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A characterization of proper regular mappings

by T. Krasiński and S. Spodzieja (Łódź)

Abstract. Let X, Y be complex affine varieties and $f : X \to Y$ a regular mapping. We prove that if $\dim X \geq 2$ and f is closed in the Zariski topology then f is proper in the classical topology.

1. Introduction. Let X, Y be complex affine varieties (i.e. irreducible algebraic subsets of complex linear spaces) and $f : X \to Y$ a regular mapping. From the Constructibility Theorem of Chevalley ([Ł$_2$], VII.8.3, [M], Proposition 2.31) it easily follows that if f is proper in the classical topology then f is closed in the Zariski topology (i.e. for any algebraic subset V of X the image $f(V)$ is an algebraic subset of Y). In this paper we prove that the converse is true provided $\dim X \geq 2$ (cf. [RS] for polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^k$).

Theorem 1.1. *Let X, Y be complex affine varieties*, $\dim X \geq 2$ *and $f : X \to Y$ a non-constant regular mapping. If f is closed in the Zariski topology then f is proper in the classical topology.*

From this theorem and well known facts we obtain the following characterization of finite mappings for affine varieties:

Corollary. *Let X, Y be complex affine varieties*, $\dim X \geq 2$ *and $f : X \to Y$ a non-constant regular mapping. Then the following conditions are equivalent*:

(i) *f is finite*, *i.e. $\mathbb{C}[X]$ is integral over $f^*(\mathbb{C}[Y])$*,
(ii) *f is proper in the classical topology*,
(iii) *f is closed in the classical topology*,
(iv) *f is closed in the Zariski topology.*

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Proof. (i)⇒(ii) is a well known fact (see e.g. [B], Satz 11.22 applied to $f : X \to f(X)$).

(ii)⇒(iii) again is a well known topological fact.

(iii)⇒(iv) follows from the Constructibility Theorem of Chevalley (see e.g. [B], Korollar 11.25).

(iv)⇒(ii) is Theorem 1.1.

(ii)⇒(i) is an easy fact for affine varieties.

The assumption in Theorem 1.1 that $\dim X$ is greater than 1 is essential, because for $X := \{(x,y) \in \mathbb{C}^2 : xy^2 + y + 1 = 0\}$, $Y := \mathbb{C}$ and $f : X \to Y$, $f(x,y) := x$ we see that f is closed in the Zariski topology and f is not proper.

The proof of Theorem 1.1 will be carried out in Section 5. In fact, we will prove a little stronger version of it. Namely, we will only assume that f maps algebraic curves onto algebraic sets. The crucial role in the proof will be played by the Łojasiewicz exponent at infinity of regular mappings on algebraic sets (Section 3). The key fact is a theorem on selection of an algebraic curve on which the Łojasiewicz exponent at infinity is attained (Theorem 3.5).

In what follows we shall use two topologies in complex linear spaces: the classical topology and the Zariski topology. To avoid confusion we agree that, unless otherwise specified, all topological notions will refer to the classical topology.

After our announcing the result, Z. Jelonek and independently J. Kollár communicated to us a new proof of Theorem 1.1. Its main idea is given at the end of the paper.

2. Meromorphic mappings at infinity. Let $D(r) = \{t \in \mathbb{C} : |t| > r\}$, $r > 0$, be the exterior of a closed disc in $\mathbb{C}$. A holomorphic mapping $\psi = (\psi_1, \ldots, \psi_n) : D(r) \to \mathbb{C}^n$ is called *meromorphic at infinity* if each ψ_j, $j = 1, \ldots, n$, is meromorphic at infinity. One can write any such mapping $\psi \neq 0$ in the form

$$(1) \qquad \psi(t) = a_d t^d + a_{d-1} t^{d-1} + \ldots, \qquad a_i \in \mathbb{C}^n,\ i \le d,\ a_d \neq 0,\ d \in \mathbb{Z}.$$

Then d is called the *degree* of ψ and denoted by $\deg \psi$. Additionally we put $\deg 0 := -\infty$.

Similarly, for a meromorphic mapping $\psi = (\psi_1, \ldots, \psi_n)$ at a point $a \in \mathbb{C}$ we define the *order* of ψ at a:

$$\operatorname{ord}_a \psi = \min\{\operatorname{ord}_a \psi_1, \ldots, \operatorname{ord}_a \psi_n\}.$$

Now, we give a version at infinity of the theorem on continuity of roots. First, we recall two known lemmas.

