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The Real Jacobian Conjecture for polynomials of degree 3

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Abstract. We show that every local polynomial diffeomorphism (f, g) of the real plane such that $\deg f \le 3$, $\deg g \le 3$ is a global diffeomorphism.

1. Introduction. In [4] Pinchuk presented a polynomial mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that F is not a global diffeomorphism although $\mathrm{Jac}(F) > 0$ everywhere in $\mathbb{R}^2$. Components of Pinchuk's mapping have degrees 10 and 35. It is an interesting question what is the lowest degree of a polynomial map in an example like this. In this note we prove that it should be at least 4.

THEOREM 1. *Every polynomial mapping* $(f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ *with a positive Jacobian such that* $\deg f \le 3$, $\deg g \le 3$ *is a global diffeomorphism.*

Recall that $\mathrm{Jac}(f, g)$ is given by $\mathrm{Jac}(f, g) = f'_x g'_y - f'_y g'_x$. The condition $\mathrm{Jac}(f, g) > 0$ guarantees that (f, g) is a local diffeomorphism. For the proof of our main result we need a sequence of lemmas.

2. Lemmas

LEMMA 1. *Let* $(f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ *be a polynomial mapping with a positive Jacobian. If for all* $t \in \mathbb{R}$ *the level sets* $\{f = t\}$ *are connected then* (f, g) *is a global diffeomorphism.*

Proof. Every injective polynomial mapping from $\mathbb{R}^2$ to itself is bijective (see [1], [5]). Therefore it suffices to show that (f, g) is an injection.

Suppose to the contrary that $(f, g)(p_1) = (f, g)(p_2) = (t, s)$ for $p_1 \ne p_2$. Let T be a segment of a curve $\{f = t\}$ joining points p_1 and p_2. Take another point $p_3 \in T$ such that $g(p_3) = \max_{p \in T} g(p)$ or $g(p_3) = \min_{p \in T} g(p)$. From Lagrange's multipliers method it follows that the derivatives $df(p_3)$ and

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$dg(p_3)$ are linearly dependent. Hence $\mathrm{Jac}(f,g)(p_3)=0$. This contradiction finishes the proof. ∎

LEMMA 2. *Let $(f,g):\mathbb{R}^2\to\mathbb{R}^2$ be a polynomial mapping with a positive Jacobian. Let f^+, g^+ denote the leading homogeneous forms of the polynomials f, g. If $(f^+,g^+)^{-1}(0)=\{(0,0)\}$ then (f,g) is a diffeomorphism.*

Proof. Under the above assumptions the mapping (f,g) is proper and hence is a diffeomorphism. For the details we refer the reader to [2], Proposition 2.1. ∎

Let $f\in\mathbb{C}[x,y]$ be a polynomial with a finite number of critical points. Set $\mu(f)=\sum_{p\in\mathbb{C}^2}(f'_x,f'_y)_p$ where $(f,g)_p$ denotes the intersection multiplicity of polynomials f and g at a point p.

LEMMA 3. *Let $f\in\mathbb{R}[x,y]$ be a polynomial with a finite number of complex critical points. If f has no real critical point then $\mu(f)$ is even.*

Proof. The sum $\sum_{p\in\mathbb{C}^2}(f'_x,f'_y)_p$ extends over all solutions of the system $f'_x=f'_y=0$. Since both partial derivatives have real coefficients, together with any complex solution p the system has the conjugate complex solution $\overline{p}$. From the definition of intersection multiplicity (see [6]) it follows that $(f'_x,f'_y)_p=(f'_x,f'_y)_{\overline{p}}$. Therefore it suffices to count the terms of the above sum in pairs to get the lemma. ∎

