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Length 2 variables of $A[x,y]$ and transfer

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Abstract. We construct and study length 2 variables of $A[x,y]$ (A is a commutative ring). If A is an integral domain, we determine among these variables those which are tame. If A is a UFD, we prove that these variables are all stably tame. We apply this construction to show that some polynomials of $A[x_1,\dots,x_n]$ are variables using transfer.

1. Introduction. Let A be a commutative ring. The automorphism group $\mathrm{GA}_n(A)$ is anti-isomorphic to $\mathrm{Aut}\,\mathbb{A}^n_A$ and has an obvious geometric significance. A naïve question about automorphisms is: "What do they look like?" This is already a difficult problem for $n=2$ and we focus on this case.

The simplest automorphisms belong to the three subgroups $\mathrm{Af}_2(A)$, $\mathrm{BA}_2(A)$ and $\mathrm{N}_n(A)$ (see Notations 1). When A is a field the situation is well understood thanks to Jung–van der Kulk's theorem (see Theorem 1): all automorphisms are tame, i.e. belong to the subgroup generated by $\mathrm{Af}_2(A)$ and $\mathrm{BA}_2(A)$. But there exist non-tame automorphisms as soon as A is no longer a field.

Since $n=2$, the study of automorphisms boils down to the study of variables (see Corollary 2). When $\mathbb{Q}\subset A$ a general result (see Theorem 2) gives a criterion for a polynomial to be a variable. Without assumptions on A, we define the length of a variable (see Notation 3). Length 1 variables were described by Russell and Sathaye (see Theorem 3).

The study of length 2 variables has begun only recently in the work of Drensky and Yu [DY], but they use characteristic zero techniques; in fact, they suppose $A=K[z]$ with K a field of characteristic zero and in this case the general criterion can be applied. We give a construction of length 2 variables without any assumption on A (see Theorem 4) and we study these variables. If A is an integral domain, we determine among these variables those which are tame (see Proposition 3) using Jung–van der Kulk's

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theorem. If A is a UFD, we prove that these variables are all stably tame (see Theorem 5). This result was obtained with the help of D. Wright and is a strong generalization of M. Smith's result (see [S]). Finally, we apply this construction to show that some polynomials of $A[x_1,\dots,x_n]$ are variables using transfer (see Theorems 7 and 8).

2. Notations and background. Throughout this paper, A and k denote commutative rings, $A^{\times}$ is the set of non-zero divisors of A and $\operatorname{qt} A$ denotes the total quotient ring of A, i.e. $\operatorname{qt} A = (A^{\times})^{-1}A$.

NOTATIONS 1. We use the following classical notations:

- $\mathrm{GA}_n(A) = \operatorname{Aut}_A A[x_1,\dots,x_n]$ is the *automorphism group* of the A-algebra $A[x_1,\dots,x_n]$,
- $\mathrm{Af}_n(A)$ is the *affine automorphism subgroup*,
- $\mathrm{BA}_n(A)$ is the *triangular automorphism subgroup* ($\beta \in \mathrm{GA}_n(A)$ is triangular if $\beta(x_i) = a_i x_i + P_i(x_{i+1},\dots,x_n)$ where a_i are units in A and $P_i \in A[x_{i+1},\dots,x_n]$ for $1 \le i \le n$),
- $\mathrm{N}_n(A)$ is the *subgroup of nilpotency* ($\beta \in \mathrm{N}_n(A)$ if $\beta(x_i) = x_i + P_i$ with P_i nilpotent elements in $A[x_1,\dots,x_n]$)
- $\mathrm{TA}_n(A)$ is the *tame automorphism subgroup* (i.e. the subgroup of $\mathrm{GA}_n(A)$ generated by $\mathrm{Af}_n(A)$, $\mathrm{BA}_n(A)$ and $\mathrm{N}_n(A)$),
- $\mathrm{ST}_n(A)$ is the *stably tame automorphism subgroup* ($\sigma \in \mathrm{GA}_n(A)$ is stably tame if there exists $m \ge 1$ such that $\sigma_m \in \mathrm{TA}_{n+m}(A)$ where $\sigma_m \in \mathrm{GA}_{n+m}(A)$ is defined by $\sigma_m(x_i) = \sigma(x_i)$ if $1 \le i \le n$ and $\sigma_m(x_i) = x_i$ if $n+1 \le i \le n+m$).

