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Commutativity of flows and injectivity of nonsingular mappings

by M. Sabatini (Trento)

Abstract. A relationship between jacobian maps and the commutativity properties of suitable couples of hamiltonian vector fields is studied. A theorem by Meisters and Olech is extended to the nonpolynomial case. A property implying the Jacobian Conjecture in $\mathbb{R}^2$ is described.

Introduction. Let $z \mapsto \Lambda(z) = (f(z), g(z))$, $z = (x, y)$, be a differentiable map of the real plane $\mathbb{R}^2$ into itself. Assume $\Lambda(z)$ to be *nonsingular*, i.e. assume that its jacobian determinant $d(z) = \det J_\Lambda(z)$ never vanishes. By the local inversion theorem, Λ is locally invertible at every point of $\mathbb{R}^2$. Local invertibility does not imply global invertibility. It is easy to find examples of maps which are locally invertible at every point of $\mathbb{R}^2$, but not globally invertible, as $(x, y) \mapsto (e^x \cos(y), e^x \sin(y))$. It is even possible to find polynomial maps which are locally invertible everywhere, but are not globally invertible [P]. The problem of finding suitable additional conditions that imply the global invertibility of locally invertible maps has attracted several researchers, working in different fields of pure and applied mathematics.

The same question can be considered with $\mathbb{R}^2$ replaced by the complex plane $\mathbb{C}^2$. In this case, if one restricts to polynomial maps, the jacobian determinant does not vanish if and only if it is a nonzero constant. Nonsingular maps with constant jacobian determinant are called *jacobian maps*. So far, there are no examples of complex polynomial jacobian maps which are locally invertible but not globally invertible. Keller conjectured that the constancy of the jacobian determinant is a sufficient condition for the global invertibility of a complex polynomial map. Keller's conjecture was studied

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in several different settings (see [BCW, D] for more information on the subject) but it is still unsolved even in dimension two, both for real and for complex maps.

Among nonpolynomial maps, one finds examples of real nonglobally invertible jacobian maps, as $(x,y) \mapsto (e^{x/2}\cos(ye^{-x}), e^{x/2}\sin(ye^{-x}))$ (see [M]). In the real bidimensional case, the constancy of the determinant allows us to establish an unusual connection between the injectivity properties of a map and the qualitative properties of a suitable hamiltonian system [S3]. Such a relationship resembles a similar one studied in connection with another jacobian conjecture [O]. In the present paper we study another relationship between jacobian maps and the dynamic properties of suitable plane hamiltonian systems.

Let us consider the hamiltonian differential systems

$$\begin{cases} \dot{x} = f_y/d, \\ \dot{y} = -f_x/d, \end{cases} \qquad \begin{cases} \dot{x} = g_y/d, \\ \dot{y} = -g_x/d, \end{cases}$$

where f and g are the components of Λ. Let $\phi(t,z)$ be the solution of the first system starting at a point z, and $\psi(s,z)$ the solution of the second system starting at a point z. The flows of the two systems commute. In other words, under a suitable hypothesis on the interval of existence of solutions, one has

$$\psi(s,\phi(t,z)) = \phi(t,\psi(s,z)).$$

Commutativity has very strong consequences on the behaviour of the orbits of commuting systems (for an account about the qualitative properties of commuting systems, see [S2]). In particular, plane centres are isochronous. In [S3] isochronicity was used to prove injectivity of jacobian maps, under suitable additional assumptions. In the present paper, we use commutativity in order to extend some of the results of [S3] to nonsingular maps, as well as a result presented in [MO]. Starting with f and g, we construct two families of couples of commuting systems. Λ comes out to be a linearization or a rectification for such systems. When such systems are complete (this is the additional hypothesis we use in this paper) the behaviour of their solutions is quite simple. This, and arguments similar to those used in [S1, S3], allow us to prove the injectivity of Λ.

