1)(p_1 - p_0)(p_0x_0 + y_0).$$

If (x_0,y_0) has period k then $x_k = x_0$ and $y_k = y_0$ and hence since $y_k = -p_{k-1}x_k$ we get $y_0 = -p_{k-1}x_0$ and

$$x_0 = (p_{k-1} - p_{k-2})(p_{k-2} - p_{k-3})\ldots(p_2 - p_1)(p_1 - p_0)(p_0 - p_{k-1})x_0,$$

which gives the desired conditions. ■

It is clear from the above lemma that it is not possible to get two-periodic points but we can get k-periodic points for $k \geq 3$. A simple example can be obtained by taking $k = 4$ with $p_1 = p_3 = 1$ and $p_0 = p_2 = 0$ and it is given in the next proposition.

PROPOSITION 2.3. *Let $T(x) = A(x)x$ where*

$$A(x) = \begin{pmatrix} x_1^2 + x_1x_2 & 1 & 0 & \ldots & 0 \\ -(x_1^2 + x_1x_2)^2 & -(x_1^2 + x_1x_2) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Then T satisfies condition A_1 and it has periodic points of period 4. *In particular the origin is not a global attractor of* (1).

Proof. It is easy to see that $A(x)$ has all eigenvalues zero for all $x \in \mathbb{R}^n$ and that $(1, -1, 0, \ldots, 0)$ is a periodic point of $T(x)$ of period 4. ■

Concerning condition A_2 we consider $A(x, y)$ of the form

$$A(x,y) = \begin{pmatrix} a & 0 \\ r(x,y) & b \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and $r(x, y) \in \mathbb{R}[x, y]$. The matrix $|A(x, y)|$ satisfies condition A_2 if $|a| < 1$ and $|b| < 1$. We can see that the hyperbola $xy = 1$ is invariant under $T(x, y) = A(x, y)(x, y)^T$ if and only if $ax^2 r(x, 1/x) = 1 - ab$. Hence, if we choose $r(x, y) = \frac{1-ab}{a} xy^3$ then the origin is not a global attractor of the corresponding map.

Proposition 2.4. *Let $T(x) = A(x)x$ where*

$$A(x) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \ldots & 0 \\ \frac{3}{2}xy^3 & \frac{1}{2} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Then T satisfies condition A_2 and $\{(x, 1/x, 0, \ldots, 0)\}$ is invariant under T. In particular the origin is not a global attractor of (1).

Proof. Clearly the eigenvalues of $|A(x)|$ are $\lambda_1 = \lambda_2 = 1/2$ and $\lambda_j = 0$ for $j = 3, \ldots, n$. Furthermore $T(x, 1/x, 0, \ldots, 0) = (x/2, 2/x, 0, \ldots, 0)$, which proves the assertion. ■

As mentioned in the introduction, condition B_1 does not imply that the origin is a global attractor, even in dimension two. The example which proves this assertion is also useful to study condition B_2.

Proposition 2.5. *Define*

$$T(x) = \left(\frac{-kx_2^3}{1 + x_1^2 + x_2^2}, \frac{kx_1^3}{1 + x_1^2 + x_2^2}, \frac{1}{2}x_3, \ldots, \frac{1}{2}x_n \right) \quad \textit{where} \quad k \in \left(1, \frac{2}{3}\right).$$

Then $T(x)$ satisfies conditions B_1 and B_2 and it has periodic points of period 4. *In particular the origin is not a global attractor.*

Proof. The Jacobian matrix of T at $x \in \mathbb{R}^n$, $T'(x)$, has the following form:

$$T'(x) = \begin{pmatrix} \dfrac{2kx_1x_2^3}{1+x_1^2+x_2^2} & \dfrac{kx_2^2(3+3x_1^2+x_2^2)}{1+x_1^2+x_2^2} & 0 & \dots & 0 \\ \dfrac{kx_1^2(3+x_1^2+3x_2^2)}{1+x_1^2+x_2^2} & \dfrac{2kx_1^3x_2}{1+x_1^2+x_2^2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2} \end{pmatrix}.$$

This matrix satisfies condition B_1 (see [5]). We are going to see that $T'(x)$ also satisfies condition B_2.