If $n \le 3$ we will use the following notations: $x = x_1$, $y = x_2$ and $z = x_3$.

The following result is well known (see for example [N] or [MW]):

THEOREM 1 ([Jung, van der Kulk]). *If A is a field then*

$$\mathrm{GA}_2(A) = \mathrm{TA}_2(A) = \mathrm{Af}_2(A) * \mathrm{BA}_2(A)$$

where $*$ is the amalgamated product of $\mathrm{Af}_2(A)$ and $\mathrm{BA}_2(A)$ along their intersection.

NOTATIONS 2. We introduce new notations:

$$\mathrm{TA}_2^n(A) = \{\sigma \in \mathrm{TA}_2(A) : \sigma = b_n a_n \dots b_1 a_1 b_0,\ b_i \in \mathrm{BA}_2(A),\ a_i \in \mathrm{Af}_2(A)\},$$
$$\mathrm{GA}_2^n(A) = \mathrm{GA}_2(A) \cap \mathrm{TA}_2^n(\operatorname{qt} A).$$

The Jung–van der Kulk theorem has the following useful consequence:

COROLLARY 1. *If A is an integral domain then*

$$\mathrm{GA}_2(A) = \bigcup_{n\in\mathbb{N}} \mathrm{GA}_2^n(A) \quad \text{and} \quad \mathrm{GA}_2^n(A) \cap \mathrm{TA}_2(A) = \mathrm{TA}_2^n(A).$$

Proof. Because A is an integral domain, $\mathrm{qt}\,A$ is a field and we can apply the theorem in $\mathrm{qt}\,A$. The first equality follows from $\mathrm{GA}_2(\mathrm{qt}\,A) = \mathrm{TA}_2(\mathrm{qt}\,A)$. The inclusion $\mathrm{TA}_2^n(A) \subset \mathrm{GA}_2^n(A) \cap \mathrm{TA}_2(A)$ is obvious, the opposite one follows from $\mathrm{TA}_2(A) = \mathrm{Af}_2(A) * \mathrm{BA}_2(A)$ and the two inclusions $\mathrm{Af}_2(A) \setminus \mathrm{BA}_2(A) \subset \mathrm{Af}_2(\mathrm{qt}\,A) \setminus \mathrm{BA}_2(\mathrm{qt}\,A)$ and $\mathrm{BA}_2(A) \setminus \mathrm{Af}_2(A) \subset \mathrm{BA}_2(\mathrm{qt}\,A) \setminus \mathrm{Af}_2(\mathrm{qt}\,A)$.

One can easily describe $\mathrm{GA}_1(A)$ (cf. [N], Proposition 3.1):

PROPOSITION 1. *We have*

$$\mathrm{GA}_1(A) = \{ax + b(x) + c : a \text{ unit in } A,\ b \text{ nilpotent in } A[x],\ c \in A\}.$$

COROLLARY 2. *Let* $\sigma, \sigma' \in \mathrm{GA}_2(A)$. *If* $\sigma(y) = \sigma'(y)$ *then* $\sigma^{-1}\sigma' \in \mathrm{BA}_2(A)\mathrm{N}_2(A)$.

This corollary shows that all the information about appearance and tameness of an automorphism of $A[x, y]$ is contained in one of its two components, i.e. in a variable.

NOTATIONS 3. We define the following subsets of $A[x, y]$:

$$\begin{aligned}
\mathrm{TV}_2(A) &= \{F \in A[x,y] : (\exists \sigma \in \mathrm{TA}_2(A))\sigma(y) = F\},\\
\mathrm{TV}_2^n(A) &= \{F \in A[x,y] : (\exists \sigma \in \mathrm{TA}_2^n(A))\sigma(y) = F\},\\
\mathrm{VA}_2(A) &= \{F \in A[x,y] : (\exists \sigma \in \mathrm{GA}_2(A))\sigma(y) = F\},\\
\mathrm{VA}_2^n(A) &= \{F \in A[x,y] : (\exists \sigma \in \mathrm{GA}_2^n(A))\sigma(y) = F\},\\
\mathrm{SV}_2(A) &= \{F \in A[x,y] : (\exists \sigma \in \mathrm{ST}_2(A))\sigma(y) = F\}.
\end{aligned}$$

