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AK-invariant, some conjectures, examples and counterexamples

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Abstract. In my talk I am going to remind you what is the AK-invariant and give examples of its usefulness. I shall also discuss basic conjectures about this invariant and some positive and negative results related to these conjectures.

Let us start with definitions. Though most of the definitions below make sense in greater generality, we are going to consider in this talk only domains over the field $\mathbb{C}$ of complex numbers.

Definition of the AK-invariant

Derivations and related notions. Let A be a commutative algebra over the field $\mathbb{C}$. Then a $\mathbb{C}$-homomorphism ∂ of A is called a *derivation* of A if it satisfies the Leibniz rule: $\partial(ab) = \partial(a)b + a\partial(b)$. It follows immediately from the Leibniz rule that it is sufficient to know a derivation on a generating set of the algebra A.

Let us denote the set of all derivations of A by $\operatorname{Der}(A)$. It is well known (and easy to check) that $\operatorname{Der}(A)$ is a Lie algebra relative to the addition and "commutator" operations in the algebra of homomorphisms of A. Also, $\operatorname{Der}(A)$ is a left A-module.

Any derivation ∂ determines two subalgebras of A. One is the kernel of ∂, which is usually denoted by A^∂ and called the *ring of ∂-constants*. The other is $\operatorname{Nil}(\partial)$, the *ring of nilpotency* of ∂. It is determined by $\operatorname{Nil}(\partial) = \{a \in A \mid \partial^n(a) = 0,\ n \gg 1\}$. In other words, $a \in \operatorname{Nil}(\partial)$ if for a sufficiently large natural number n we have $\partial^n(a) = 0$. Both A^∂ and $\operatorname{Nil}(\partial)$ are subalgebras of A because of the Leibniz rule.

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Let us call a derivation *locally nilpotent* if $\mathrm{Nil}(\partial) = A$. Let us denote by $\mathrm{LND}(A)$ the set of all locally nilpotent derivations. Sometimes we abbreviate locally nilpotent derivation as Lnd.

Here are several examples of locally nilpotent derivations. The best one is the partial derivatives on a polynomial ring. Next, let us consider a ring A_P with generators x, y, z and with one relation $xy = P(z)$ where $P(z)$ is a polynomial of positive degree. Then ∂_1 which is given by $\partial_1(x) = P'$, $\partial_1(y) = 0$, $\partial_1(z) = y$ is a locally nilpotent derivation of A_P. Since our relation is x-y symmetric we can switch x and y and obtain another locally nilpotent derivation $\partial_2(x) = 0$, $\partial_2(y) = P'$ and $\partial_2(z) = x$. The ring A_P is isomorphic to the polynomial algebra $\mathbb{C}[u, v]$ only if $\deg(P) = 1$. Otherwise it is not a unique factorization domain and so is not isomorphic to $\mathbb{C}[u, v]$.

Now let us take the ring $A_{P,n}$ which is given by a relation $x^n y = P(z)$. If $n > 1$ then $\partial(x) = 0$, $\partial(y) = P'$, $\partial(z) = x^n$ still define a locally nilpotent derivation. But since the relation is not symmetric any more we cannot easily produce a second derivation. In fact, it is possible to prove that all locally nilpotent derivations of $A_{P,n}$ are in some sense equivalent to ∂. We will give a precise definition later.

It is not easy to describe locally nilpotent derivations. Even for rings of polynomials we know the description only for rings with one generator (an exercise) and for two generators (see [Re]). For three generators Miyanishi proved that the kernel of an Lnd is isomorphic to $\mathbb{C}[u, v]$ (see [Mi1] and [Fr]) and Freudenburg, Daigle, and Daigle and Russell described all homogeneous Lnd (see the talk of D. Daigle). But the complete description of Lnd for $\mathbb{C}[x, y, z]$ is not known yet.

For the rings of polynomials with a larger number of generators the kernels of locally nilpotent derivations may even fail to be finitely generated (see the talk of G. Freudenburg).

The AK-invariant. The intersection of the rings of constants of all locally nilpotent derivations is called the *ring of absolute constants* and denoted by $\mathrm{AK}(A)$.