By easy computations we find that the eigenvalues of the matrix $|T'(x)|$ are

$$\lambda_1 = \frac{3k|x_1x_2|}{x_1^2+x_2^2+1},$$
$$\lambda_2 = \frac{-k|x_1x_2|(x_1^2+x_2^2+3)}{(x_1^2+x_2^2+1)^2},$$
$$\lambda_j = \frac{1}{2} \quad \text{for any } j = 3, \dots, n.$$

Since $|x_1x_2| \le (x_1^2+x_2^2)/2$ and $k < 2/3$ we have

$$|\lambda_1| = \lambda_1 < \frac{3k}{2} < 1,$$
$$|\lambda_2| = -\lambda_2 < \frac{k(x_1^2+x_2^2)(3x_1^2+3x_2^2+3)}{2(x_1^2+x_2^2+1)^2} < \frac{3k}{2} < 1.$$

On the other hand it is also easy to see that the point $(1/\sqrt{k-1}, 0, \dots, 0)$ has period 4 under T. ∎

Concerning condition C we consider

$$A(x,y) = \begin{pmatrix} C + p(x,y) & 1 \\ -p^2(x,y) & C - p(x,y) \end{pmatrix},$$

which has $\lambda_1 = \lambda_2 = C$ for all $(x,y) \in \mathbb{R}^2$. We choose $C = -1$. The associated differential system is

$$\begin{cases} \dot{x} = (p(x,y) - 1)x + y, \\ \dot{y} = -p^2(x,y)x - (p(x,y)+1)y. \end{cases}$$

We can see that the straight line $x = k$ is invariant under the flow generated by the system if and only if $p(k,y) = 1 - \frac{1}{k}y$. Hence, if we choose

$p(x,y) = 1 - \frac{1}{k}y$ then the origin is not a global attractor of the corresponding system.

PROPOSITION 2.6. *Consider the differential system $\dot{x} = A(x)x$ in $\mathbb{R}^n$, where*

$$A(x) = \begin{pmatrix} -x_2 & 1 & 0 & \dots & 0 \\ -(1-x_2)^2 & x_2 - 2 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix}.$$

Then $T(x) = A(x)x$ satisfies condition C and the hyperplane $x_1 = 1$ is invariant under the flow generated by $\dot{x} = A(x)x$. In particular the origin is not a global attractor of the above system.

Proof. The system $\dot{x} = A(x)x$ can be written as

$$\begin{cases} \dot{x}_1 = x_2 - x_1x_2, \\ \dot{x}_2 = -x_1(1-x_2)^2 + x_2(x_2 - 2), \\ \dot{x}_k = -x_k, \quad k \geq 3. \end{cases}$$

Clearly the eigenvalues of $A(x)$ are $\lambda = -1$ for all $x \in \mathbb{R}^n$. The hyperplane $x_1 = 1$ is invariant under the flow if and only if $\dot{x}_1 = 0$ on $x_1 = 1$, which is trivially satisfied. ■

3. Proof of the Theorem. Before proving the main result of this section we need some definitions.

We denote by $\mathbb{K}$ one of the fields $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Let $M = M(x)$ be an $n \times n$ matrix with coefficients in $\mathbb{K}[x]$ where $x = (x_1, \dots, x_m)$. A *diagonal minor* of M is a minor with diagonal contained in the diagonal of M. Denote by $a_i^j = a_i^j(x)$ the entry of M in column j and row i. An *elementary product of length k* of M is an element which can be written as

$$a_{\sigma(i_1)}^{i_1} a_{\sigma(i_2)}^{i_2} \dots a_{\sigma(i_k)}^{i_k},$$

where $i_1, \dots, i_k$ are k different elements of $I = \{1, \dots, n\}$ and $\sigma \in S_k$, the symmetric group of k elements. We will say that M is *block-triangular* if there exists a partition $I_1, \dots, I_k$ of I such that

$$a_i^j \begin{cases} = 0 & \text{if } i \in I_r,\ j \in I_s \text{ and } r > s, \\ \in \mathbb{K} & \text{if } i, j \in I_r \text{ for some } r \leq k. \end{cases}$$