In order to state subsequent lemmas we need a few notations. Let $f=\sum_{\alpha\in\mathbb{N}^2}a_\alpha x^{\alpha_1}y^{\alpha_2}$ be a polynomial. We call the set

$$\Delta_f=\mathrm{conv}(\{\alpha\in\mathbb{N}^2:a_\alpha\neq 0\})$$

the *Newton polygon* of f. Here $\mathrm{conv}(A)$ denotes the convex hull of a set A. For a compact subset Δ of $\mathbb{R}^2$ and $\xi\in\mathbb{R}^2$ we define $l(\Delta,\xi)=\max_{\alpha\in\Delta}\langle\xi,\alpha\rangle$ and $\Delta^\xi=\{\,\alpha\in\Delta:\langle\xi,\alpha\rangle=l(\Delta,\xi)\,\}$. We call the polynomial $f^\xi=\sum_{\alpha\in\Delta_f^\xi}a_\alpha x^{\alpha_1}y^{\alpha_2}$ the *leading part* of f with respect to ξ. It is a quasi-homogeneous polynomial of weight $\mathrm{w}(f)=l(\Delta_f,\xi)$ provided that $\mathrm{w}(x)=\xi_1$ and $\mathrm{w}(y)=\xi_2$.

To shorten notation we write $f>0$ if $f(x,y)$ is positive for all $(x,y)\in\mathbb{R}^2$.

LEMMA 4. *Let $f\in\mathbb{R}[x,y]$, $f>0$. Then $f^\xi\geq 0$ for every $\xi\in\mathbb{R}^2$.*

Proof. Fix $(x,y)\in\mathbb{R}^2$. Expanding the polynomial $f(t^{\xi_1}x,t^{\xi_2}y)$ with respect to powers of t we get $f(t^{\xi_1}x,t^{\xi_2}y)=f^\xi(x,y)t^{l(\Delta_f,\xi)}+$ terms of lower degrees. For large t the sign of the right-hand side is determined by the sign of $f^\xi(x,y)$. Hence $f^\xi(x,y)\geq 0$. ∎

COROLLARY 1. *If $f\in\mathbb{R}[x,y]$ is everywhere positive, then the polygon Δ_f has vertices at points with even coordinates only.*

LEMMA 5. *Let* $(f,g):\mathbb{R}^2\to\mathbb{R}^2$ *be a polynomial mapping. Assume that* $\xi\in\mathbb{R}^2$ *is such that* $\Delta_f^\xi=\{\alpha\}$, $\Delta_g^\xi=\{\beta\}$ *and* α, β *are linearly independent. Then for* $J=\mathrm{Jac}(f,g)$ *we have* $\Delta_J^\xi=\{\alpha+\beta-(1,1)\}$.

Proof. Lemma 5 is a consequence of the property $J^\xi=\mathrm{Jac}(f^\xi,g^\xi)$ *provided that* $\mathrm{Jac}(f^\xi,g^\xi)\neq 0$. Indeed, under our assumptions f^ξ and g^ξ are monomials $a_\alpha x^{\alpha_1}y^{\alpha_2}$, $b_\beta x^{\beta_1}y^{\beta_2}$ and we see that $\mathrm{Jac}(f^\xi,g^\xi)=a_\alpha b_\beta(\alpha_1\beta_2-\alpha_2\beta_1)x^{\alpha_1+\beta_1-1}y^{\alpha_2+\beta_2-1}$ is nonzero. ∎

COROLLARY 2. *Under the assumptions of Lemma* 5, *if* $\alpha+\beta$ *has an even coordinate then the polynomial* $\mathrm{Jac}(f,g)$ *changes sign.*

Now we formulate Kouchnirenko's theorem (see [3]) in the form suitable for our purposes.

THEOREM 2. *Let* $f\in\mathbb{C}[x,y]$ *be a polynomial such that* $a=\deg f(x,0)>0$, $b=\deg f(0,y)>0$ *and* $f(0,0)\neq 0$. *If edges of the Newton polygon* Δ_f *intersect the lattlice* $\mathbb{N}^2$ *at vertices only then* $\mu(f)=2\,\mathrm{Area}(\Delta_f)-a-b+1$.