Degree function and equivalence relation. For a locally nilpotent derivation ∂ acting on a ring A, one can define $\deg_\partial(f) = \max(n \mid \partial^n(f) \neq 0)$ if $f \in A^* = A \setminus 0$ and $\deg_\partial(0) = -\infty$. This is a degree function (see for example [FLN]), i.e., $\deg_\partial(a + b) \leq \max(\deg_\partial(a), \deg_\partial(b))$ and $\deg_\partial(ab) = \deg_\partial(a) + \deg_\partial(b)$.

Let us call two Lnd *equivalent* if they define the same degree function.

Several lemmas. Let F be the field of fractions of A and let ∂ be a nonzero Lnd of A. We can extend ∂ to F. Let F^∂ be the subring of constants of ∂ on F and let $\mathrm{Nil}_F(\partial)$ be the ring of nilpotency of ∂ on F. Finally, let K be the field of fractions of A^∂.

LEMMA 1. $F^{\partial} = K$.

Proof (O. Hadas). If $f \in F^{\partial}$ then $f = ab^{-1}$ where $a, b \in A$. Since $\partial(f) = (\partial(a)b - a\partial(b))b^{-2} = 0$ we have $ab^{-1} = \partial(a)\partial(b)^{-1}$. But $\deg_{\partial}(\partial(a)) = \deg_{\partial}(a) - 1$, so we can see that there is a representation of f in which both the numerator and denominator are in A^{∂}.

LEMMA 2. *There exists an element $t \in F$ such that* $\mathrm{Nil}_F(\partial) = K[t]$.

Proof. Since $A \neq A^{\partial}$ there exists an $r \in A \setminus A^{\partial}$ such that $\partial(r) \in A^{\partial}$. Indeed, let $a \in A \setminus A^{\partial}$. Then we can take $r = \partial^n(a)$ where $n = \deg_{\partial}(a) - 1$. Let $s = \partial(r)$. If we take $t = rs^{-1}$ then $\partial(t) = 1$. Let $f \in \mathrm{Nil}_F(\partial)$. Let us use induction on $\deg_{\partial}(f) = n$ to show that $f = \sum_{i=0}^{n} f_i t^{n-i}$ where $f_i \in F^{\partial}$. If $\deg_{\partial}(f) = 0$ then $f \in F^{\partial}$. Let us make the step from $\deg_{\partial}(f) = n - 1$ to $\deg_{\partial}(f) = n$. If $\deg_{\partial}(f) = n$ then $\deg_{\partial}(\partial(f)) = n - 1$ and by induction $\partial(f) = \sum_{i=0}^{n-1} f_i t^{n-1-i}$ for some $f_i \in F^{\partial}$. Let $g = \sum_{i=0}^{n-1}(n-i)^{-1} f_i t^{n-i}$. Then $\partial(g) = \partial(f)$. So $\partial(f - g) = 0$, which means that $f = g + f_n$ where $f_n \in F^{\partial}$.

LEMMA 3. *Two Lnd of A are equivalent if and only if their kernels are the same.*

Proof. It is clear that if two Lnd are equivalent then their kernels are the same. Let us assume now that ∂_1 and ∂_2 are non-zero Lnd and that their kernels are the same. Then by Lemma 1 the kernels of their extensions on F are the same and by Lemma 2 there exists a $t \in F$ such that $\partial_1(t) = 1$ and $t \in \mathrm{Nil}_F(\partial_2)$. Let us take $a \in A \setminus A^{\partial}$. Then $a = \sum_{i=0}^{m} a_i t^{m-i}$ where $m > 0$ and $a_i \in K = F^{\partial_1}$. So $\partial_2(a) = \sum_{i=0}^{m}(m-i)a_i t^{m-i-1}\partial_2(t)$. If $\partial_2(t) \notin K$ then $\partial_2(a) \notin K$ and $\partial_2(A) \cap K = 0$. But, as we saw in the proof of Lemma 2, this is impossible. So $\partial_2(t) \in K$ and the degree of a relative to ∂_2 is also m.

Similarly we can prove the following lemma.

LEMMA 4. *If ∂ is an Lnd and $\partial(A) \subset bA$ for some $b \in A$ then $\partial(b) = 0$.*

Proof. If $\partial(b) \neq 0$ then $\partial(A) \cap A^{\partial} = 0$ and this is impossible.