Notice that after a reordering of indices a block-triangular matrix (a_i^j) looks like

$$\begin{pmatrix} \begin{pmatrix} b_1^1 & \dots & b_p^1 \\ \vdots & \ddots & \vdots \\ b_1^p & \dots & b_p^p \end{pmatrix} & 0 & 0 & 0 \\ \vdots & \begin{pmatrix} b_{p+1}^{p+1} & \dots & b_{p+q}^{p+1} \\ \vdots & \ddots & \vdots \\ b_{p+1}^{p+q} & \dots & b_{p+q}^{p+q} \end{pmatrix} & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & \begin{pmatrix} b_{n-m}^{n-m} & \dots & b_n^{n-m} \\ \vdots & \ddots & \vdots \\ b_{n-m}^{n} & \dots & b_n^n \end{pmatrix} \end{pmatrix},$$

where $b_i^j = a_{\sigma(i)}^{\sigma(j)}$ for some permutation σ, and the entries of the diagonal submatrices are elements of $\mathbb{K}$.

The following lemma relates the above notions to condition B_2.

PROPOSITION 3.1. *Let $M = M(x)$ be an $n \times n$ matrix with entries in $\mathbb{K}[x]$, where $x = (x_1, \dots, x_m)$. Then the following conditions are equivalent*:

(i) *There exists $\alpha \in \mathbb{R}$ such that for each $x \in \mathbb{K}^m$ and for each eigenvalue $\lambda(x)$ of $|M(x)|$, $|\lambda(x)| < \alpha$.*

(ii) *Every elementary product of M belongs to $\mathbb{K}$.*

(iii) $\det N \in \mathbb{K}$ *for every diagonal minor N of M.*

(iv) *The matrix M is block-triangular.*

(v) *The characteristic polynomial of $|M(x)|$ has coefficients in $\mathbb{K}$.*

(vi) *The eigenvalues of $|M(x)|$ do not depend on x.*

Proof. We will prove that (i)⇒(ii)⇒(iv)⇒(v) and (iii)⇒(ii). The proofs of (v)⇒(vi)⇒(i) and (ii)⇒(iii) are obvious.

(i)⇒(ii). We prove by induction on k that each elementary product of length k belongs to $\mathbb{K}$. For $k = 1$ we have to prove that a_i^i belongs to $\mathbb{K}$ for any $i \in \{1, \dots, n\}$. To see this consider $t_1(x) = |a_1^1| + |a_2^2| + \dots + |a_n^n|$, which is a symmetric polynomial in the eigenvalues of $|M(x)|$. Since by hypothesis the eigenvalues are bounded functions of x we conclude that $t_1(x)$ is also bounded. Since a_i^i are polynomials we see that they all belong to $\mathbb{K}$. Now assume that the result holds for each l, $l < k \le n$. Let $t_k(x)$ be the sum of the determinants of all diagonal minors of order k. Then $t_k(x)$ is a symmetric polynomial in the eigenvalues of $|M(x)|$ and hence $t_k(x)$ is a bounded function. On the other hand $t_k(x)$ can be written as

$$\sum_{\substack{1 \le i_1 < \dots < i_k \le n \\ \sigma \in S_k}} (-1)^{\varepsilon(\sigma)} |a_{\sigma(i_1)}^{i_1} a_{\sigma(i_2)}^{i_2} \dots a_{\sigma(i_k)}^{i_k}|.$$

