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Recent progress on Hilbert's Fourteenth Problem via triangular derivations

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Abstract. We give an overview of recent results concerning kernels of triangular derivations of polynomial rings. In particular, we examine the question of finite generation in dimensions 4, 5, 6, and 7.

1. Introduction. The subject of this paper is the following special case of Hilbert's Fourteenth Problem.

Let k be a field of characteristic 0, *and let* $B = k[x_1, \ldots, x_n]$, *the polynomial ring in n variables over k. If $D : B \to B$ is a k-derivation*, *is the kernel of D always finitely generated over k?*

In other words, do we get an analogue of the Hilbert Basis Theorem when we consider derivations of B in place of ring homomorphisms? The importance of this special case to Hilbert's original formulation is seen in the following result of Nowicki [7].

Theorem 1. *If a connected algebraic group G acts algebraically on k^n, there exists a k-derivation δ of B such that* $\ker \delta = B^G$.

For example, Nagata's famous first counterexample to Hilbert Fourteen [6] was realized by Derksen [3] as the kernel of a derivation of $k[x_1, \ldots, x_{32}]$.

The answer to the question above is known in the following cases:

- Yes if $n \leq 3$ (Zariski, 1954 [9]).
- No if $n \geq 7$ (Roberts, 1990 [8]; van den Essen and Janssen, 1995 [4]).
- No if $n = 6$ (Freudenburg, 1998 [5]).
- No if $n = 5$ (Daigle and Freudenburg, 1999 [1]).
- Yes if $n = 4$ and D is triangular (Daigle and Freudenburg, 1999, preprint).

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So only the case $n = 4$, D non-triangular remains open. Here, *triangular* means $DX_i \in k[X_1, \ldots, X_{i-1}]$ for $i \geq 2$, and $DX_1 \in k$. This paper will provide an overview of the recent work done in dimensions 7, 6, 5, and 4.

2. Dimensions 5, 6, and 7. A family of counterexamples to Hilbert's Fourteenth Problem was published by Roberts [8] in 1990. The importance of these examples lay not only in lowering the dimension of known counterexamples (to $n = 7$), but also in providing counterexamples which were relatively simple to describe.

Given $m \geq 2$, define on the polynomial ring $B = k[X, Y, Z, S, T, U, V]$ the triangular derivation

$$\mathcal{D} = (X^{m+1})\frac{\partial}{\partial S} + (Y^{m+1})\frac{\partial}{\partial T} + (Z^{m+1})\frac{\partial}{\partial U} + (XYZ)^m\frac{\partial}{\partial V}.$$

Roberts proved that the kernel $\ker \mathcal{D}$ is not finitely generated. In his proof, he shows that there exists a sequence $\lambda_n \in \ker \mathcal{D}$ of the form $\lambda_n = XV^n +$ (lower-degree V-terms). By using the fact that $\mathcal{D}$ and the λ_n are homogeneous with respect to a certain grading of B, he succeeds in showing that no finitely generated subring of $\ker \mathcal{D}$ can contain all λ_n.

In an effort to generalize Roberts' method, we gave in [1] the following criterion for non-finite generation of kernels.

LEMMA 1. *Let $K = \bigoplus_{i \in \mathbb{N}} K_i$ be a graded k-domain such that $K_0 = k$, and let δ be a homogeneous locally nilpotent k-derivation of K* [1]. *For $\alpha \in \ker \delta$ which is not in the image of δ, let $\widetilde{\delta}$ be the extension of δ to $K[T]$ (T a variable) defined by $\widetilde{\delta}T = \alpha$. Suppose ϕ_n is a sequence of non-zero elements of $\ker \widetilde{\delta}$ having leading T-coefficients $b_n \in K$. If $\deg b_n$ is bounded, but $\deg_T \phi_n$ is not bounded, then $\ker \widetilde{\delta}$ is not finitely generated over k.*

This criterion can be used to show the existence of counterexamples in dimensions 5 and 6, as follows.

Let $R = k[a, b, s, t, u]$, a polynomial ring in 5 variables over k, and define a triangular derivation

$$\Delta = a\frac{\partial}{\partial s} + bs\frac{\partial}{\partial t} + bt\frac{\partial}{\partial u}.$$

Define a sequence $t_n \in R$ by

$$t_1 = a, \quad t_2 = b, \quad t_3 = ab, \quad \text{and} \quad t_n = t_{n-3} \quad \text{for } n \geq 4.$$

The central result of [5] is:

THEOREM 2. *There exist $w_n \in R$ ($n \geq 0$) such that $w_0 = 1$, $w_1 = s$, and $\Delta w_n = t_n \cdot w_{n-1}$ for all $n \geq 1$.*

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[1] Recall that *locally nilpotent* means that, for each $f \in K$, $\delta^s f = 0$ for $s \gg 0$. In particular, triangular implies locally nilpotent in the case of polynomial rings.