If A is a finitely generated domain then A is isomorphic to $\mathbb{C}_n/I$ where $\mathbb{C}_n$ is a polynomial ring in n variables and I is a prime ideal.

There is an important class of derivations on $\mathbb{C}_n$, the so-called *Jacobian derivations*. They are defined as follows. Let $f_1, \ldots, f_{n-1}$ be some elements of $\mathbb{C}_n$. Then $\partial(f) = \partial_{f_1,\ldots,f_{n-1}}(f)$ is the determinant of the corresponding Jacobi matrix (Jacobian).

Similarly we can define Jacobian derivations on $\mathbb{C}_n/I$. Let $\partial \in \mathrm{Der}(\mathbb{C}_n)$ and let $\partial(I) \subset I$. Then ∂ defines a derivation on $\mathbb{C}_n/I$. Indeed, $\partial(a + I) = \partial(a) + I$ is well defined on $\mathbb{C}_n/I$ and it is easy to check that the resulting homomorphism is a derivation. We shall call this derivation of $\mathbb{C}_n/I$ *Jacobian* if ∂ is a Jacobian derivation.

It has been known for some time that an Lnd of $\mathbb{C}_n$ is equivalent to a Jacobian derivation. But in fact this is true for any Lnd on an affine domain.

LEMMA 5 (with Kaliman). *Let $A = \mathbb{C}_n/I$ where I is a prime ideal and let $\partial \in \mathrm{LND}(A)$. There exists a set of elements $f_1, \dots, f_{n-1}$ in $\mathbb{C}_n$ such that the derivation $\partial_{f_1,\dots,f_{n-1}}$ defines a derivation on A which is equivalent to ∂.*

Unfortunately, the proof is a bit too involved to be presented here, but you can request a preprint from me if you want.

Because of this lemma we can see that $\mathrm{AK}(A) = \bigcap_{\partial \in \mathrm{JLnd}(A)} A^{\partial}$ where $\mathrm{JLnd}(A)$ denotes the Lnd of Jacobian type. This really helps in computations of AK.

What is AK good for? As with any invariant, if we can compute it for a ring then we can tell that this ring is different from a ring with a different invariant.

For example $\mathrm{AK}(\mathbb{C}_n) = \mathbb{C}$ because though we do not know all Lnd of $\mathbb{C}_n$ we know that all partial derivatives are Lnd and this is enough to see that $\mathrm{AK}(\mathbb{C}_n) = \mathbb{C}$.

In many situations it is important to characterize polynomial rings. If A is very small, that is, if the transcendence degree of A is one, then $\mathrm{AK}(A)$ is either A or $\mathbb{C}$. And if it is $\mathbb{C}$ then $A \simeq \mathbb{C}[t]$.

If $\mathrm{trdeg}(A) = 2$ then $\mathrm{AK}(A) = \mathbb{C}$ does not imply that $A \simeq \mathbb{C}[x,y]$. We already saw an example of a ring which is not isomorphic to $\mathbb{C}[x,y]$ with AK equal to $\mathbb{C}$. This is any ring A_P with the degree of P at least two. Then $\mathrm{AK}(A_P) = \mathbb{C}$ because the intersection of the kernels of ∂_1 and ∂_2 is already $\mathbb{C}$. But if $\mathrm{AK}(A) = \mathbb{C}$ and A is a UFD (unique factorization domain) then Miyanishi proved that $A \simeq \mathbb{C}[x,y]$ (see [Mi2]).

Now, if $\mathrm{trdeg}(A) = 3$, then $\mathrm{AK}(A) = \mathbb{C}$ and A being a UFD is not sufficient to make A a polynomial ring. For example sl_2 which is given by $xy - uv = 1$ satisfies these conditions but is not isomorphic to $\mathbb{C}[x,y,z]$. So what condition should be added? For example, if A also admits three non-equivalent *commuting* Lnd then $A \simeq \mathbb{C}[x,y,z]$. But this condition is too strong. So I do not know what the right additional condition is. It may be that it is of a geometric nature and I invite you to formulate it.