Lemma 2.1 (Puiseux Theorem at Infinity, [A], [CK$_1$]). *Let*

$$Q(t,y) = y^m + q_1(t)y^{m-1} + \ldots + q_m(t)$$

be a polynomial in y with coefficients $q_i : D(r) \to \mathbb{C}$ meromorphic at infinity, $i = 1,\ldots,m$. Then there exist $N \in \mathbb{N}$ and functions λ_i, $i = 1,\ldots,m$, meromorphic at infinity such that

$$Q(t^N, y) = \prod_{i=1}^{m} (y - \lambda_i(t)).$$

Moreover, it suffices to take $N = m!$.

Lemma 2.2 (Theorem on Continuity of Roots, [Ł$_1$], Sect. 16, Lemma 1). *Let $C > 1$, $\delta > 0$. If $|c_i| \le C$ and $|\widetilde{c}_i - c_i| < \delta$ for $i = 1,\ldots,m$ and $\xi \in \mathbb{C}$ satisfy the equation*

$$\xi^m + c_1\xi^{m-1} + \ldots + c_m = 0,$$

then there exists $\widetilde{\xi} \in \mathbb{C}$ satisfying

$$\widetilde{\xi}^m + \widetilde{c}_1\widetilde{\xi}^{m-1} + \ldots + \widetilde{c}_m = 0 \quad \textit{and} \quad |\widetilde{\xi} - \xi| < 3C\delta^{1/m}.$$

Proposition 2.3 (Theorem on Continuity of Roots at Infinity). *Let*

$$P(t,y) = y^m + p_1(t)y^{m-1} + \ldots + p_m(t),$$
$$Q(t,y) = y^m + q_1(t)y^{m-1} + \ldots + q_m(t)$$

be polynomials in y with coefficients $p_i, q_i : D(r) \to \mathbb{C}$ meromorphic at infinity, $i = 1,\ldots,m$, and let $L \in \mathbb{Z}$, $L \ge \max_{i=1,\ldots,m} \deg p_i$. By Lemma 2.1 there exists $N \in \mathbb{N}$ such that

$$P(t^N, y) = \prod_{i=1}^{m} (y - \varphi_i(t)), \qquad Q(t^N, y) = \prod_{j=1}^{m} (y - \psi_j(t)),$$

where $\varphi_i, \psi_j : D(r') \to \mathbb{C}$, $r' \ge r$, are meromorphic at infinity. If for some $K \in \mathbb{Z}$,

$$\deg(p_i - q_i) \le K,$$

then for any φ_i there exists ψ_j such that

$$\deg(\varphi_i - \psi_j) \le (L + K/m)N.$$

Proof. By assumptions there exist $C > 0$, $r'' \ge r'$ such that for $|t| > r''$,

$$|p_i(t^N) - q_i(t^N)| \le C|t|^{KN}, \qquad i = 1,\ldots,m,$$

and

$$|p_i(t^N)| \le C|t|^{LN}.$$

By Lemma 2.2, for any $|t| > r''$ and each φ_i,

$$\min_j |\varphi_i(t) - \psi_j(t)| \le 3C^{1+1/m}|t|^{(L+K/m)N}.$$

Then for each φ_i there exists ψ_j such that

$$\deg(\varphi_i - \psi_j) \le (L + K/m)N.$$

PROPOSITION 2.4. *Let $f : \mathbb{C}^n \to \mathbb{C}^k$ be a non-constant polynomial mapping and $\varphi : D(r) \to \mathbb{C}^n$ be a mapping meromorphic at infinity*, $\deg\varphi > 0$. *Then for any $L \in \mathbb{Z}$ and any mapping $\psi : D(r) \to \mathbb{C}^n$ meromorphic at infinity such that*

$$\deg(\varphi - \psi) < L,$$

we have

$$\deg(f \circ \varphi - f \circ \psi) < L + (\deg f - 1)\deg\varphi.$$

Proof. It suffices to prove this theorem in the case of a polynomial function $f : \mathbb{C}^n \to \mathbb{C}$. Let $L \in \mathbb{Z}$ and $\deg(\varphi - \psi) < L$. Let $\varphi = (\varphi_1, \ldots, \varphi_n)$, $d := \deg\varphi$ and

$$\varphi_j(t) = a_d^j t^d + a_{d-1}^j t^{d-1} + \ldots, \qquad a_i^j \in \mathbb{C},\ j = 1, \ldots, n.$$

Let further

$$f \circ \varphi(t) = b_m t^m + b_{m-1} t^{m-1} + \ldots, \qquad b_s \in \mathbb{C},\ m \in \mathbb{Z}.$$