To obtain a counterexample in dimension 6, let x and y be integral elements over R such that $x^3 = a$ and $y^3 = b$, and let v be transcendental over R. Then $B := R[x, y, v] = k[x, y, s, t, u, v]$ is a polynomial ring in 6 variables over k. If D is the triangular derivation on B defined by

$$D = (x^3)\frac{\partial}{\partial s} + (y^3 s)\frac{\partial}{\partial t} + (y^3 t)\frac{\partial}{\partial u} + (x^2 y^2)\frac{\partial}{\partial v},$$

then $D|_R = \Delta$. Therefore, $Dw_n = t_n \cdot w_{n-1}$. The function

$$\pi(f) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}(D^i f)\frac{v^i}{(Dv)^i}$$

defines a homomorphism from B to $(\ker D)_{Dv}$, the localization of $\ker D$ at Dv. It can be shown using Theorem 2 that $\pi(xw_{3n})$ is a polynomial for all n. Direct calculation shows that $\pi(xw_{3n}) = c_n x v^{3n}$ + (lower-degree v-terms), where $c_n \in k^*$.

Now the lemma above may be applied: If $K = k[x, y, s, t, u]$, then $D|_K$ is homogeneous with respect to the grading on $K = \bigoplus_{i \geq 0} K_i$ defined by $k = K_0$, $x, y \in K_1$, $s \in K_3$, $t \in K_6$, and $u \in K_9$. Moreover, $Dv = x^2 y^2$ does not lie in the image of $D|_K$. Using the sequence $\phi_n := \pi(xw_{3n})$, we deduce from the lemma that $\ker D$ is not finitely generated.

To obtain a counterexample in dimension 5, we now simply set $y = 1$ in the above example. More precisely, if $\overline{B} = B \bmod (y-1)$ and $\overline{D} = D \bmod (y-1)$, then $\overline{B} = k[x, s, t, u, v]$, a polynomial ring in 5 variables, and $\overline{D}$ is the triangular derivation

$$\overline{D} = x^3\frac{\partial}{\partial s} + s\frac{\partial}{\partial t} + t\frac{\partial}{\partial u} + x^2\frac{\partial}{\partial v}.$$

We see that, if $\overline{K} = k[x, s, t, u]$, then $\overline{D}|_{\overline{K}}$ is homogeneous with respect to the grading on $\overline{K}$ for which $\deg x = 1$ and $\deg s = \deg t = \deg u = 3$. The elements $\overline{\phi}_n$ of $\ker \overline{D}$ are of the form $c_n x v^{3n}$+(lower-degree v-terms). Since $\overline{D}(v) = x^2$ is not in the image of $\overline{D}|_{\overline{K}}$, we conclude by the lemma above that $\ker \overline{D}$ is not finitely generated.

Note that this last example can be simplified by changing coordinates in $k[x, s, t, u, v]$. If σ fixes x, t, u, and v, and maps s to $s + xv$, then

$$\sigma\overline{D}\sigma^{-1} = x^2\frac{\partial}{\partial v} + (xv + s)\frac{\partial}{\partial t} + t\frac{\partial}{\partial u}.$$

The dimension 6 and dimension 5 counterexamples may thus be summarized as follows.

THEOREM 3 (see [5]). *Let $B = k[x, y, s, t, u, v]$ be the polynomial ring in 6 variables over k, and let D be the triangular derivation on B defined by*

$$D = (x^3)\frac{\partial}{\partial s} + (y^3 s)\frac{\partial}{\partial t} + (y^3 t)\frac{\partial}{\partial u} + (x^2 y^2)\frac{\partial}{\partial v}.$$

Then the kernel of D is not finitely generated as a k-algebra.

Theorem 4 (see [1]). *Let* $A = k[a,b,x,y,z]$ *be the polynomial ring in* 5 *variables over* k, *and let* d *be the triangular derivation on* A *defined by*

$$d = a^2\frac{\partial}{\partial x} + (ax+b)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$$

Then the kernel of d *is not finitely generated as a* k*-algebra.*

3. Dimension 4. A natural question to ask next is: If $B = k[W,X,Y,Z]$ is the polynomial ring in four variables over k, and if T is a *triangular* k-derivation of B, is $\ker T$ finitely generated over k? To answer this question, the following result was proved very recently by the author and Daigle, using a recent result of Sathaye (both manuscripts are in preparation).

Theorem 5. *Let* k *be an algebraically closed field of characteristic zero, and let* R *be a* k*-affine Dedekind domain or a localization of such a ring. The kernel of any triangular* R*-derivation of* $R[X,Y,Z]$ *is finitely generated as an* R*-algebra.*

This result easily implies a positive answer to our question when k is algebraically closed: Since T is triangular, we may assume, with no loss of generality, that $TW = 0$. Thus, T is a triangular R-derivation of $R[X,Y,Z]$, where $R = k[W]$, and the theorem implies that $\ker T$ is finitely generated.

Finite generation notwithstanding, $\ker T$ may be very complicated. In [2] we construct, for each integer $n \geq 3$, a triangular derivation of $k[W,X,Y,Z]$ whose kernel cannot be generated by fewer than n elements. The actual construction is a bit complicated, and the reader should see the article for details.

Finally, the reader should note that Theorem 5 fails for more general rings R. For example, if $R = k[a,b]$, a polynomial ring in two variables over k, then the derivation d of Theorem 4 is a triangular R-derivation of $R[x,y,z]$ with non-finitely generated kernel.

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