EXAMPLE. Let the polynomial f have the Newton diagram C_3 (see Figure 1). Then f satisfies the assumptions of Kouchnirenko's theorem. We have $\mathrm{Area}(\Delta_f)=3$, $a=2$, $b=2$ and consequently $\mu(f)=2\,\mathrm{Area}(\Delta_f)-a-b+1=3$. Similarly, if $\Delta_f=C_4$ then $\mu(f)=2(5/2)-1-2+1=3$.

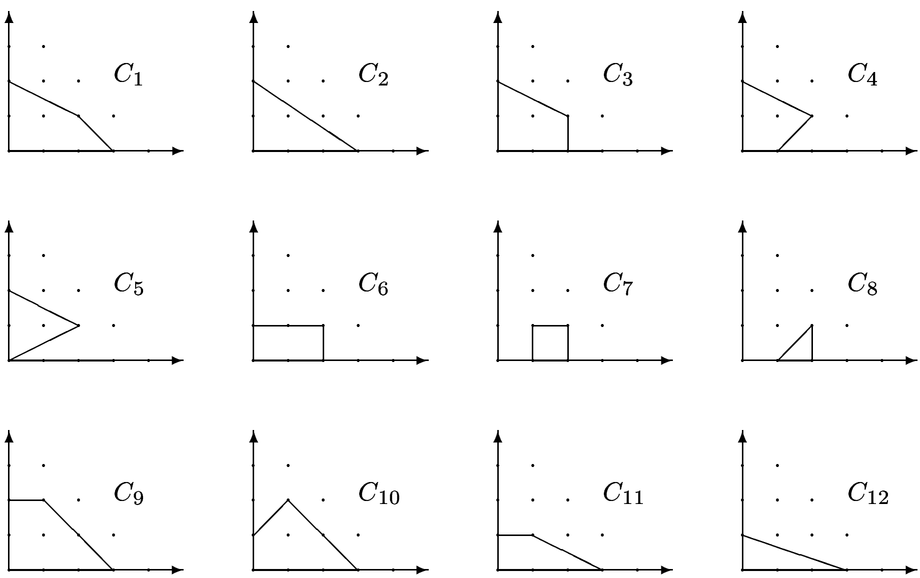

Fig. 1

3. Proof of Theorem 1. In the course of the proof we will often replace a mapping (f,g) by $(\widetilde{f},\widetilde{g})=L_1\circ(f,g)\circ L_2$ where $L_1,L_2:\mathbb{R}^2\to\mathbb{R}^2$ are affine orientation preserving automorphisms. Clearly, Theorem 1 holds for (f,g) iff it does for $(\widetilde{f},\widetilde{g})$. In the proof we construct a number of Newton polygons. They are each drawn in a separate figure.

First consider the special case when one of the polynomials f, g has degree 1 or 2. If $\deg f=1$ then all level sets $\{f=t\}$ are connected and the theorem follows by Lemma 1. If $\deg f=2$ then a suitable affine change of coordinates reduces f to $f=y-x^2$ or $f=ax^2+by^2+c$. In the first case all level sets $\{f=t\}$ are connected and so by Lemma 1, (f,g) is a diffeomorphism. In the second case $(0,0)$ is a critical point of f, which is impossible because $\operatorname{Jac}(f,g)>0$.

From now on we assume that $\deg f=\deg g=3$. If the leading homogeneous parts of the polynomials f and g do not have a common nonzero root then by Lemma 2, (f,g) is a diffeomorphism. If they have, then without loss of generality we can assume that their common root is $(0,1)$ (we apply a linear change of coordinates). Then we have $f^+=axy^2+bx^2y+cx^3$ and $g^+=Axy^2+Bx^2y+Cx^3$. Moreover, we may assume that $a=0$. Indeed, for $A\neq 0$ we can replace f by $\widetilde{f}=f-(a/A)g$ and for $A=0$ we change the roles of f and g. This gives $\Delta_f\subset C_1$ and $\Delta_g\subset C_9$.

Two cases can occur: (1) $(2,1)\in\Delta_f$, and (2) $(2,1)\notin\Delta_f$.