Finally, we show that in order to prove the Jacobian Conjecture in the real plane it is sufficient to prove that, for every polynomial jacobian map Λ, there exist an integer n_Λ and a real constant h_Λ such that the following inequality holds, outside a suitable compact subset of the plane:

$$(fg_y - gf_y)^2 + (-fg_x + gf_x)^2 \le h_\Lambda(x^2+y^2)(f^2+g^2)^{2n_\Lambda}d.$$

Definitions and results. In this section we denote by $z = (x,y)$ a point of the plane. Let us consider a couple of differential systems defined on an open, connected subset U of the plane

$$(S_V) \qquad \dot{z} = V(z), \qquad V = (v_1, v_2) \in C^2(U, \mathbb{R}^2),$$
$$(S_W) \qquad \dot{z} = W(z), \qquad W = (w_1, w_2) \in C^2(U, \mathbb{R}^2).$$

We denote by $\phi(t,z)$ (resp. $\psi(s,z)$) the solution of (S_V) (resp. (S_W)) such that $\phi(0,z) = z$ (resp. $\psi(0,z) = z$). In general, we do not assume the solutions to be defined for all real numbers t or s. Hence, we shall consider the *local flows* defined by the two differential systems. We say that a solution starting at z is *complete* if $\phi(t,z)$ exists for all $t \in \mathbb{R}$. We say that a solution starting at z is *positively complete* if $\phi(t,z)$ exists for all $t \geq 0$. We say that a vector field is *complete* if the solutions of the corresponding differential system are defined for all $t \in \mathbb{R}$. We say that a vector field is *positively complete* if the solutions of the corresponding differential system are defined for all $t \geq 0$. If a vector field is complete, the flow it induces is said to be *global*.

Let T, S be positive numbers, and $P = [0,T] \times [0,S]$ be a rectangle, which will be called a *parametric rectangle*. We say that the local flows $\phi(t,z)$ and $\psi(s,z)$ *commute* if, for every parametric rectangle P such that both $\phi(t,\psi(s,z))$ and $\psi(s,\phi(t,z))$ exist whenever $(t,s) \in P$, one has

$$\phi(t,\psi(s,z)) = \psi(s,\phi(t,z)).$$

By a classical result, two local flows commute if and only if the Lie brackets $[V,W]$ of V and W vanish identically on U (see [A, Olv]):

$$\begin{cases} [V,W]_1 = \left(v_1 \dfrac{\partial w_1}{\partial x} - w_1 \dfrac{\partial v_1}{\partial x} \right) + \left(v_2 \dfrac{\partial w_1}{\partial y} - w_2 \dfrac{\partial v_1}{\partial y} \right) = 0, \\[2ex] [V,W]_2 = \left(v_1 \dfrac{\partial w_2}{\partial x} - w_1 \dfrac{\partial v_2}{\partial x} \right) + \left(v_2 \dfrac{\partial w_2}{\partial y} - w_2 \dfrac{\partial v_2}{\partial y} \right) = 0. \end{cases}$$

In this case we say that V and W commute, or that W is a *commutator* of V. Equivalently, we shall say that (S_V) and (S_W) commute. V and W are said to be *nontrivial commutators* if they are transversal at nonsingular points. By Theorem 1.5 in [S2], we know that if V and W are transversal at a point z, then they are transversal at every point that can be connected to z by means of finitely many arcs of orbits of (S_V) and (S_W).

We say that a map $\Lambda : U \to \mathbb{R}^2$, $\Lambda(x,y) = (f(x,y), g(x,y))$, $f, g \in C^2(U, \mathbb{R})$, is a *nonsingular map* if its jacobian determinant $d(z) = \det J_\Lambda(z)$ never vanishes. We say that it is a *jacobian map* if its jacobian determinant $d(z)$ is a nonzero constant. In the latter case, possibly multiplying by a suitable constant, we can reduce to the case $d(z) \equiv 1$. Moreover, we can assume $\Lambda(0,0) = (0,0)$, possibly replacing $\Lambda(z)$ with $\Lambda(z) - \Lambda(0,0)$. In general, we do not assume $\Lambda(z)$ to be a polynomial map.