DEFINITIONS. An element of $\mathrm{TV}_2(A)$ (resp. $\mathrm{VA}_2(A)$, resp. $\mathrm{SV}_2(A)$) is called a *tame variable* (resp. a *variable*, resp. a *stably tame variable*). A *length n variable* is an element of $\mathrm{VA}_2^n(A) \setminus \mathrm{VA}_2^{n-1}(A)$.

Here is a powerful criterion for proving that some polynomials are variables:

THEOREM 2. *Suppose* $\mathbb{Q} \subset A$ *and let* $F \in A[x, y]$. *Then* $F \in \mathrm{VA}_2(A)$ *if and only if the following two assumptions hold*:

(i) $F \in \mathrm{VA}_2(\mathrm{qt}\,A)$,
(ii) $1 \in (\partial_x F, \partial_y F)$.

This theorem is based on the theory of locally nilpotent derivations developed recently by Daigle and Freudenburg when A is a UFD (see [DF], Proposition 2.3 and Theorem 2.5), Bhatwadekar and Dutta when A is a (normal) noetherian integral domain (see [BD], Theorem 4.7) and Berson, van den Essen and Maubach in the general situation, i.e. when $\mathbb{Q} \subset A$ (see [BEM], Theorem 3.7).

If $\mathbb{Q} \not\subset A$ then Theorem 2 is not true if A is not a field.

If $\mathrm{char}(A) = p > 0$, let $q \in A^\times$ and suppose q is not a unit in A. We consider $F = qx + y + y^p \in \mathrm{VA}_2(\mathrm{qt}\, A)$. Let $\mathcal{M}$ be a maximal ideal such that $q \in \mathcal{M}$. We have $F = y(1 + y^{p-1}) \bmod \mathcal{M}$, hence $\overline{F} \notin \mathrm{VA}_2(A/\mathcal{M})$ and $F \notin \mathrm{VA}_2(A)$. Nevertheless, assumptions (i) and (ii) do hold.

If $\mathrm{char}(A) = 0$, let $q \in A^\times$ and suppose q is an integer and is not a unit in A. We consider $F = qx + y + y^q$ and we conclude as above.

On the other hand, we have (see [R]) the following old result (without the assumption $\mathbb{Q} \subset A$) which describes $\mathrm{VA}_2^1(A)$:

Theorem 3 (Russell, Sathaye). *Let $F \in A[x, y]$. Then $F \in \mathrm{VA}_2^1(A)$ if and only if $F(x, y) = px + \sum g_i y^i$ with $p \in A^\times$ and $\sum g_i y^i \in A[y]$ such that g_1 (resp. g_i for $i \geq 2$) is a unit (resp. are nilpotent) mod pA.*

Remark. If $\mathbb{Q} \subset A$ then Theorem 3 follows from Theorem 2.

Corollary 1 gives:

Proposition 2. *If A is an integral domain then*

$$\mathrm{VA}_2^1(A) \cap \mathrm{TV}_2(A) = \Big\{px + \sum g_i y^i \in \mathrm{VA}_2^1(A) : (\forall i \geq 2)\ g_i = 0 \bmod pA\Big\}.$$

Remark. The Nagata polynomial is $N = z^2x + y + zy^2 \in A[x, y]$ with $A = k[z]$. Theorem 3 and Proposition 2 show that $N \in \mathrm{VA}_2^1(A) \setminus \mathrm{TV}_2(A)$, which is a result due to Nagata (see [N]).

3. Length 2 variables of $A[x, y]$

Notation. We denote by $\mathrm{LV}_2^2(A)$ the the set of polynomials $y + H(px + G(y))$ with $p \in A^\times$, $G(y) = \sum g_i y^i \in A[y]$ and $H(y) = \sum h_i y^i \in A[y]$ such that h_i is nilpotent mod pA for $i \geq 1$.

Theorem 4. *We have the following inclusion*: $\mathrm{LV}_2^2(A) \subset \mathrm{VA}_2^2(A)$.