If $\sigma \in S_k$ decomposes into a product of disjoint cycles of length less than k then the corresponding elementary product $a^{i_1}_{\sigma(i_1)} a^{i_2}_{\sigma(i_2)} \dots a^{i_k}_{\sigma(i_k)}$ also decomposes into elementary products of length less than k. So by the induction hypothesis $a^{i_1}_{\sigma(i_1)} a^{i_2}_{\sigma(i_2)} \dots a^{i_k}_{\sigma(i_k)}$ belongs to $\mathbb{K}$. Otherwise σ is a cycle of length k and $\varepsilon(\sigma) = k+1$. Denote by R_k the set of all cycles of length k in S_k. We have

$$\begin{aligned} t_k(x) = (-1)^{k+1} & \sum_{\substack{1\leq i_1<\ldots<i_k\leq n \\ \sigma\in R_k}} |a^{i_1}_{\sigma(i_1)} a^{i_2}_{\sigma(i_2)} \dots a^{i_k}_{\sigma(i_k)}| \\ & + \sum_{\substack{1\leq i_1<\ldots<i_k\leq n \\ \sigma\in S_k\setminus R_k}} (-1)^{\varepsilon(\sigma)} |a^{i_1}_{\sigma(i_1)} a^{i_2}_{\sigma(i_2)} \dots a^{i_k}_{\sigma(i_k)}|. \end{aligned}$$

Since $t_k(x)$ is bounded and the second summand of this equality is, by the induction hypothesis, a complex number, we deduce that

$$\sum_{\substack{1\leq i_1<\ldots<i_k\leq n \\ \sigma\in R_k}} |a^{i_1}_{\sigma(i_1)} a^{i_2}_{\sigma(i_2)} \dots a^{i_k}_{\sigma(i_k)}|$$

is also bounded. Since each summand is the modulus of a polynomial we conclude that each summand is a complex number. Thus each indecomposable elementary product of length k belongs to $\mathbb{K}$.

(ii)$\Rightarrow$(iv). Assume that (ii) holds. We say that $a^{j_1}_{j_2} a^{j_2}_{j_3} \dots a^{j_{k-1}}_{j_k}$ is a polynomial word beginning at j_1 if it is a non-constant polynomial and there are no repetitions among the j_i's. Clearly the length of a polynomial word is bounded by $n-1$. Let K_i be the set of polynomial words beginning at i. Let $w = a^{j_1}_{j_2} a^{j_2}_{j_3} \dots a^{j_{k-1}}_{j_k} \in K_{j_1}$. The cardinality of the set $\{i : a^{j_i}_{j_{i+1}}$ is a non-constant polynomial$\}$ will be called the *rank* of w, and denoted by $r(w)$. Let $i \in \{1, \dots, n\}$ and define

$$n(i) = \begin{cases} 0 & \text{if } K_i = \emptyset, \\ \max\{r(w) : w \in K_i\} & \text{otherwise.} \end{cases}$$

For $j = 0, \dots, n-1$ let $I_j = \{i \in \{1, \dots, n\} : n(i) = j\}$. Note that for some j, I_j may be empty. Clearly $I_0, \dots, I_{n-1}$ is a partition of I. We claim that M is block-triangular with respect to this partition.

Fix a^i_j with $i \in I_r$, $j \in I_s$ and $r < s$ and assume that $a^i_j \neq 0$. Let $w = a^j_{j_1} \dots a^{j_k}_{j_{k+1}}$ be a polynomial word beginning at j with $r(w) = n(j) = s > n(i) = r$ and consider the word $a^i_j w$. If $a^i_j w$ is a polynomial word we obtain $r = n(i) \geq n(j) = s$, which is a contradiction. Therefore there exists t such that $j_t = i$. Then, by hypothesis, $a^i_j a^j_{j_1} \dots a^{j_{t-1}}_{j_t}$ is constant and all non-constant terms in w appear after $a^{j_t}_{j_{t+1}} = a^i_{j_{t+1}}$. This implies that $n(i) \geq n(j)$; again a contradiction. Hence $a^i_j = 0$.