Denoting by h_x, h_y the partial derivatives of a function $h(x,y)$ with respect to x and y, let us consider the differential systems

$$(H_f) \quad \begin{cases} \dot{x} = f_y, \\ \dot{y} = -f_x, \end{cases} \qquad (H_g) \quad \begin{cases} \dot{x} = g_y, \\ \dot{y} = -g_x, \end{cases}$$

$$(H_m) \quad \begin{cases} \dot{x} = ff_y + gg_y, \\ \dot{y} = -ff_x - gg_x, \end{cases} \qquad (H_a) \quad \begin{cases} \dot{x} = \dfrac{fg_y - gf_y}{f^2+g^2}, \\ \dot{y} = \dfrac{-fg_x + gf_x}{f^2+g^2}. \end{cases}$$

We denote by V_f, V_g, V_m, V_a the corresponding vector fields. The first three systems are the hamiltonian systems generated by the hamiltonian functions f, g, $m := (f^2+g^2)/2$. The fourth one is, locally, the hamiltonian system generated by the multivalued function $\arg(f,g)$. The systems (H_f) and (H_g) have been previously considered in [MO, S1] in order to study the injectivity properties of Λ.

It is useful to find conditions under which V_f commutes with V_g, and V_m commutes with V_a. In the next lemma, we show that the Lie bracket of V_f and V_g is itself a hamiltonian vector field. Let us denote by V_d the hamiltonian vector field $(d_y, -d_x)$.

Lemma 1. *Let $f, g \in C^2(U, \mathbb{R})$. Then*

$$[V_f, V_g] = V_d.$$

Proof. Computing the commutator gives

$$\begin{aligned} [V_f, V_g] &= (f_{yx}g_y - f_{yy}g_x - g_{yx}f_y + g_{yy}f_x, -f_{xx}g_y + f_{xy}g_x + g_{xx}f_y - g_{xy}f_x) \\ &= (d_y, -d_x) = V_d. \ \blacksquare \end{aligned}$$

Now we can characterize the jacobian mappings as those for which (H_f) and (H_g) commute.

Lemma 2. *Let $f, g \in C^2(U, \mathbb{R})$. Assume that there exists $z^* \in U$ such that $V_f(z^*) \neq (0,0)$. Then V_f and V_g are nontrivial commutators on U if and only if Λ is a jacobian map.*

Proof. Assume that V_f and V_g are nontrivial commutators. By Lemma 1, $V_d = [V_f, V_g] = (0,0)$, hence ∇d vanishes identically. Since U is open and connected, d is constant. Moreover, $d(z^*) \neq 0$ since V_f and V_g are transversal at noncritical points. Hence $d(z)$ is a nonzero constant on U.

Conversely, assume Λ to be a jacobian map. The jacobian determinant of Λ is a nonzero constant, hence V_f and V_g are transversal at every point. By Lemma 1 we have $[V_f, V_g] = V_d = (0,0)$, hence V_f and V_g commute. $\blacksquare$

A diffeomorphism is said to be a *canonical transformation* if it preserves the 2-form $dx \wedge dy$. Canonical transformations preserve the form of hamiltonian systems. By a classical result, a diffeomorphism is a canonical transformation if and only if its jacobian determinant is identically 1 (see [A]). Lemma 2 shows that the hamiltonian systems (H_f) and (H_g) commute if and only if Λ is, up to a multiplicative constant, a canonical transformation.

Let $\Sigma \subset U$ be the set of zeros of $|\Lambda|$.