Proof. Let $p \in A^\times$, $G(y) = \sum g_i y^i \in A[y]$ and $H(y) = \sum h_i y^i \in A[y]$ such that h_i $(i \geq 1)$ is nilpotent mod pA.

Lemma. *Let $\mathcal{H}$ be the ideal of $A[y]$ generated by h_i $(i \geq 1)$. Define the family $(Q_n)_{n\in\mathbb{N}}$ in $A[y]$ by $Q_0(y) = 0$ and $Q_{n+1}(y) = G(y - H(Q_n(y)))$ for $n \in \mathbb{N}$. Then*:

(1) $(\forall n \in \mathbb{N})$ $Q_n(y + H(G(y))) = G(y) \bmod \mathcal{H}^n$,
(2) $(\forall n \in \mathbb{N})$ $Q_{n+1}(y) = Q_n(y) \bmod \mathcal{H}^n$.

Proof of the lemma. We prove (1) and (2) by induction. For $n = 0$ this is trivial (indeed, $\mathcal{H}^0 = A$). Let $n \in \mathbb{N} \setminus \{0\}$ and suppose (1) and (2) hold at step $n - 1$. Then modulo $\mathcal{H}^n$:

$$Q_n(y + H(G(y))) = G(y + H(G(y)) - H(Q_{n-1}(y + H(G(y))))) = G(y)$$

and

$$Q_{n+1}(y) = G(y - H(Q_n(y))) = G(y - H(Q_{n-1}(y))) = Q_n(y).$$

We return to the proof of Theorem 4. Let l be such that $\mathcal{H}^l \subset pA$ and let $Q(y) = Q_l(y)$. If $F = y + H(px + G(y))$ (resp. $F_1 = y - H(px + Q(y))$) then by (1) (resp. (2)) of the lemma $Q(F) = G(y) \bmod pA[x,y]$ (resp. $G(F_1) = Q(y) \bmod pA[x,y]$), i.e. there exists $K \in A[x,y]$ (resp. $K_1 \in A[x,y]$) such that $Q(F) = G(y) - pK(x,y)$ (resp. $G(F_1) = Q(y) - pK_1(x,y)$). We define $\sigma = (x + K, F)$ and $\sigma_1 = (x + K_1, F_1)$. We have $\sigma\sigma_1(y) = F_1(x + K, F) = F - H(px + pK + Q(F) = y$ and $\sigma\sigma_1(x) = x + K + K_1(x + K, F)$. Since $p(K + K_1(x + K, F)) = Q(F) - G + Q(F) - G(F_1(x + K, F)) = 0$ we have $K + K_1(x + K, F) = 0$ because $p \in A^\times$, hence $\sigma\sigma_1(x) = x$. A similar computation shows that $\sigma_1\sigma(y) = y$ and $\sigma_1\sigma(x) = x$, hence $\sigma \in \mathrm{GA}_2^2(A)$ and $F \in \mathrm{VA}_2^2(A)$.

Remark. If $\mathbb{Q} \subset A$ then Theorem 4 follows from Theorem 2. The first part of Theorem 3.10 in [DY] follows from Theorem 4 and also from Theorem 2.

Remark. The lemma gives an algorithm to compute $Q_l(y)$ and find a $\sigma \in \mathrm{GA}_2^2(A)$ such that $\sigma(y) = y + H(px + G(y))$ (i.e. find $\sigma(x)$) but we do not know any general formula giving $Q_l(y)$ in terms of p, G and H. However, here is an example of such a formula:

Example. Let $p \in A^\times$, and $G(y) = ay^j$, $H(y) = ry^k$ with $j, k \geq 1$, $r \in A$ nilpotent mod pA and $a \in A$. We can compute the sequence defined in the lemma:

$$(\forall n \in \mathbb{N})\ Q_n(y) = ay^j \sum_{i=0}^{n-1} c_{i,j,k}(-ra^k y^{jk-1})^i \bmod r^n A[y]$$

where the coefficients $c_{i,j,k}$ are defined by the formula $C(T) = \sum_{i=0}^{\infty} c_{i,j,k} T^i$ where $C(T) \in A[[T]]$ is defined by $C(T) = (1 + TC(T)^k)^j$.