AK also works another way. If we know that $\mathrm{AK}(A) \neq \mathbb{C}$ then, of course, we know that $A \not\simeq \mathbb{C}_n$. But this information is also good for another purpose.

Here is an example. Let us take $A = A_{n,P}$ which is given by $x^n y = P(z)$. Let us assume that $n > 1$ and $\deg(P) > 1$. It is possible to show that $\mathrm{AK}(A) = \mathbb{C}[x]$. Let α be an automorphism of A. Then α induces an automorphism of $\mathbb{C}[x]$. So we have a rather strong restriction on α: $\alpha(x) = ax + b$ where $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. In fact, it is possible to show that $\alpha(x) = ax$ and to get a complete description of the automorphisms of A. Similarly one can answer when $A_{n,P}$ is isomorphic to $A_{m,Q}$ since $\mathrm{AK}(A_{n,P})$ goes onto $\mathrm{AK}(A_{m,Q})$ (see [ML1]).

I hope that these examples have persuaded you that AK is a useful invariant and it is interesting to find how it behaves.

Conjectures. The first natural question is how $\mathrm{AK}(A)$ and $\mathrm{AK}(A_n)$, where $A_n = A[x_1, \ldots, x_n]$, are connected. It is rather clear that $\mathrm{AK}(A_n) \subseteq \mathrm{AK}(A)$. Indeed, any $\partial \in \mathrm{LND}(A)$ can be extended to an Lnd of A_n by $\partial(x_i) = 0$ for all i. Also, all partial derivatives are in $\mathrm{LND}(A_n)$. The intersection of the kernels of all Lnd extended from A and all partial derivatives is already $\mathrm{AK}(A)$. And we may have more Lnd on A_n. But can we have $\mathrm{AK}(A_n)$ smaller than $\mathrm{AK}(A)$?

So is it true that $\mathrm{AK}(A[x]) = \mathrm{AK}(A)$? I checked this conjecture in two cases. If $\mathrm{trdeg}(A) = 1$ then $\mathrm{AK}(A_n) = \mathrm{AK}(A)$ for any n (see [ML2]). Also, if $\mathrm{AK}(A) = A$ then $\mathrm{AK}(A[x]) = A$ (see [ML3]).

It was my original conjecture that $\mathrm{AK}(A[x]) = \mathrm{AK}(A)$. I wanted this conjecture to be true to such a degree that I forgot about counterexamples which I knew.

These counterexamples were mentioned yesterday by Peter van Rossum, but let us talk about them in greater detail. They belong to Danielewski. As we already saw, $\mathrm{AK}(A_P) = \mathbb{C}$ for any P. We also mentioned that $\mathrm{AK}(A_{2,P}) = \mathbb{C}[x]$ if the degree of P is more than 2. But Danielewski [Dan] has shown that $A_P[t] \simeq A_{2,P}[t]$ for $P = z^2 - 1$. So $\mathrm{AK}(A_P[t]) \subseteq \mathrm{AK}(A_P) = \mathbb{C}$ and therefore $\mathbb{C} = \mathrm{AK}(A_P[t]) = \mathrm{AK}(A_{2,P}[t]) \neq \mathrm{AK}(A_{2,P}) = \mathbb{C}[x]$.

Here are the formulae. Let $R = A_{2,P}[t]$. Then $\partial/\partial t$ and ∂ which is given by $\partial(x) = 0$, $\partial(y) = 2z$, $\partial(z) = x^2$, and $\partial(t) = 0$ are the expected Lnd. Here is an additional one. Let us take a derivation on $\mathbb{C}[x, y, z, t]$ which is given by $\varepsilon(r) = J(x^2y - z^2, t^2x + 2tz + xy, t^3x + 3t^2z + 3txy + yz, r)$ where J denotes the Jacobian. It is not difficult to check that $\varepsilon(I) \subset I$ where I is the principal ideal generated by $x^2y - z^2 - 1$, that the resulting derivation is nilpotent, and that x is not in its kernel.

So life is more difficult than I hoped and my original conjecture is wrong. Let me then modify it to

Conjecture 1. $\mathrm{AK}(A) = \mathrm{AK}(A[x])$ *if A is a UFD.*

At least I do not know counterexamples to this one.