LEMMA 3. *Let $\Lambda \in C^2(U, \mathbb{R}^2)$ be a jacobian map. Then V_m and V_a are nontrivial commutators on $U \setminus \Sigma$.*

Proof. It is sufficient to prove that $[V_m(z), V_a(z)] = 0$ for all $z \in U \setminus \Sigma$. If $z^* \in U \setminus \Sigma$, then at least one among f and g does not vanish at z^*. Without loss of generality, we can assume that $f(z^*) \neq 0$. Then there exists a neighbourhood U^* of z^* such that the function $a^*(z) := \arg(f(z), g(z))$ is single-valued on U^*. The hamiltonian vector field associated with $a^*(z)$ is just V_a. Now we can apply Lemma 2 to the map $z \mapsto (m(z), a^*(z))$. A straightforward computation shows that its jacobian determinant is a nonzero constant, hence V_m and V_a commute on U^*, and $[V_m(z), V_a(z)] = 0$ on U^*. ∎

REMARK 1. The commutativity of the couples (V_f, V_g) and (V_m, V_a) can be proved in a different way. If Λ is a jacobian map, then it transforms locally the four systems (H_f), (H_g), (H_m) and (H_a) into the systems

$$(L_f) \quad \begin{cases} \dot f = 0, \\ \dot g = -1, \end{cases} \qquad (L_g) \quad \begin{cases} \dot f = 1, \\ \dot g = 0, \end{cases}$$

$$(L_m) \quad \begin{cases} \dot f = g, \\ \dot g = -f, \end{cases} \qquad (L_a) \quad \begin{cases} \dot f = \dfrac{f}{f^2+g^2}, \\ \dot g = \dfrac{g}{f^2+g^2}. \end{cases}$$

The commutativity of (L_f) and (L_g) is evident. Since the commutativity is invariant under a change of variables, this entails the commutativity of (H_f) and (H_g). Similarly for the couples (L_m), (L_a), and (H_m), (H_a).

Incidentally, we observe that Λ is a local *rectification* for (H_f) and (H_g), and a local *linearization* for (H_m) and (H_a) (see [I] for details about rectifications and linearizations). If Λ is injective, then it is a global rectification (respectively, a global linearization).

Given a plane differential system with a singular point O, we say that O is a *centre* if it has a punctured neighbourhood filled with cycles. Let us denote by N_O the largest open connected set covered with cycles surrounding O. We say that a centre is *isochronous* if every cycle contained in N_O has the same minimal period. In [S3], a different terminology was used: isochronous centres were named *totally isochronous centres*.

The fact that Λ is a local linearization was used in [S3] to prove that the origin O is an isochronous centre of (H_m). In [S3, Corollary 2.2] it was proved that the restriction of Λ to N_O is injective. Since the central region of a hamiltonian isochronous centre is unbounded (see [S3], Lemma 2.1), this means that every point of the plane belongs to an unbounded injectivity region (one can replace $\Lambda(z)$ with $\Lambda(z) - \Lambda(z_0)$, so that every conclusion concerning N_O holds as well for N_{z_0}, where $z_0 \in \mathbb{R}^2$). However, if Λ is not injective, all such central regions have finite area, as shown in the next corollary. We recall that m is a first integral of (V_m): it takes constant value on every orbit γ of (V_m). If γ is an orbit of (V_m), we denote by m_γ the value of m at the points of γ. If ∂N_O is nonempty, we write $m_{\partial N_O}$ for the value of m on the boundary of N_O.

THEOREM 1. *Let $\Lambda \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ be a jacobian map with* $\det J_\Lambda \equiv 1$. *Then*

(1) *if γ is a cycle contained in N_O, then the area of the region bounded by γ is $2\pi m_\gamma$;*

(2) *if $N_O \neq \mathbb{R}^2$, then the area of N_O is $2\pi m_{\partial N_O}$;*

(3) *if Λ is a polynomial map, then it is invertible if and only if the area of N_0 is infinite.*

Proof. (1) If γ is a cycle in N_O, then Λ transforms γ into the circle of equation $f^2 + g^2 = 2m_\gamma$ in the (f, g)-plane. Λ preserves area, since its jacobian determinant is 1, so that the area of the region bounded by γ coincides with the area of the circle $\{(f, g) \in \mathbb{R}^2 : f^2 + g^2 \leq 2m_\gamma\}$. Hence the area of the region bounded by γ is $2\pi m_\gamma$.