In fact, by induction, modulo $r^{n+1}A[y]$:

$$\begin{aligned} Q_{n+1}(y) &= G(y - H(Q_n(y))) \\ &= a\Big(y - r\Big[ay^j \sum_{i=0}^{n-1} c_{i,j,k}(-ra^k y^{jk-1})^i\Big]^k\Big)^j \\ &= ay^j\Big(1 - ra^k y^{jk-1}\Big[\sum_{i=0}^{n-1} c_{i,j,k}(-ra^k y^{jk-1})^i\Big]^k\Big)^j \\ &= ay^j \sum_{i=0}^{n} c_{i,j,k}(-ra^k y^{jk-1})^i. \end{aligned}$$

So we can take one of the following automorphisms (if $b \in A$ and $b = 0$ mod pA then $p^{-1}b$ means one of $c \in A$ such that $b = pc$):

$$\begin{cases} \sigma(x) = x + p^{-1}a\Big(y^j - \sigma(y)^j \sum_{i=1}^{l-1} c_{i,j,k}(-ra^k\sigma(y)^{jk-1})^i\Big), \\ \sigma(y) = y + r(px + ay^j)^k. \end{cases}$$

Example. With $A = k[z,t]$, $p = z$, $G(y) = ty$ and $H(y) = zy^2$, we have the following automorphism σ of $k[z,t][x,y]$:

$$\begin{cases} \sigma(x) = x - t(zx + ty)^2, \\ \sigma(y) = y + z(zx + ty)^2. \end{cases}$$

This is Popov's second automorphism [P], a variant of which is used by van den Essen and Hubbers in [EH] as a counterexample to several conjectures.

Example. With $A = k[z]$, $p = z^2$, $G(y) = y^2$ and $H(y) = zy^2$, we have the following automorphism σ of $k[z][x,y]$:

$$\begin{cases} \sigma(x) = x + z^{-2}(y^2 - \sigma(y)^2 - 2z\sigma(y)^5), \\ \sigma(y) = y + z(z^2x + y^2)^2. \end{cases}$$

Questions. One can ask the following questions (the answers should depend on A):

1. Let $\mathrm{U}(A) = \{(x, ay) : a \text{ a unit in } A\}$. Do we have $\mathrm{VA}_2^2(A) \setminus \mathrm{VA}_2^1(A) \subset U(A)\mathrm{LV}_2^2$?

2. Does there exist an integer n such that $\mathrm{GA}_2(A)$ is generated by $\mathrm{Af}_2(A)$, $\mathrm{GA}_2^n(A)$ and $\mathrm{N}_2(A)$?

Remark. The lemma of Theorem 5 is a partial answer to Question 1.

4. Tameness properties.

Corollary 1 gives:

Proposition 3. *If A is an integral domain, let $F = y + H(px + G(y)) \in \mathrm{LV}_2^2(A)$. Then $F \in \mathrm{TV}_2(A)$ if and only if the following three conditions are satisfied*:

(1) $g_i = 0 \bmod pA$ *for all* $i \geq 2$,
(2) *there exists* $s \in A$ *such that* $pA + g_1A = sA$,
(3) $sk_i = 0 \bmod pA$ *for all* $i \geq 2$ *where* $H(sy + g_0) = \sum k_i y^i$.

Remark. The "moreover" part of Theorem 3.10 in [DY] follows from Proposition 3.

Theorem 5. *If A is a UFD, then* $\mathrm{LV}_2^2(A) \subset \mathrm{SV}_2(A)$.

Lemma. *Suppose A is a UFD. Let $p \in A^\times$, $G(y) = \sum g_i y^i \in A[y]$ and $H(y) = \sum h_i y^i \in A[y]$. If $\gcd(p, g_i : i \geq 1)$ is a unit and $y + H(px + G(y)) \in \mathrm{VA}_2^2(A)$ then for all $i \geq 1$, h_i is nilpotent mod pA.*

Proof of the lemma. We write $p = \prod p_j^{n_j}$ with p_j irreducible. For all j, $y + \overline{H}(\overline{G}(y))$ is a variable in $(A/p_jA)[y]$. Since $\deg \overline{G} \geq 1$, we have $\deg \overline{H} \leq 1$, i.e. $h_i = 0 \bmod p_jA$ for all $i \geq 1$. Hence for all $i \geq 1$ we have $h_i = 0 \bmod \prod p_jA$, which implies h_i is nilpotent mod pA.