Now consider a^i_j with $i,j \in I_s$ and assume that a^i_j is a non-constant polynomial. Let $w = a^j_{j_1} \dots a^{j_k}_{j_{k+1}}$ be a polynomial word beginning at j with $r(w) = n(j) = n(i) = s$ and consider the word $a^i_j w$. If $a^i_j w$ is a polynomial word we obtain $n(i) > n(j)$, a contradiction. Therefore there exists t such that $j_t = i$. Then, by hypothesis, $a^i_j a^j_{j_1} \dots a^{j_{t-1}}_{j_t}$ is constant. Since a^i_j is a non-constant polynomial we deduce that $a^j_{j_1} \dots a^{j_{t-1}}_{j_t} = 0$, which gives a contradiction. Thus, a^i_j must be constant.

(iv)⇒(v). Assume that (iv) holds, that is, the matrix $|M(x)| = (a^i_j)_{i,j\in I}$ is block-triangular. Let $I_1, \dots, I_k$ be its associated partition. For each $l \in \{1, \dots, k\}$ we choose a total order in I_l and we set $M_l = (a^i_j)_{i,j\in I_l}$. By hypothesis each entry in M_l is an element of $\mathbb{K}$. Denote by p and p_l the characteristic polynomials of $|M|$ and M_l. Then by (iv) we have $p = p_1 \dots . p_k$. Since $p_1, \dots, p_k$ are polynomials with coefficients in $\mathbb{K}$ we obtain the desired result.

(iii)⇒(ii). Assume that M satisfies (iii). We prove (ii) by induction on the length of the elementary products. For $k = 1$ an elementary product is a diagonal minor so it belongs to $\mathbb{K}$. Assume that any elementary product of length less than k belongs to $\mathbb{K}$ and let $a = a^{i_1}_{\sigma(i_1)} a^{i_2}_{\sigma(i_2)} \cdots a^{i_k}_{\sigma(i_k)}$. Note that if σ decomposes into cycles of length less than k then $a \in \mathbb{K}$ by the induction hypothesis. So we assume that σ is a cycle of length k. Reordering indices we can assume that $a = a^1_2 a^2_3 \dots a^k_1$. Consider the diagonal minor D formed by the first k columns and rows. By hypothesis, $\det D \in \mathbb{K}$. On the other hand we have

$$\det D = (-1)^{k+1} \sum_{\sigma \in R_k} a^1_{\sigma(1)} a^{\sigma(1)}_{\sigma^2(1)} \cdots a^{\sigma^{k-1}(1)}_1 + \sum_{\sigma \in S_k \setminus R_k} (-1)^{\varepsilon(\sigma)} a^1_{\sigma(1)} a^2_{\sigma(2)} \cdots a^k_{\sigma(k)}$$

where R_k is the set of cycles of length k. Suppose that $a \notin \mathbb{K}$. Since $\det D$ and the second summand of the above equality belong to $\mathbb{K}$ we conclude that there exists $\sigma \in R_k$ with $\sigma(i) \neq i+1 \pmod k$ such that $b = a^1_{\sigma(1)} a^{\sigma(1)}_{\sigma^2(1)} \cdots a^{\sigma^{k-1}(1)}_1 \notin \mathbb{K}$. We will see that $ab \in \mathbb{K}$, which gives a contradiction. Without loss of generality we suppose that $\sigma(1) = l \neq 2$. Then

$$\begin{aligned} ab &= a^1_2 a^2_3 \dots a^k_1 a^1_l a^l_{\sigma(l)} \cdots a^{\sigma^{k-2}(l)}_1 \\ &= (a^1_l a^l_{l+1} \dots a^k_1)(a^l_{\sigma(l)} a^{\sigma(l)}_{\sigma^2(l)} \cdots a^{\sigma^{k-2}(l)}_1 a^1_2 a^2_3 \dots a^{l-1}_l). \end{aligned}$$