It is easy to see that the coefficients b_s depend only on the coefficients a_i^j up to the index $i := s - (\deg f - 1)\deg\varphi$ (cf. [RS], proof of Proposition 1). Hence

$$\deg(f \circ \varphi - f \circ \psi) < L + (\deg f - 1)\deg\varphi.$$

3. The Łojasiewicz exponent of regular mappings. Let $X \subset \mathbb{C}^M$ be an algebraic set and let $f : X \to \mathbb{C}^k$ be a regular mapping. Let $S \subset X$ be an unbounded set. We define

$$N(f|S) := \{\nu \in \mathbb{R} : \exists_{C>0,\, R>0}(z \in S \wedge |z| > R) \ \Rightarrow\ |f(z)| \ge C|z|^\nu\},$$

where $|\cdot|$ denotes the policylindric norm. The *Łojasiewicz exponent at infinity* $\mathcal{L}_\infty(f|S)$ of f on S is defined by

$$\mathcal{L}_\infty(f|S) := \sup N(f|S).$$

If $N(f|S) = \emptyset$, we put $\mathcal{L}_\infty(f|S) = -\infty$. If $S = X$, we write $N(f)$ and $\mathcal{L}_\infty(f)$.

Now, we shall consider the problem of finding algebraic curves on which the Łojasiewicz exponent at infinity is attained.

Let $\mathcal{U} \subset \mathbb{C}^n$ be a neighbourhood of infinity (i.e. $\mathcal{U} = \mathbb{C}^n \setminus H$, where $H \subset \mathbb{C}^n$ is a compact set). Let $\Gamma \subset \mathcal{U}$ be an analytic set of dimension 1. If there exists a mapping meromorphic at infinity ψ of the form (1), holomorphic on $\overline{D(r)}$, such that $\deg\psi > 0$ and $\Gamma = \psi(D(r))$, then Γ is called an *analytic curve meromorphic at infinity*, and ψ its *description*.

From the definition we immediately obtain

LEMMA 3.1. *Let $X \subset \mathbb{C}^M$ be an algebraic set, $f : X \to \mathbb{C}^k$ be a regular mapping, $\#f^{-1}(0) < \infty$, and $\Gamma \subset X$ be an analytic curve meromorphic at infinity. Then*

$$\mathcal{L}_\infty(f|\Gamma) = \frac{\deg(f \circ \psi)}{\deg \psi} \in N(f|\Gamma),$$

where ψ is a description of Γ.

PROPOSITION 3.2. *Let X be a complex affine variety, $\dim X > 0$ and let $f : X \to \mathbb{C}^k$ be a regular mapping such that $\#f^{-1}(0) < \infty$. Then for any algebraic subset $W \subset X$, $\dim W < \dim X$, there exists an analytic curve $\Gamma \subset X$ meromorphic at infinity such that $\Gamma \cap W = \emptyset$ and*

$$\mathcal{L}_\infty(f) = \mathcal{L}_\infty(f|\Gamma).$$

In particular, $\mathcal{L}_\infty(f) \in N(f)$.

Proof (cf. Proof of Proposition 1 in [CK$_2$]). By the Tarski–Seidenberg Theorem (cf. [BR], Remark 3.8) the set

$$B := \{x \in X \setminus W : |f(x)| \le 2 \min_{|z|=|x|,\, z\in X} |f(z)|\}$$

is semi-algebraic. Since X is an irreducible algebraic set, and $\dim W < \dim X$, it follows that $X \setminus W$ is dense in X. Hence, since $\#f^{-1}(0) < \infty$, we easily deduce that B is an unbounded set. By a version of the Curve Selection Lemma (cf. [NZ], Lemma 2), we see that there exists a real-analytic curve $\kappa : (r, \infty) \to B$, $r > 0$, of the form

$$\kappa(t) = \alpha_d t^d + \alpha_{d-1} t^{d-1} + \ldots, \qquad \alpha_i \in \mathbb{C}^N,\ \alpha_d \ne 0,\ d > 0.$$

