A class of counterexamples to the Cancellation Problem for arbitrary rings

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Abstract. We present a class of counterexamples to the Cancellation Problem over arbitrary commutative rings, using non-free stably free modules and locally nilpotent derivations.

1. Introduction. The Cancellation Problem for algebraic varieties asks the following question.

Problem 1 (Cancellation Problem, geometric formulation). Let \( V \) be an algebraic variety over a field \( k \) and \( n \in \mathbb{N}^* \). Does \( V \times k \cong k^n \) imply that \( V \cong k^{n-1} \)?

This question can be reformulated as follows.

Problem 2 (Cancellation Problem, algebraic formulation). Let \( B \) be an affine domain over a field \( k \) and \( n \in \mathbb{N} \). Assume that \( B[T] \cong_k k[X_1, \ldots, X_n] \). Does it then follow that \( B \cong_k k[X_1, \ldots, X_{n-1}] \)?

See also the paper by Kraft ([Kra89]) for background on these and other cancellation problems in algebraic geometry. This paper considers this question not for a field \( k \), but for an arbitrary commutative ring \( A \).

Problem 3 (General Cancellation Problem). Let \( A \) be a commutative ring, \( B \) an \( A \)-domain, and \( n \in \mathbb{N} \). Assume that \( B[T] \cong_A A[X_1, \ldots, X_n] \). Does it then follow that \( B \cong_A A[X_1, \ldots, X_{n-1}] \)?

This paper shows how to construct a whole class of counterexamples to this problem.

The construction in Section 3 has two ingredients. On the one hand, it uses the existence of commutative rings \( A \) with a unimodular row \((a_1, \ldots, a_n)\) over \( A \) that cannot be completed to an invertible square matrix. In other words, it uses the existence of commutative rings \( A \) for which

2000 Mathematics Subject Classification: 14R10, 13B25.

Key words and phrases: cancellation problem, locally nilpotent derivations.
there exists a stably free module of type 1 that is not free. This was in fact also a basic ingredient in a paper by Hochster ([Hoc72]) to construct a counterexample to the Biregular Cancellation Problem. He considered the ring $\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ and the unimodular row $(x, y, z)$. On the other hand, our construction uses the notion of locally nilpotent derivations. Section 2 contains a brief overview of the required facts about these derivations.

2. Derivations. Let $k$ be a field of characteristic zero and let $A$ be a commutative $k$-algebra. A $k$-derivation on $A$ is a $k$-linear map $D : A \to A$ satisfying the Leibniz rule, $D(ab) = a(Db) + (Da)b$ for all $a, b \in A$. It is said to be locally nilpotent if for all $a \in A$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$. The kernel of such a derivation $D$ is denoted by $A^D$. A slice of $D$ is an element $s \in A$ such that $D(s) = 1$.

If $D$ is locally nilpotent and $t \in A$, then we can define a map $\phi_t : A \to A$ by $\phi_t(a) := \sum_{i=0}^{\infty} (1/i!)(D^i(a)t^i)$. If $D$ also has a slice $s$, this map can be used to easily describe the kernel of $D$.

**Proposition 4** ([Ess93], Proposition 2.1). Let $D$ be a locally nilpotent derivation on a finitely generated commutative $k$-algebra $A = k[a_1, \ldots, a_n]$. Assume that $D$ has a slice $s \in A$. Then

$$A^D = \phi_{-s}(A) = k[\phi_{-s}(a_1), \ldots, \phi_{-s}(a_n)].$$

**Proposition 5** ([Wri81], Proposition 2.1). Let $D$ be a locally nilpotent derivation on a commutative $k$-algebra $A$ and assume that $D$ has a slice $s \in A$. Then

1. $A = A^D[s]$;
2. $s$ is algebraically independent over $A$ [and therefore $A = A^D[s]$ is a polynomial ring in one variable over $A^D$];
3. $D = d/ds$.

**Remark 6.** Note that if, in the above situation, $A$ is a domain and $\text{trdeg}_k Q(A)$ is finite, it follows that $\text{trdeg}_k Q(A^D) = \text{trdeg}_k Q(A) - 1$. In particular, if $A$ is of the form $A = B[X_1, \ldots, X_n]$ for some domain $B$ whose quotient field is of finite transcendence degree over $k$ and $A^D = B[F_1, \ldots, F_{n-1}]$ for certain polynomials $F_1, \ldots, F_{n-1} \in A$, then the $F_i$ are algebraically independent over $k$.