(2) Now, if $N_O \neq \mathbb{R}^2$, then the boundary ∂N_O is nonempty. The hamiltonian m takes constant value $m_{\partial N_O}$ on the boundary. Then one can repeat the above argument, observing that $\Lambda(N_O)$ is contained in a circle of radius $\sqrt{2m_{\partial N_O}}$, and contains every circle of smaller radius. Hence the area of N_O is $2\pi m_{\partial N_O}$.

(3) Finally, if Λ is polynomial, by [S3, Theorem 2.3] it is invertible if and only if $N_O = \mathbb{R}^2$. Hence, if Λ is invertible, then $N_O = \mathbb{R}^2$, and the area of N_O is infinite.

Conversely, if Λ is not invertible, we can repeat the argument of the previous point, showing that the area of N_O is finite. ∎

REMARK 2. As a consequence, when Λ is polynomial, the area of N_O is finite if and only if the area of N_{z_0} is finite for every $z_0 \in \mathbb{R}^2$.

The argument of Remark 1 suggests to consider a wider class of systems, for which one can prove commutativity by considering the action of the map Λ. Let us consider the systems

$$(H_f^d)\quad \begin{cases} \dot x = f_y/d, \\ \dot y = -f_x/d, \end{cases} \qquad (H_g^d)\quad \begin{cases} \dot x = g_y/d, \\ \dot y = -g_x/d, \end{cases}$$

$$(H_m^d)\quad \begin{cases} \dot x = \dfrac{ff_y + gg_y}{d}, \\[2ex] \dot y = -\dfrac{ff_x + gg_x}{d}, \end{cases} \qquad (H_a^d)\quad \begin{cases} \dot x = \dfrac{fg_y - gf_y}{(f^2+g^2)d}, \\[2ex] \dot y = \dfrac{-fg_x + gf_x}{(f^2+g^2)d}. \end{cases}$$

We denote by V_f^d, V_g^d, V_m^d, V_a^d the corresponding vector fields.

Lemma 4. *Let $\Lambda \in C^2(\mathbb{R}^2,\mathbb{R}^2)$ be a nonsingular map. Then (H_f^d) and (H_g^d) are nontrivial commutators, as are (H_m^d) and (H_a^d).*

Proof. The map Λ transforms (H_f^d), (H_g^d), (H_m^d), (H_a^d) respectively into the systems (L_f), (L_g), (L_m), (L_a). Hence (H_f^d) commutes with (H_g^d) and (H_m^d) commutes with (H_a^d). Since Λ preserves transversality, both couples are couples of nontrivial commutators. ∎

In the next theorem we show how commutativity can be used in order to prove the injectivity of Λ. First we prove a lemma that extends Corollary 2.3 of [S3].

Lemma 5. *Let $\Lambda \in C^2(\mathbb{R}^2,\mathbb{R}^2)$ be a nonsingular map. Then*

(1) *Λ is injective on the central region of (H_m^d) surrounding the origin*;
(2) *every point of the plane belongs to an unbounded region of injectivity.*

Proof. (1) Λ transforms (H_m^d) into (L_m). Then one can apply the same argument as in [S3, Corollary 2.2].

(2) In order to prove this point, let us recall that the orbits of (H_m^d) are the same as those of (H_m). The origin O is a centre of commuting (H_m^d) and (H_m), with the same central region N_O. By [S4, Theorem 2], O is an isochronous centre of (H_m^d), hence ∂N_O contains no singular points of (H_m^d). Assume, by absurd, that N_O is bounded. Then also ∂N_O is bounded, and one can repeat the argument of [S3, Lemma 2.1] in order to prove that this is not compatible with (H_m) being a hamiltonian system.