Since $\lim_{t\to\infty} |\kappa(t)| = \infty$, the set $\Gamma' := \kappa(r, \infty)$ is unbounded. We have

$$\mathcal{L}_\infty(f) = \mathcal{L}_\infty(f|\Gamma'). \tag{2}$$

In fact, the inequality $\mathcal{L}_\infty(f) \le \mathcal{L}_\infty(f|\Gamma')$ is obvious. To prove the opposite inequality, take $R > 0$, $C > 0$ such that for any $z \in \Gamma'$,

$$|z| > R \ \Rightarrow\ |f(z)| \ge C|z|^{\mathcal{L}_\infty(f|\Gamma')}.$$

We may assume that for any $x \in X$, $|x| > R$, there exists $z \in \Gamma'$ such that $|x| = |z|$. Take any $x \in X$, $z \in \Gamma'$ such that $|z| = |x| > R$. By the definition of B we have

$$|f(x)| \ge \tfrac{1}{2}|f(z)| \ge \tfrac{1}{2}C|z|^{\mathcal{L}_\infty(f|\Gamma')} = \tfrac{1}{2}C|x|^{\mathcal{L}_\infty(f|\Gamma')},$$

so $\mathcal{L}_\infty(f|\Gamma') \in N(f)$ and

$$\mathcal{L}_\infty(f) \ge \mathcal{L}_\infty(f|\Gamma'),$$

which proves (2).

Let $\varphi : D(r) \to X$ be the complexification of κ. Let $r' > r$ and $\Gamma = \varphi(D(r'))$. Since φ is a mapping meromorphic at infinity, we easily see that

$\Gamma \cap W$ is finite. So, increasing r' we deduce that Γ is an analytic curve meromorphic at infinity and $\Gamma \cap W = \emptyset$.

Since
$$\mathcal{L}_\infty(f) = \mathcal{L}_\infty(f|\Gamma') = \frac{\deg(f \circ \kappa)}{\deg \kappa} = \frac{\deg(f \circ \varphi)}{\deg \varphi},$$
Γ satisfies the required conditions. This ends the proof.

A mapping $\psi = (\psi_1, \ldots, \psi_n) : \mathbb{C} \setminus \{0\} \to \mathbb{C}^n$ is called a *Laurent polynomial mapping* if any component ψ_i is a Laurent polynomial, i.e. $\psi_i \in \mathbb{C}[t, t^{-1}]$.

PROPOSITION 3.3. *Let $\psi : \mathbb{C} \setminus \{0\} \to \mathbb{C}^n$ be a Laurent polynomial mapping. If $\deg \psi > 0$ and $\operatorname{ord}_0 \psi < 0$, then the set $\psi(\mathbb{C} \setminus \{0\})$ is an algebraic curve.*

Proof. Since the graph of ψ is an algebraically constructible set in $\mathbb{C} \times \mathbb{C}^n$ and $\psi(\mathbb{C} \setminus \{0\})$ is the projection of this graph onto $\mathbb{C}^n$, by the Chevalley Theorem $\psi(\mathbb{C} \setminus \{0\})$ is algebraically constructible in $\mathbb{C}^n$. From the assumptions that $\deg \psi > 0$ and $\operatorname{ord}_0 \psi < 0$ we see that ψ is a proper mapping, thus $\psi(\mathbb{C} \setminus \{0\})$ is closed. In consequence we have the assertion.

Let us give one more property of Laurent polynomial mappings.

PROPOSITION 3.4. *Let $\psi : \mathbb{C} \setminus \{0\} \to \mathbb{C}^n$, $n \geq 2$, be a Laurent polynomial mapping, $\deg \psi > 0$. Let $W \subset \mathbb{C}^n$ be an algebraic set, $\dim W \leq n - 2$ and $d < \min\{0, \operatorname{ord}_0 \psi\}$. Then, for generic $\xi \in \mathbb{C}^n$, $\xi \neq 0$ (i.e. for ξ outside a proper algebraic set), and for the Laurent polynomial mapping $\widetilde{\psi}(t) := \psi(t) + \xi t^d$ the set $V := \widetilde{\psi}(\mathbb{C} \setminus \{0\})$ is algebraic and $V \cap W = \emptyset$.*

Proof. Consider the algebraic set
$$Y := \{(t, \xi, z) \in \mathbb{C} \times \mathbb{C}^n \times W : t^{-d}\psi(t) + \xi = t^{-d} z\}.$$
It is easy to see that $\dim Y \leq n - 1$. Since $n \geq 2$, for generic $\xi \in \mathbb{C}^n$, $\xi \neq 0$, we have $Y \cap (\mathbb{C} \times \{\xi\} \times \mathbb{C}^n) = \emptyset$. For such ξ, putting $\widetilde{\psi}(t) := \psi(t) + \xi t^d$ we find that $V = \widetilde{\psi}(\mathbb{C} \setminus \{0\})$ does not intersect W. Since $\deg \widetilde{\psi} = \deg \psi > 0$ and $\operatorname{ord}_0 \widetilde{\psi} = d < 0$, by Proposition 3.3 the set V is algebraic. This ends the proof.