For more information on locally nilpotent derivations see, for instance, [Ess00], Chapter 1, or [Now94].

3. Counterexamples. Let $k$ be a field of characteristic zero and let $A$ be a finitely generated commutative $k$-algebra without zero divisors. Let $(a_1, \ldots, a_n)$ be a unimodular row over $A$, say $b_1, \ldots, b_n \in A$ with $b_1a_1 +$
... + b_n a_n = 1, and assume that it cannot be completed to an invertible square matrix.


$$D := b_1 \frac{\partial}{\partial X_1} + \ldots + b_n \frac{\partial}{\partial X_n}.$$ 

This derivation is locally nilpotent and has a slice, namely $s := a_1 X_1 + \ldots + a_n X_n$. Letting $B := A[X]^D$ be the kernel of the derivation, we deduce from Proposition 5 that $A[X] = B[s]$, a polynomial ring over $B$ in $s$, and from Proposition 4 that $B = A[X_1 - b_1 s, \ldots, X_n - b_n s]$.

NOTATION 7. For $F \in A[X]$ and $j \in \mathbb{N}$ we denote by $F_{(j)}$ the homogeneous part of $F$ of degree $j$. So $F = F_{(0)} + F_{(1)} + \ldots + F_{(d)}$ where $d := \deg F$.

Lemma 8. Let $F_1, \ldots, F_{n-1} \in A[X]$ and assume $B = A[F_1, \ldots, F_{n-1}]$. Take $f_i := F_{i(1)}$ (i.e. the linear part of $F_i$). Then $B = A[f_1, \ldots, f_{n-1}]$.

Proof. “$\subseteq$” We may assume, without loss of generality, that the polynomials $F_1, \ldots, F_{n-1}$ do not have a constant term. Now consider $X_i - b_i s \in B = A[F_1, \ldots, F_{n-1}]$. Then there is a polynomial $p(T_1, \ldots, T_{n-1}) \in A[T_1, \ldots, T_{n-1}]$ such that $X_i - b_i s = p(F_1, \ldots, F_{n-1})$. Then

$$X_i - b_i s = (p(F_1, \ldots, F_{n-1}))_{(1)} \quad \text{(because $X_i - b_i s$ is linear)}$$

$$= (p_{(1)}(F_1, \ldots, F_{n-1}))_{(1)} \quad \text{(because $F_1, \ldots, F_{n-1}$ have no constant term)}$$

$$= p_{(1)}(F_{1(1)}, \ldots, F_{n-1(1)}) \quad \text{(because $p_{(1)}$ is linear)}$$

$$= p_{(1)}(f_1, \ldots, f_{n-1}) \in A[f_1, \ldots, f_{n-1}].$$

“$\supseteq$” Because $F_i \in B = A[X]^D$, every homogeneous part $F_{i(j)}$ of $F_i$ is also in $B$. In particular, $f_i \in B$. 

Lemma 9. Let $f_1, \ldots, f_m \in A[X]$ be linear polynomials. Then

$$A[f_1, \ldots, f_m] \cap AX_1 \oplus \ldots \oplus AX_n = Af_1 + \ldots + Af_m$$

[i.e. every polynomial expression $p(f_1, \ldots, f_m)$ in the $f_i$ which is linear in the $X_i$ is in fact an $A$-linear combination of the $f_i$].

Proof. “$\subseteq$” Take $p(T_1, \ldots, T_m) \in A[T_1, \ldots, T_m]$ and let $g := p(f_1, \ldots, f_m)$ be a polynomial expression in the $f_i$. Assume that $g$ is in fact linear in the $X_i$. Then, using essentially the same argument as in the proof of the previous lemma, we get
\[ g = (p(f_1, \ldots, f_m))(1) = (p(1)(f_1, \ldots, f_m))(1) \\
= p(1)(f_1, \ldots, f_m) \in Af_1 + \ldots + Af_n. \]

“\(\supseteq\)” This is obvious. \(\blacksquare\)

**Lemma 10.** Let \(f_1, \ldots, f_{n-1} \in A[X]\) be linear polynomials and assume that \(B = A[f_1, \ldots, f_{n-1}].\) Then

\[ As \oplus Af_1 \oplus \ldots \oplus Af_{n-1} = AX_1 \oplus \ldots \oplus AX_n \]

[i.e. every linear polynomial in \(A[X]\) can be written in a unique way as an \(A\)-linear combination of \(s, f_1, \ldots, f_{n-1}\).]