How can one generalize this question? As was observed in one of the talks yesterday, $A[x] = A \otimes_{\mathbb{C}} \mathbb{C}[x]$. So Conjecture 1 can be rewritten as $\mathrm{AK}(A \otimes_{\mathbb{C}} \mathbb{C}[x]) = \mathrm{AK}(A) \otimes_{\mathbb{C}} \mathrm{AK}(\mathbb{C}[x])$ if A is a UFD.

If we recall that for two affine varieties V_1 and V_2 the ring of regular functions $O(V_1 \times V_2) = O(V_1) \otimes_{\mathbb{C}} O(V_2)$, we may want to relate $\mathrm{AK}(A \otimes_{\mathbb{C}} B)$ and $\mathrm{AK}(A) \otimes_{\mathbb{C}} \mathrm{AK}(B)$. It is again quite easy to show that $\mathrm{AK}(A \otimes_{\mathbb{C}} B) \subseteq$

$\mathrm{AK}(A) \otimes_{\mathbb{C}} \mathrm{AK}(B)$ since any $\partial \in \mathrm{LND}(A)$ can be extended to $A \otimes_{\mathbb{C}} B$ by $\partial(B) = 0$ and this extension is an Lnd. So if $r = \sum a_i \otimes b_i \in \mathrm{AK}(A \otimes_{\mathbb{C}} B)$, $\{b_i\}$ are linearly independent over $\mathbb{C}$, and ∂ is an extension of an Lnd of A then $0 = \partial(r) = \sum \partial(a_i) \otimes b_i$ implies that $\partial(a_i) = 0$ for all i. So $a_i \in \mathrm{AK}(A)$. Similarly, any Lnd of B can be extended to an Lnd of $A \otimes_{\mathbb{C}} B$ and $b_i \in \mathrm{AK}(B)$.

Of course, the Danielewski examples show that in general $\mathrm{AK}(A \otimes_{\mathbb{C}} B) \neq \mathrm{AK}(A) \otimes_{\mathbb{C}} \mathrm{AK}(B)$. So it is interesting to find out when we have equality.

As I mentioned before, we have $\mathrm{AK}(A \otimes_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]) = \mathrm{AK}(A) \otimes_{\mathbb{C}} \mathrm{AK}(\mathbb{C}[x_1, \ldots, x_n])$ if $\mathrm{trdeg}(A) = 1$. In fact, if $\mathrm{trdeg}(A) = 1$ and $A \not\simeq \mathbb{C}[t]$ then $\mathrm{AK}(A \otimes_{\mathbb{C}} B) = \mathrm{AK}(A) \otimes_{\mathbb{C}} \mathrm{AK}(B)$ for any B with finite transcendence degree (see [ML2]).

Another thing which was also already mentioned is that $\mathrm{AK}(A \otimes_{\mathbb{C}} \mathbb{C}[x]) = \mathrm{AK}(A) \otimes_{\mathbb{C}} \mathrm{AK}(\mathbb{C}[x])$ if $\mathrm{AK}(A) = A$.

So here is

CONJECTURE 2. *If* $\mathrm{AK}(A) = A$ *then* $\mathrm{AK}(A \otimes_{\mathbb{C}} B) = \mathrm{AK}(A) \otimes_{\mathbb{C}} \mathrm{AK}(B)$.

Even the special case when $B = \mathbb{C}[x_1, \ldots, x_n]$ is very interesting. If it is proved for A with transcendence degree 2, it gives a new proof of the Zariski Cancellation Conjecture for surfaces (see [Fu], [MS], and [Su]). The same is true for Conjecture 1.

Let me finish with

CONJECTURE 3. *If* A *and* B *are UFD then*

$$\mathrm{AK}(A \otimes_{\mathbb{C}} B) = \mathrm{AK}(A) \otimes_{\mathbb{C}} \mathrm{AK}(B).$$

Of course, all these conjectures are motivated by affine algebraic geometry. Any substantial progress with them gives us results on the Zariski Cancellation Conjecture. But it is a dubious blessing. Since the majority of experts now think that the Zariski Cancellation Conjecture is wrong, the question is what happens first: proofs or counterexamples?

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