Now, we prove a theorem on selection of an algebraic curve at which the Łojasiewicz exponent at infinity is attained.

THEOREM 3.5. *Let X be a complex affine variety and let $f : X \to \mathbb{C}^k$ be a regular mapping such that $\# f^{-1}(0) < \infty$. For any algebraic subset $Z \subset X$ with $\dim Z \leq \dim X - 2$, there exists an algebraic curve $V \subset X$ such that $V \cap Z = \emptyset$ and*
$$\mathcal{L}_\infty(f) = \mathcal{L}_\infty(f|V) \in N(f).$$

Proof. Let $X \subset \mathbb{C}^M$ and $n = \dim X$. If $M = n$, i.e. $X = \mathbb{C}^M$, then the assertion follows immediately from Propositions 3.2, 2.4 and 3.4. So, let $M > n$. We shall use the classical description of irreducible algebraic sets (the Rückert Lemma, see e.g. [Ł$_2$], VII.9.3). So, there exists a linear change of coordinates in $\mathbb{C}^M$ such that in the new coordinates $(x', x'') \in \mathbb{C}^n \times \mathbb{C}^{M-n}$, $x' = (x_1, \ldots, x_n)$, $x'' = (x_{n+1}, \ldots, x_M)$ there exist

(i) $P \in \mathbb{C}[x', x_{n+1}]$, $P(x', x_{n+1}) = x_{n+1}^m + p_1(x')x_{n+1}^{m-1} + \ldots + p_m(x')$,
(ii) $\delta \in \mathbb{C}[x']$, $\delta \not\equiv 0$,
(iii) $Q_j \in \mathbb{C}[x', x_j]$, $j = n+2, \ldots, M$,

such that

$$(3) \quad X \setminus W = \{(x', x'') \in \mathbb{C}^M : P_{n+1}(x', x_{n+1}) = 0, \\ \delta(x')x_j = Q_j(x', x_{n+1}),\ j = n+2, \ldots, M\},$$

where $W = \{(x', x'') \in X : \delta(x') = 0\}$. Moreover, we may assume (see [Ł$_2$], VII.7.1) that

$$(4) \quad X \subset \{(x', x'') \in \mathbb{C}^n \times \mathbb{C}^{M-n} : |x''| \le C(1 + |x'|)\},$$

for some constant $C > 0$. Then the projection $\pi : X \ni (x', x'') \mapsto x' \in \mathbb{C}^n$ is proper. Hence the sets $\pi(Z)$ and $\pi^{-1}(\pi(Z))$ are algebraic of dimension at most $n - 2$.

By Proposition 3.2, there exists an analytic curve $\Gamma \subset X \setminus (W \cup \pi^{-1}(\pi(Z)))$ meromorphic at infinity such that

$$\mathcal{L}_\infty(f) = \mathcal{L}_\infty(f|\Gamma).$$

Let $\varphi = (\varphi', \varphi'') : D(r') \to X \subset \mathbb{C}^n \times \mathbb{C}^{M-n}$ be a description of Γ, $\deg \varphi > 0$. By (4),

$$(5) \quad \deg \varphi' = \deg \varphi.$$

Since $\Gamma \cap W = \emptyset$, we have $\delta \circ \varphi' \not\equiv 0$.

Take any $L < \min\{0, \deg \delta \circ \varphi'\}$. By Proposition 2.4 there exists a Laurent polynomial mapping $\psi = (\psi', \psi'') : \mathbb{C} \setminus \{0\} \to \mathbb{C}^n \times \mathbb{C}^{M-n}$ such that

$$(6) \quad \deg(\varphi - \psi) < L,$$

$$(7) \quad \deg(p_i \circ \varphi' - p_i \circ \psi') < L, \qquad i = 1, \ldots, m,$$

$$(8) \quad \deg(\delta \circ \varphi' - \delta \circ \psi') < L,$$

$$(9) \quad \mathrm{ord}_0\, \psi' < 0.$$

From (6) and (5) we have

$$(10) \quad \deg \varphi = \deg \psi = \deg \psi',$$

and from (8),

$$(11) \quad \deg \delta \circ \psi' = \deg \delta \circ \varphi' > -\infty.$$

By (9) and Proposition 3.3,

$$V_1 := \psi'(\mathbb{C} \setminus \{0\}) \tag{12}$$

is an algebraic curve in $\mathbb{C}^n$. Moreover, by Proposition 3.4 (changing ψ' without affecting (6)–(9)) we may assume that