Proof of Theorem 5 (this argument follows an idea due to D. Wright ([1])). Let $F = y + H(px + G(y)) \in \mathrm{VA}_2^2(A)$; we prove that F is stably tame by induction on the number of irreducible factors of p. If p is a unit then F is tame. Suppose p is not a unit. Let $s = \gcd\{p, g_i : i \geq 1\}$.

(1) If s is not a unit then we change p to $s^{-1}p$, $G(y)$ to $s^{-1}(G(y) - g_0)$ and $H(y)$ to $H(sy + g_0)$. This does not change F and the number of irreducible factors of p decreases.

(2) If s is a unit, by the lemma $r = \gcd\{p, h_i : i \geq 1\}$ is not a unit. Let $\sigma \in \mathrm{GA}_2^2(A)$ be such that $\sigma(y) = F$ and let $z = x_3$ be a new indeterminate. We extend σ to σ_1 by $\sigma_1(x) = \sigma(x)$, $\sigma_1(y) = \sigma(y)$ and $\sigma_1(z) = z$. We write $H(y) = rH_1(y)$ (one can suppose $h_0 = 0$) with $H_1 \in A[y]$. We consider $\alpha = (x, z, y + rz), \beta = (z, y, x), \gamma = (y, -ry + z, x) \in \mathrm{Af}_3(A)$ and $\tau = (x - H_1(G(y) + pz), y, z) \in \mathrm{BA}_3(A)$. (Here $\varrho = (f, g, h)$ means $\varrho(x) = f$, $\varrho(y) = g$, $\varrho(z) = h$.) Let $\sigma_2 = \gamma\tau\beta\sigma_1\alpha$. We have $\sigma_2(z) = z$ and $\sigma_2(y) = y - H_1(px + G(-ry + z))$. Let $G^*(y) = \sum g_i^* y^i = G(w - ry) \in A[w][y]$; now r divides $s^* = \gcd\{p, g_i^* : i \geq 1\}$ and we can reduce the number of irreducible factors of p as in case (1).

Proposition 4. *Let $F \in \mathrm{VA}_2^1(A)$. Then there exists $\alpha \in \mathrm{Af}_2(A)$ such that $\alpha(F) \in \mathrm{LV}_2^2$.*

Proof. Let $F = px + G(y) \in \mathrm{VA}_2^1(A)$ with $G(y) = \sum g_i y^i$. Let $a, b \in A$ be such that $ag_1 - pb = 1$. We consider $\alpha = (g_1x - by, -px + ay) \in \mathrm{Af}_2(A)$; then $\alpha(F) \in \mathrm{LV}_2^2$.

Corollary 3. *If A is a UFD, then* $\mathrm{VA}_2^1(A) \subset \mathrm{TA}_2(A)$.

Example. Let $\varepsilon \in \mathbb{C}$ be such that $\varepsilon^2 + 3 = 0$. By Theorem 3 we have $2(1+\varepsilon)x + y + 2y^2 + (1+\varepsilon)y^3 \in \mathrm{VA}_2^1(\mathbb{Z}[\varepsilon])$, and the proof of Theorem 5 cannot be applied to this variable because the terms $2y^2$ and $(1 + \varepsilon)y^3$ cannot be cancelled simultaneously. So we do not know whether this variable is stably tame or not.

Remark. Corollary 3 implies that the Nagata polynomial is stably tame, which is a result due to M. Smith [S].

Definition. If $\mathbb{Q} \subset A$, we say that $\sigma \in \mathrm{GA}_2(A)$ is a *Smith automorphism* if there exists a triangular derivation D of $A[x, y]$ and $W \in \ker D$ such that $\sigma = \exp(WD)$; we say that $F \in A[x, y]$ is a *Smith variable* if there exists a Smith automorphism σ such that $\sigma(y) = F$.

([1]) Private communication.