Since $l \neq 2$ the first parenthesis above is an elementary product of length less than k. So by the induction hypothesis it belongs to $\mathbb{K}$. Concerning the second parenthesis note that there exists $i \in \{1, \dots, k-2\}$ such that

$\sigma^i(l) = 2$. Then $a_{\sigma^{i+1}(l)}^{\sigma^i(l)} \dots a_1^{\sigma^{k-2}(l)} a_2^1$ is an elementary product of length less than k and by the induction hypothesis it belongs to $\mathbb{K}$. The remaining term is $a_{\sigma(l)}^l \dots a_2^{\sigma^{i-1}(l)} a_3^2 \dots a_l^{l-1}$, which decomposes into a product of elementary products of length less than k because the superscript 1 does not appear in the expression. So we conclude that $ab \in \mathbb{K}$, which is a contradiction. This ends the proof of the lemma. ∎

REMARK 3.2. Observe that the main property that we have used to prove the above proposition is that if the product of two polynomials is an element of $\mathbb{K}$ then either one of them is zero or both are constant. When we consider a ring R without zero divisors it is also true that if the product of two elements is invertible or zero then either one of them is zero or both are invertible. Therefore the equivalence (ii)⇒(iii)⇒(iv)⇒(ii) remains true if we replace the matrices with polynomial entries by matrices over a domain R and we give a suitable definition of a block-triangular matrix (substituting the role of the elements of $\mathbb{K}$ by the zero or invertible elements of R).

We also need the following lemma which is essentially proved in [8]. Remember that the spectral radius of a square matrix A is defined as $\varrho(A) = \max_{\{\lambda \in \sigma(A)\}} |\lambda|$.

LEMMA 3.3. *Let A be an $n \times n$ real matrix. Then*

$$\varrho(A) \leq \varrho(|A|).$$

Proof. Following [8] we say that A is *reducible* if there exists a permutation of its rows (and the same permutation of its columns) which puts it into the form

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B and D are square matrices. A matrix which is not reducible is called *irreducible*. For irreducible matrices a statement stronger than the one of our lemma is proved in Lemma 2 of Section 2 of [8]. So we just have to prove our lemma for reducible matrices. Observe that $\sigma(A) = \sigma(B) \cup \sigma(D)$. Therefore we can reduce our study to smaller matrices. Each of the new matrices is again either reducible or irreducible. In the second case the proof is finished. In the first case we continue the process until we arrive at an irreducible matrix. ∎

Proof of the Theorem. Let T be such that all the eigenvalues of $|T'(x)|$ have modulus smaller than 1. Proposition 3.1 shows that the matrix $T'(x)$ is block-triangular. Furthermore Lemma 3.3 implies that all its eigenvalues also have modulus smaller than 1. Therefore there exists a basis of $\mathbb{R}^n$ such that the map $T(x)$ can be written as

$$\begin{aligned}(3)\quad T(x) &= (T_1(x_1), T_2(x_1,x_2), \ldots, T_{m-1}(x_1,\ldots,x_{m-1}), T_m(x_1,\ldots,x_m))\\ &= (\Lambda_1 x_1, \Lambda_2 x_2 + t_2(x_1), \Lambda_3 x_3 + t_3(x_1,x_2), \ldots,\\ &\quad \Lambda_{m-1}x_{m-1} + t_{m-1}(x_1,\ldots,x_{m-2}), \Lambda_m x_m + t_m(x_1,\ldots,x_{m-1})),\end{aligned}$$

where $x_i \in \mathbb{R}^{n_i}$, $x = (x_1,\ldots,x_m) \in \mathbb{R}^n$, Λ_i are $n_i \times n_i$ real matrices and t_i are polynomials. Furthermore all the eigenvalues of each Λ_i have modulus smaller than 1. The above expression of $T(x)$ is the key to our proof that the origin is a global attractor. This proof is similar to the one of Theorem A of [5] for the case of $n_i = 1$ for each $i = 1,\ldots,m$.