3.1. *Analysis of case* (1). We have $f^+=bx^2y+cx^3=bx^2(y+(c/b)x)$, $b\neq 0$. A linear substitution $y=\widetilde{y}-(c/b)x$ gives $f^+=bx^2\widetilde{y}^2$. Hence we may assume that $(3,0)\notin\Delta_f$ and so $\Delta_f\subset C_3$. By Kouchnirenko's theorem if $\Delta_f=C_3$ or $\Delta_f=C_4$ then $\mu(f)=3$ (see the Example at the end of the previous section). By Lemma 3 in both cases a polynomial f has a real critical point. For $\Delta_f=C_5$ direct easy computations show that f has a real critical point. All these cases are excluded. Hence $\Delta_f\subset C_6$.

Let us write $f=f_1(x)y+f_2(x)$. If $f_1(x)$ has a constant sign then the level sets $\{f=t\}$ are connected, because they have equations $y=(t-f_2(x))/f_1(x)$, and by Lemma 1, (f,g) is a diffeomorphism. If there is x_0 such that $f_1(x_0)=0$ then without the loss of generality we may replace (f,g) with $(f(x+x_0,y)-f(x_0,0),g(x+x_0,y))$. This reduces the Newton polygon of f to $\Delta_f\subset C_7$. If $(1,1)\in\Delta_f$ then an easy computation shows that f has a critical point, which is impossible. Therefore $\Delta_f\subset C_8$.

Consider the Newton polygon Δ_g. We have $\Delta_g\subset C_9$. Suppose that $(0,2)\in\Delta_g$. Then by Corollary 2 applied to $(0,2)\in\Delta_g$ and $(2,1)\in\Delta_f$ the Jacobian $\operatorname{Jac}(f,g)$ would change sign. Thus $(0,2)\notin\Delta_g$ and $\Delta_g\subset C_{10}$. Note that $(0,1)\in\Delta_g$ because otherwise $\operatorname{Jac}(f,g)(0,0)=0$. Fix a direction $\xi=(-1,1)$ and put $J=\operatorname{Jac}(f,g)$. We can write $f^\xi=ax+bx^2y$, $a\neq 0$, $b\neq 0$ and $g^\xi=Ay+Bxy^2$, $A\neq 0$. Hence $J^\xi=\operatorname{Jac}(f^\xi,g^\xi)=3bB(xy)^2+2(Ab+aB)xy+aA$. Since a discriminant $(2(Ab+aB))^2-4(3bB)(aA)$ can be written as a sum of squares $3(Ab-aB)^2+(Ab+aB)^2$ it is a positive number. Therefore J^ξ changes its sign and so does J by Lemma 4. This contradiction finishes the proof of case (1).

3.2. *Analysis of case* (2). We have $\Delta_f \subset C_2$ and $\Delta_g \subset C_9$. By Corollary 2 it is impossible that both $(0,2) \in \Delta_f$ and $(1,2) \in \Delta_g$.

Assume first that $(1,2) \notin \Delta_g$. Then $\Delta_g \subset C_1$. If $(2,1) \in \Delta_g$ then we can put $\widetilde{f} = g$, $\widetilde{g} = -f$ and the proof for a pair $(\widetilde{f}, \widetilde{g})$ has already been given. If $(2,1) \notin \Delta_g$ then $\Delta_g \subset C_2$. Comparing the Newton polygons of f and g we see that there is a constant c such that the polynomial $\widetilde{g} = g - cf$ is of degree 1 or 2. We have checked this case at the beginning of the proof.

Assume now that $(0,2) \notin \Delta_f$. Then $\Delta_f \subset C_{11}$. If $(1,1) \in \Delta_f$ then it is easily seen that f has a critical point, which is impossible. Hence $\Delta_f \subset C_{12}$.

If $(0,1) \in \Delta_f$ then all level sets $\{f = t\}$ are connected and by Lemma 1 the map (f,g) is a diffeomorphism.

If $(0,1) \notin \Delta_f$ then the polynomial f depends on the variable x only. Moreover, f has no critical point and consequently the level sets of f are (single) straight lines. Hence by Lemma 1, (f,g) is a diffeomorphism. ∎

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