Remark. M. Smith's result [S] asserts that a Smith automorphism is stably tame.

Theorem 6. *If* A *is a UFD and* $\mathbb{Q} \subset A$, *let* $F = y + H(px + G(y)) \in \mathrm{LV}_2^2(A)$ *and suppose* $H \notin A$. *Then* F *is a Smith variable if and only if* $H(G(y))G'(y) = 0 \bmod pA[y]$.

Proof. We write $p = \prod_{i\in I} p_i^{l_i}$ (p_i irreducible in A) and for $i \in I$ we denote by v_i the p_i-adic valuation in $A[y]$. We have the following equivalence ($K \in A[y]$):

$$(*) \qquad K = 0 \bmod pA[y] \Leftrightarrow (\forall i \in I)\ v_i(K) \ge l_i.$$

(1) Suppose that $y+H(px+G(y))$ is a Smith variable. Then there exists a triangular derivation D of $A[x,y]$ and $W \in \ker D$ such that $\exp(WD)(y) = y + H(px + G(y))$. We write $D = B(y)D_x + bD_y$ with $B \in A[y]$ and $b \in A$ and we obtain $bW = H(px + G(y))$. We have $v_i(b) \le v_i(H(px + G(y)))$ for $i \in I$ and $W = b^{-1}H(px + G(y))$. Hence $0 = D(W) = B(y)b^{-1}pH'(px + G(y))+G'(y)H'(px+G(y))$ and $B(y)p+bG'(y) = 0$ because $H \notin A$. Finally, $v_i(p) \le v_i(b) + v_i(G'(y)) \le v_i(px + H(G(y))G'(y))$ for $i \in I$ and we obtain $H(G(y))G'(y) = H(px + G(y))G'(y) = 0 \bmod pA[y]$ using $(*)$.

(2) Let $h = \gcd(H(G(y)), p)$ and suppose that $H(G(y))G'(y) = 0 \bmod pA[y]$. Then $-p^{-1}hG'(y) \in A[y]$. Let $D = -p^{-1}hG'(y)D_x + hD_y$, which is a triangular derivation of $A[x,y]$, and let $W = h^{-1}H(px + G(y))$. We have $D(W) = -G'(y)H'(px + G(y)) + G'(y)H'(px + G(y)) = 0$, i.e. $W \in \ker(D)$.

We have $(WD)(y) = H(px + G(y))$ and $(WD)^j(y) = 0$ for $j \ge 2$, hence $\exp(WD)(y) = y + H(px + G(y))$ and $y + H(px + G(y))$ is a Smith variable.

Example. If k is a UFD and $\mathbb{Q} \subset k$, then $y + z(zx + ty)^2$ is a Smith variable in $A = k[z,t]$ and $y+z(z^2x+y^2)^2$ is not a Smith variable in $A = k[z]$.

Proposition 5. *If* A *is a UFD and* $\mathbb{Q} \subset A$ *then all Smith variables are variables in* $\mathrm{LV}_2^2(A)$.

Proof. Let D be a triangular derivation of $A[x,y]$ and $W \in \ker D$. We write $D = B(y)D_x + bD_y$ with $B \in A[y]$ and $b \in A$. Further, we denote by $\int_0 B(y)$ the unique polynomial $C \in A[y]$ such that $C'(y) = B(y)$ and $C(0) = 0$. We choose $p = b$, $G(y) = \int_0 B(y)$ and we remark that $bW(b^{-1}x - b^{-1}\int_0 B(y), y)$ is in $A[x]$ because $D_y(bW(b^{-1}x - b^{-1}\int_0 B(y), y)) = 0$ since $DW = 0$, hence we can choose $H(x) = bW(b^{-1}x - b^{-1}\int_0 B(y), y)$. Now we have $\exp(WD)(y) = y + H(px + G(y)) \in \mathrm{LV}_2^2(A)$.

Remark. Proposition 5 shows that Theorem 5 generalizes M. Smith's criterion.

5. Transfer. Theorems 3 and 4 can be used to find variables of $\mathrm{VA}_n(A)$ ($n \ge 2$) by induction on n.

NOTATIONS. Let Y be a non-empty set of indeterminates, y be an element of Y and $Z = Y \setminus \{y\}$.