First note that $\mathbb{R}^n$ can be written as the direct sum of $\mathbb{R}^{n_i}$ for $i = 1,\ldots,m$ and in each $\mathbb{R}^{n_i}$ we can consider a norm $|\ |_i$ such that

$$(4)\qquad |\Lambda_i x_i|_i \le k_i |x_i|_i \qquad \text{with } k_i < k < 1$$

for each $i = 1,\ldots,m$. These norms induce a norm in the whole space that we will denote by $|\ |$.

From now on we fix an $x^{(0)} \in \mathbb{R}^n$ and we will prove that $|x^{(j)}| = |T^j(x^{(0)})|$ tends to zero as j goes to infinity. We argue by induction on the number of components of $T(x)$. In fact we prove by induction the following statement (which of course implies our theorem):

Induction Hypothesis. *There exist $M > 0$ and $0 \le K < 1$ such that for any natural number j, $|x_i^{(j)}| \le MK^j$ for each $i = 1,\ldots,s$.*

For $s = 1$ the proof is trivial by (4). Assume that it is true for $s-1$ and we prove it for s. By the induction hypothesis we know that for all j, for all $i < s$ and for all $t \in [0,1]$ the vectors $(x_1^{(j)}, x_2^{(j)}, \ldots, tx_i^{(j)}, 0, \ldots, 0)$ lie in a compact set L. Consider

$$\begin{aligned}|x_s^{(j)}| &= |T_s(x^{(j-1)})|\\ &= \bigg|\int_0^1 \frac{\partial}{\partial t} T_s(x_1^{(j-1)}, \ldots, tx_s^{(j-1)})\,dt + \int_0^1 \frac{\partial}{\partial t} T_s(x_1^{(j-1)}, \ldots, tx_{s-1}^{(j-1)}, 0)\,dt\\ &\qquad + \ldots + \int_0^1 \frac{\partial}{\partial t} T_s(tx_1^{(j-1)}, 0, \ldots, 0)\,dt\bigg|\\ &\le k|x_s^{(j-1)}| + S\{|x_{s-1}^{(j-1)}| + |x_{s-2}^{(j-1)}| + \ldots + |x_1^{(j-1)}|\}\\ &\le k|x_s^{(j-1)}| + (s-1)SMK^{j-1},\end{aligned}$$

where S is the maximum of the norms of the continuous functions DT_s/Dx_1, $DT_s/Dx_2, \ldots, DT_s/Dx_{s-1}$ over the compact set L. Hence the above expression gives

$$|x_s^{(j)}| \le k|x_s^{(j-1)}| + NK^{j-1},$$

for some constant N and with $0 \le K, k < 1$. From this result (see again [5]) it is easy to prove that there exist M' and $\max(k, K) \le K' < 1$ such that $|x_s^{(j)}| \le M'(K')^j$ for any j. Therefore the theorem is proved. ■

REMARK 3.4. The results of [6] imply that if a polynomial map T from $\mathbb{C}^n$ into itself is such that all the principal diagonal minors of T' are non-zero constants then it is invertible (remember that the principal diagonal minors are the $i \times i$ minors formed by the first i rows and columns for $i = 1, \ldots, n$). From Proposition 3.1 and the expression (3) used in the proof of our main Theorem it is easy to deduce the following related, but different, result: If a polynomial map T from $\mathbb{C}^n$ into itself is such that all the diagonal minors of T' are constants (maybe zero) and $\det(T'(x))$ is a non-zero constant then T is invertible.

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