Hence N_O is unbounded, and by point (1), Λ is injective on N_O. ∎

Theorem 2. *Let $\Lambda \in C^2(\mathbb{R}^2,\mathbb{R}^2)$ be a nonsingular map. If one of the vector fields V_f^d, V_g^d, V_m^d is complete, then Λ is injective.*

Proof. Let us assume V_f^d to be complete. Then, by Theorem 2.1 of [S2], the flow of (H_f^d) is parallelizable. Since such a flow has the same orbits as that of (H_f), by Theorem 2.1 of [S1], the map is injective.

A similar reasoning applies if V_g^d is complete.

Now, assume V_m^d to be complete. The origin is an isochronous centre of (H_m^d), and by Theorem 2.2 in [S2], the central region of the origin is the whole plane. The system (H_m) has the same orbits as (H_m^d), hence also the central region of the origin, with respect to (H_m), is the whole plane. Then the conclusion comes from Lemma 5. ∎

In [MO, Corollary 1′], the authors show that if Λ is polynomial and nonsingular, then the completeness of V_f^d and V_g^d implies the injectivity of Λ. Theorem 2 extends their result to the case of nonpolynomial maps, assuming the completeness of just one system.

Let us now consider the family of systems

$$(H_a^{dn}) \qquad \begin{cases} \dot{x} = \dfrac{fg_y - gf_y}{(f^2+g^2)^n d}, \\ \dot{y} = \dfrac{-fg_x + gf_x}{(f^2+g^2)^n d}, \end{cases}$$

where n is an integer. We denote by V_a^{dn} the corresponding vector fields. If Λ is a polynomial jacobian map, and $n \le 0$, then (H_a^{dn}) is a polynomial system. If Λ is a polynomial map, then there exist infinitely many $n \ge 0$ such that (H_a^{dn}) is unbounded at the origin. Hence there could exist solutions of (H_a^{dn}) which are not complete. That is why in the next theorem we take into account only positive completeness.

Theorem 3. *Let $\Lambda \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ be a nonsingular map. If there exists n_Λ such that $(H_a^{dn_\Lambda})$ is positively complete, then Λ is injective.*

Proof. In this proof we denote by $\phi(t,z)$ and $\psi(s,z)$ the local flows defined by $(H_a^{dn_\Lambda})$ and, respectively, (H_m^d).

The map Λ transforms $(H_a^{dn_\Lambda})$ into the system

$$(L_a^{n_\Lambda}) \qquad \begin{cases} \dot{f} = \dfrac{f}{(f^2+g^2)^{n_\Lambda}}, \\ \dot{g} = \dfrac{g}{(f^2+g^2)^{n_\Lambda}} \end{cases}$$

that has a negatively asymptotically stable point at O. Let us observe that, for every integer n, the vector field V_a^{dn} commutes with V_m^d. In fact, by Lemma 2 of [S4], multiplying a commutator of V_m^d (in this case, V_a^d) by a first integral of V_m^d (in this case, an integer power of f^2+g^2) one gets another commutator of V_m^d.

By assumption, $\phi(t,z)$ exists for all $t \ge 0$. We want to prove that V_m^d is complete, so that O is a global centre for (H_m^d), and, by Lemma 5, Λ is injective. By absurd, assume V_m^d to be noncomplete, so that the solutions of (H_m^d) are not all periodic. As a consequence, $N_O \ne \mathbb{R}^2$, $\partial N_O \ne \emptyset$ and ∂N_O is unbounded. Let us choose arbitrarily a point $z^* \in \partial N_O$. Let δ be the connected component of ∂N_O passing through z^*. The solution $\psi(s,z^*)$