THEOREM 7. *Let $p \in A^\times$, let a be a unit mod pA, and let $G, F \in A[Y]$. If all the coefficients of G are nilpotent mod pA and if F is a variable in $A[Y]$ then $aF(Y) + G(Y) + px$ is a variable in $A[x, Y]$.*

Proof. Let $\alpha \in \mathrm{Aut}_A A[Y]$ such that $\alpha(y) = F(Y)$. By Theorem 3, there exists $\sigma \in \mathrm{Aut}_{A[Z]} A[Z][x, y]$ such that $\sigma(y) = ay + \alpha^{-1}G(Y) + px$. We extend α and σ to automorphisms of $A[x, Y]$ by $\alpha(x) = x$ and $\sigma(z) = z$ for all $z \in Z$. Then $\alpha\sigma(y) = \alpha(ay + \alpha^{-1}G(Y) + px) = aF(Y) + G(Y) + px$.

COROLLARY 4 (generalised Choudary–Dimca's hypersurfaces). *Let $m \in \mathbb{N} \setminus \{0\}$. Let $p_i \in A[z_i]$ and $G_i \in A[x_0, \ldots, x_{i-1}, z_1, \ldots, z_i]$ for $1 \le i \le m$. Let $r_i \in A[z_i]$ be nilpotent mod $p_iA[z_i]$ ($1 \le i \le m$). Then $x_0 + \sum_{i=1}^{m} (r_iG_i + x_ip_i)$ is a variable in $A[x_0, \ldots, x_m, z_1, \ldots, z_m]$.*

Proof. We proceed by induction on m. Theorem 3 applied to the ring $A[z_1]$ implies that $x_0 + r_1G_1 + p_1x_1$ is a variable in $A[x_0, x_1, z_1]$ ($x_1 = x$, $x_0 = y$).

If $x_0 + \sum_{i=1}^{m} (r_iG_i + x_ip_i)$ is a variable in $A[x_0, \ldots, x_m, z_1, \ldots, z_m]$ then Theorem 7 applied to the ring $A[z_{m+1}]$ with $Y = \{x_0, \ldots, x_m, z_1, \ldots, z_m\}$ implies that $x_0 + \sum_{i=1}^{m+1} (r_iG_i + x_ip_i)$ is a variable in $A[x_0, \ldots, x_{m+1}, z_1, \ldots, z_{m+1}]$.

REMARK. In the case $A = \mathbb{C}$, $p_i = z_i^{d-1}$, $G_i = x_{i-1}^{d-1}$ and $r_i = z_i$ for $1 \le i \le m$, these are Choudary–Dimca's hypersurfaces (cf. [CD]) and this corollary gives an answer to Question 1 in [CD].

THEOREM 8. *Let $p \in A^\times$, let $H(y) = \sum h_iy^i \in A[y]$, and let $G, F \in A[Y]$. If h_i ($i \ge 1$) is nilpotent mod pA and if F is a variable in $A[Y]$ then $F(Y) + H(px + G(Y))$ is a variable in $A[x, Y]$.*

Proof. Let $\alpha \in \mathrm{Aut}_A A[Y]$ be such that $\alpha(y) = F(Y)$. By Theorem 4, there exists $\sigma \in \mathrm{Aut}_{A[Z]} A[Z][x, y]$ such that $\sigma(y) = y + H(px + \alpha^{-1}G(Y))$. We extend α and σ to automorphisms of $A[x, Y]$ by $\alpha(x) = x$ and $\sigma(z) = z$ for all $z \in Z$. Then $\alpha\sigma(y) = \alpha(y + H(px + \alpha^{-1}G(Y))) = F(Y) + H(px + G(Y))$.

EXAMPLE. Using Theorems 7 and 8 one can prove that the following polynomials of $A[x, y, z, u, v]$ are variables:

$$(u-1)(y + z(z^2x + yz + y^3)^4 + u(u+1)x^2 + u^3(u+1)^2v,$$
$$(z-1)y + z(z+1)y^2 + z^3(z+1)^2x + u(u^3v + y^3)^4,$$
$$y + z(z^2x + y^2)^2 + u(uv + xyz)^3.$$

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