$$V_1 \cap \pi(Z) = \emptyset. \tag{13}$$

Take $N \in \mathbb{N}$ (e.g. $N = m!$) such that $P(\varphi'(t^N), x_{n+1})$ and $P(\psi'(t^N), x_{n+1})$ decompose into linear factors (Lemma 2.1). Since $P(\varphi'(t^N), \varphi_{n+1}(t^N)) \equiv 0$, by (7) and Proposition 2.3 there exists a function $\lambda_{n+1} : D(r') \to \mathbb{C}$ meromorphic at infinity such that $P(\psi'(t^N), \lambda_{n+1}(t)) \equiv 0$ and

$$\deg(\varphi_{n+1}(t^N) - \lambda_{n+1}(t)) < (l + L/m)N, \tag{14}$$

where $l := \max\{1, \deg(p_1, \dots, p_m)\}$. By (6) and the obvious inequality $L \le l + L/m$ we have

$$\deg((\psi'(t^N), \lambda_{n+1}(t)) - (\varphi'(t^N), \varphi_{n+1}(t^N))) < (l + L/m)N.$$

Hence, by Proposition 2.4,

$$\begin{aligned}&\deg(Q_i(\psi'(t^N), \lambda_{n+1}(t)) - Q_i(\varphi'(t^N), \varphi_{n+1}(t^N)))\\ &\qquad < (l + L/m)N + (\deg Q_i - 1)N \deg \varphi \quad \text{for } i = n+2, \dots, M.\end{aligned}$$

Define

$$\lambda_i(t) := \frac{Q(\psi'(t^N), \lambda_{n+1}(t))}{\delta \circ \psi'(t^N)}, \qquad i = n+2, \dots, M,$$
$$\lambda(t) := (\psi'(t^N), \lambda_{n+1}(t), \dots, \lambda_M(t))$$

for sufficiently large t, say $t \in D(r'')$. The mapping $\lambda : D(r'') \to X$ is meromorphic at infinity and

$$\begin{aligned}\deg(\lambda_i(t) - \varphi_i(t^N)) &= \deg\left(\frac{Q_i(\psi'(t^N), \lambda_{n+1}(t))}{\delta \circ \psi'(t^N)} - \frac{Q_i(\varphi'(t^N), \varphi_{n+1}(t^N))}{\delta \circ \varphi'(t^N)}\right)\\ &= \deg\left(\frac{Q_i(\psi'(t^N), \lambda_{n+1}(t)) - Q_i(\varphi'(t^N), \varphi_{n+1}(t^N))}{\delta \circ \psi'(t^N)}\right.\\ &\qquad \left. - \frac{Q_i(\varphi'(t^N), \varphi_{n+1}(t^N))[\delta \circ \psi'(t^N) - \delta \circ \varphi'(t^N)]}{\delta \circ \psi'(t^N)\delta \circ \varphi'(t^N)}\right)\\ &\le N \max\{l + L/m + (\deg Q_i - 1)\deg \varphi - \deg(\delta \circ \varphi'),\\ &\qquad \deg(Q_i(\varphi', \varphi_{n+1})) + L - 2\deg \delta \circ \varphi'\}\end{aligned}$$

for $i = n+2, \dots, M$. So, decreasing L sufficiently we may assume that

$$\deg f \circ \varphi(t^N) = \deg f \circ \lambda(t).$$

Put

$$\Gamma' := \lambda(D(r'')).$$

Then $\Gamma' \subset X$ is an analytic curve meromorphic at infinity and

$$\mathcal{L}_\infty(f) = \mathcal{L}_\infty(f|\Gamma) = \frac{\deg f \circ \varphi}{\deg \varphi} = \frac{\deg f \circ \lambda}{\deg \lambda} = \mathcal{L}_\infty(f|\Gamma').$$

Take

$$V := \pi^{-1}(V_1).$$

Obviously V is an algebraic curve, $V \subset X$ and $\Gamma' \subset V$. Moreover, by (13), $V \cap Z = \emptyset$. So, from the above,

$$\mathcal{L}_\infty(f) \le \mathcal{L}_\infty(f|V) \le \mathcal{L}_\infty(f|\Gamma') = \mathcal{L}_\infty(f).$$

This ends the proof of Theorem 3.5.