**Proof.** We first show that \(As + Af_1 + \ldots + Af_{n-1} = AX_1 \oplus \ldots \oplus AX_n.\)

“\(\subseteq\)” This is obvious.

“\(\supseteq\)” Take \(g \in AX_1 \oplus \ldots \oplus AX_n.\) Then \(Dg \in A\) and therefore we have

\[ D(g - (Dg)s) = Dg - (D^2g)s - (Dg)(Ds) = Dg - Dg = 0. \]

so \(g - (Dg)s \in B \cap AX_1 \oplus \ldots \oplus AX_n = A[f_1, \ldots, f_{n-1}] \cap AX_1 \oplus \ldots \oplus AX_n = Af_1 + \ldots + Af_{n-1}\) (by Lemma 9)

and hence \(g \in As + Af_1 + \ldots + Af_{n-1}.\)

To see that \(As + Af_1 + \ldots + Af_{n-1}\) is in fact a direct sum, take \(\mu, \lambda_1, \ldots, \lambda_{n-1} \in A\) and assume that

\[ \mu s + \lambda_1 f_1 + \ldots + \lambda_{n-1} f_{n-1} = 0. \]

Applying \(D\) to both sides yields \(\mu = 0, \) so \(\lambda_1 f_1 + \ldots + \lambda_{n-1} f_{n-1} = 0.\) The \(f_i,\) however, are even algebraically independent (by Remark 6) and therefore \(\lambda_1 = \ldots = \lambda_{n-1} = 0.\) \(\blacksquare\)

**Theorem 11.** We have \(B \not\cong_A A[X_1, \ldots, X_{n-1}],\) even though \(B[s] = A[X_1, \ldots, X_{n-1}][X_n].\)

**Proof.** Assume that \(B \cong_A A[X_1, \ldots, X_{n-1}].\) Then \(B = A[F_1, \ldots, F_{n-1}]\) for certain polynomials \(F_1, \ldots, F_{n-1} \in A[X]\) and by Lemma 8 even \(B = A[f_1, \ldots, f_{n-1}]\) for certain linear polynomials \(f_1, \ldots, f_{n-1} \in A[X].\) Now Lemma 10 implies that

\[ As \oplus Af_1 \oplus \ldots \oplus Af_{n-1} = AX_1 \oplus \ldots \oplus AX_n, \]

say \(f_i = \lambda_{i1} X_1 + \ldots + \lambda_{in} X_n\) (and \(s = a_1 X_1 + \ldots + a_n X_n\)). This is an equality between free \(A\)-modules of rank \(n\) and the base transformation matrix is

\[
\begin{pmatrix}
  a_1 & \ldots & a_n \\
  \lambda_{11} & \ldots & \lambda_{1n} \\
  \vdots & \ddots & \vdots \\
  \lambda_{n-1,1} & \ldots & \lambda_{n-1,n}
\end{pmatrix}.
\]

This is an invertible matrix and hence the unimodular row \((a_1, \ldots, a_n)\) has been completed to an invertible matrix, which contradicts the assumption. \(\blacksquare\)
So, every coordinate ring $A$ of an affine variety that has a unimodular row that cannot be completed to an invertible matrix, gives rise to a counterexample to the General Cancellation Problem.

Over the real numbers, we recover Hochster’s example mentioned in the Introduction. Over the complex numbers, one can consider the “generic” example $A = \mathbb{C}[a, b, c, x, y, z]/(ax + by + cz - 1)$. The unimodular row $(\bar{x}, \bar{y}, \bar{z})$ cannot be completed to an invertible square matrix. This was shown by Raynaud [Ray68] using homological methods and, in a more general setting, by Suslin [Sus82] (Theorem 2.8) using $K$-theory.

Acknowledgments. We would like to thank Wilberd van der Kallen for pointing out the references to the results of Raynaud and Suslin.

References


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Reçu par la Rédaction le 25.2.2000