cannot exist for all $s \in \mathbb{R}$, otherwise, by continuity, ∂N_O would be a cycle, contradicting the unboundedness of the central region. Let $\eta > 0$ be such that $\phi(-\eta, z^*)$ exists, and γ be the cycle passing through the point $u^* = \phi(-\eta, z^*)$. By commutativity, for all s such that $\psi(s, z^*)$ exists, one has $\psi(s, u^*) = \psi(s, \phi(-\eta, z^*)) = \phi(-\eta, \psi(s, z^*))$. That is, the solutions of $(H_a^{dn_\Lambda})$ starting at a point of δ reach γ simultaneously, all taking the time $-\eta$. Similarly, if one considers the vector field $-V_a^{dn_\Lambda}$, the solutions of $-V_a^{dn_\Lambda}$ starting at a point of γ reach simultaneously, at $t = \eta$, the boundary ∂N_O.

Set

$$\sigma := \sup\{s \in \mathbb{R} : \psi(s, z^*) \text{ exists}\} > 0.$$

Consider the $V_a^{dn_\Lambda}$-orbit $\phi(t, \psi(\sigma, u^*))$. We claim that it does not meet ∂N_O. In fact, if $\phi(t, \psi(\sigma, u^*))$ meets δ, we should have $\phi(\eta, \psi(\sigma, u^*)) = \psi(s, z^*)$, for some $s \in (0, \sigma)$. But the $V_a^{dn_\Lambda}$-orbit starting at $\psi(s, z^*)$ meets γ at the point $\phi(-\eta, \psi(s, z^*)) = \psi(s, \phi(-\eta, z^*)) = \psi(s, u^*) \neq \psi(\sigma, u^*)$. This would contradict the uniqueness of solutions.

On the other hand, if $\phi(t, \psi(\sigma, u^*))$ meets ∂N_O outside δ, then by the continuous dependence on initial data, all the $V_a^{dn_\Lambda}$-orbits starting close enough to $\psi(\sigma, u^*)$ should do the same, while all those starting at $\psi(s, u^*)$, for $s \in [0, \sigma)$, meet δ.

Concluding, the $V_a^{dn_\Lambda}$-orbit starting at $\psi(\sigma, u^*)$ meets every cycle of the central region, but does not meet ∂N_O. This implies that the $V_a^{dn_\Lambda}$-solution starting at $\psi(\sigma, u^*)$ goes to infinity in a finite time:

$$\lim_{t \to \eta^-} |\phi(t, \psi(\sigma, u^*))| = \infty,$$

contradicting the existence for all $t \geq 0$ of the solutions of $(H_a^{dn_\Lambda})$. ∎

Corollary 1. *Let $\Lambda \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ be a nonsingular map. If there exist a compact set Ω, an integer n_Λ and $h \in \mathbb{R}$ such that for all z outside Ω,*

$$\text{(I)} \qquad (fg_y - gf_y)^2 + (-fg_x + gf_x)^2 \leq h_\Lambda (x^2 + y^2)(f^2 + g^2)^{2n_\Lambda} d^2,$$

then Λ is injective.

Proof. Since the jacobian matrix of Λ is nonsingular, the critical points of $(H_a^{dn_\Lambda})$ coincide with the zeros of $|\Lambda|$. As in the proof of Theorem 3, we can prove that all such points are repellers. Moreover, under the condition (I), the vector field of system $(H_a^{dn_\Lambda})$ is sublinear outside Ω, hence its solutions exist for all $t \geq 0$. Then one can apply Theorem 3. ∎

Remark 3. Corollary 1 provides a new possible approach to the Jacobian Conjecture. The form of inequality (I) suggests that it could be true for a wide class of polynomial maps. In fact, if the degree of Λ is k, then the degree of the right hand side is $4kn_\Lambda + 2$, while the degree of the left hand side is $4k-2$. On the other hand, *a priori* the function f^2+g^2 is not necessarily coercive, so that the possibility to choose the integer n_Λ is not sufficient to entail the validity of (I).

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Dipartimento di Matematica
Università di Trento
I-38050 Povo (TN), Italy
E-mail: sabatini@science.unitn.it

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