4. The set of points at which a polynomial mapping is not proper. Let X be a complex affine variety, $\dim X = n$, and let $f : X \to \mathbb{C}^n$ be a dominating regular mapping. We say that f is *not proper* at a point $y \in \mathbb{C}^n$ if there is no neighbourhood $U \subset \mathbb{C}^n$ of y such that $f^{-1}(\overline{U})$ is compact; equivalently, there exists a sequence $\{z^\nu\} \subset X$ such that

$$|z^\nu| \to \infty \quad \text{and} \quad f(z^\nu) \to y.$$

It is easy to see that

PROPOSITION 4.1. *Let X be a complex affine variety*, $\dim X = n$ *and let $f : X \to \mathbb{C}^n$ be a dominating polynomial mapping. Then f is a proper mapping if and only if the set S_f of points at which f is not proper is empty.*

PROPOSITION 4.2 (cf. [J$_1$], Corollary 9, [J$_2$], Theorem 3.8). *Let X be a complex affine variety*, $\dim X = n$, *and let $f : X \to \mathbb{C}^n$ be a dominating regular mapping. Then the set S_f of points at which f is not proper is either an empty set or an algebraic set of pure dimension $n - 1$.*

Let us give a connection between points at which a polynomial mapping is not proper and the Łojasiewicz exponent.

PROPOSITION 4.3. *Let X be a complex affine variety*, $\dim X = n$ *and let $f : X \to \mathbb{C}^n$ be a dominating regular mapping. Then $\mathcal{L}_\infty(f - y) < 0$ if and only if f is not proper at y.*

Proof. Immediately from the definition we see that if f is not proper at y, then $f - y$ is not proper at 0, so $\mathcal{L}_\infty(f - y) < 0$. Conversely, assume $\mathcal{L}_\infty(f - y) < 0$. Then from Proposition 3.2, there exists a meromorphic curve at infinity $\Gamma \subset X$ such that $\mathcal{L}_\infty(f - y) = \mathcal{L}_\infty((f - y)|\Gamma)$. Let φ be a description of Γ. Then, by Lemma 3.1,

$$\mathcal{L}_\infty((f - y)|\Gamma) = \frac{\deg(f - y) \circ \varphi}{\deg \varphi}.$$

Since $\deg\varphi > 0$, we have $\deg(f-y)\circ\varphi < 0$ and so $\lim_{t\to\infty} f\circ\varphi(t) - y = 0$. Since $\lim_{t\to\infty}|\varphi(t)| = \infty$, there exists a sequence $\{z^\nu\}\subset\Gamma$ such that $|z^\nu|\to\infty$ and $f(z^\nu)\to y$. This implies that f is not proper at y, and completes the proof.

5. Proof of Theorem 1.1. We start with the following

LEMMA 5.1. *Let X be a complex affine variety, and let $f : X \to \mathbb{C}^k$, $k \le \dim X$, be a non-constant polynomial mapping. If f maps algebraic curves onto algebraic sets, then $k = \dim X$ and f is a dominating mapping.*

Proof. Let $n = \dim X$. Assume to the contrary that $k < \dim X$ or f is not a dominating mapping. Let $f = (f_1,\dots,f_k)$ and $W = \overline{f(X)} \subset \mathbb{C}^k$. Then W is an algebraic set. Let $l = \dim W$. Then $0 < l < n$. After a linear change of coordinates in $\mathbb{C}^k$, we may assume that

$$W \subset \{(y',y'')\in\mathbb{C}^l\times\mathbb{C}^{k-l} : |y''| < C(1+|y'|)\},$$

where $C > 0$. Thus, the canonical projection from W onto $\mathbb{C}^l$ is proper. Hence, it is easy to see that $\widetilde{f} := (f_1,\dots,f_l) : X \to \mathbb{C}^l$ is a dominating mapping and maps algebraic curves onto algebraic sets.

Let

$$Y_\xi = \{(y_1,\dots,y_l,t)\in\mathbb{C}^{l+1} : y_1\dots y_l t - \xi = 0\},\qquad \xi\in\mathbb{C}.$$

Then the projection $\pi : Y_\xi \ni (y,t)\mapsto y\in\mathbb{C}^l$ is a dominating mapping.

Let $g : X \to \mathbb{C}$ be a regular mapping which, treated as an element of $\mathbb{C}(X)$, is not algebraic over $\mathbb{C}(f_1,\dots,f_l)\subset\mathbb{C}(X)$. Then the mapping $\widetilde{g} := (f_1,\dots,f_l,g) : X\to\mathbb{C}^{l+1}$ is dominating. Thus there exists $\xi_0\in\mathbb{C}$, $\xi_0\neq 0$, such that $Y_{\xi_0}\cap\widetilde{g}(X)$ is a dense subset of Y_{ξ_0}. Let $V := \widetilde{g}^{-1}(Y_{\xi_0})$. Then

$$\widetilde{f}(V) = \pi(\widetilde{g}(V)) = \pi(Y_{\xi_0}\cap\widetilde{g}(X)),$$

so, by the above, $\widetilde{f}(V)$ is a dense subset of $\mathbb{C}^l$. Since $\xi_0\neq 0$, we easily see that $(\{0\}\times\mathbb{C})\cap Y_{\xi_0} = \emptyset$, thus $0\notin\widetilde{f}(V)$.

Since $\widetilde{f}(V)$ is a dense subset of $\mathbb{C}^l$, there exists a linear subspace $E\subset\mathbb{C}^l$, $\dim E = 1$, such that $E\cap\widetilde{f}(V)$ is a dense subset of E. Then there exists an algebraic curve $V_1\subset V\cap\widetilde{f}^{-1}(E)$ such that $\widetilde{f}(V_1)$ is a dense subset of E. Moreover, $0\in E\setminus\widetilde{f}(V_1)$, so $\widetilde{f}(V_1)$ is not an algebraic set. This contradicts the assumption and ends the proof.

We will prove a slightly stronger version of Theorem 1.1, namely

THEOREM 5.2. *Let X, Y be complex affine varieties, $\dim X\ge 2$, and $f : X\to Y$ a non-constant regular mapping. If f maps algebraic curves $V\subset X$ onto algebraic sets, then f is a proper mapping in the classical topology.*

Proof. Let $X \subset \mathbb{C}^M$, $Y \subset \mathbb{C}^N$, $n := \dim X$, $n \geq 2$. By the Noether Normalization Lemma, there exists a linear mapping $L : \mathbb{C}^N \to \mathbb{C}^n$ such that $L \circ f : X \to \mathbb{C}^n$ maps algebraic curves onto algebraic sets. Since the properness of $L \circ f$ implies the properness of f, we may assume that $Y = \mathbb{C}^n$. By Lemma 5.1, f is dominating.

By Proposition 4.1, it suffices to prove that the set S_f of points at which f is not proper is empty. Assume to the contrary that $S_f \neq \emptyset$. By Proposition 4.2, we find that S_f is an algebraic set of pure dimension $n-1$. From [M], Corollaries 3.15 and 3.16 we deduce that the set

$$T := \overline{\{y \in \mathbb{C}^n : \dim f^{-1}(y) > 0\}}$$

is algebraic and has dimension at most $n-2$. Thus, there exists $y \in S_f \setminus T$. We may assume that $y = 0$. Let $f^{-1}(0) = \{z^1, \ldots, z^m\}$. Then, by Theorem 3.5 and Proposition 4.3, there exists an algebraic curve $V \subset X \setminus \{z^1, \ldots, z^m\}$ such that

$$\mathcal{L}_\infty(f) = \mathcal{L}_\infty(f|V) < 0.$$

Obviously, $0 \notin f(V)$. Since, by Theorem 3.5, $\mathcal{L}_\infty(f|V) \in N(f|V)$, we see that there exists a sequence $\{x^\nu\} \subset V$ such that $\lim_{\nu\to\infty} |x^\nu| = \infty$ and $\lim_{\nu\to\infty} f(x^\nu) = 0$. In consequence, 0 is an accumulation point of $f(V)$, and so $f(V)$ is not algebraic. This contradiction gives the assertion and ends the proof of Theorem 5.2.

Remark 5.3. One can prove Theorem 1.1 by another method (suggested by Z. Jelonek and J. Kollár). The main idea is as follows: Let $f : X \to Y$ satisfy the assumptions of Theorem 1.1. Let $\overline{f} : \overline{X} \to \overline{Y}$ be a projective compactification of f. Assume to the contrary that f is not proper. Then there is a point $y \in Y$ such that $\overline{f}^{-1}(y)$ is finite and $\overline{f}^{-1}(y) \cap (\overline{X} \setminus X) \neq \emptyset$. If we take an algebraic curve $\overline{\Gamma} \subset \overline{X}$ not lying in $\overline{X} \setminus X$ such that $\overline{\Gamma} \cap (\overline{f}^{-1}(y) \cap X) = \emptyset$ and $\overline{\Gamma}$ passes through one point of $\overline{f}^{-1}(y) \cap (\overline{X} \setminus X)$ then the image $f(\overline{\Gamma} \cap X)$ is not Zariski closed.

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Faculty of Mathematics
University of Łódź
S. Banacha 22
90-238 Łódź, Poland
E-mail: krasinsk@krysia.uni.lodz.pl
spodziej@imul.uni.lodz.pl

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