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Siciak's extremal function in complex and real analysis

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Abstract. The Siciak extremal function establishes an important link between polynomial approximation in several variables and pluripotential theory. This yields its numerous applications in complex and real analysis. Some of them can be found on a rich list drawn up by Klimek in his well-known monograph "Pluripotential Theory". The purpose of this paper is to supplement it by applications in constructive function theory.

The extremal function associated with a compact subset E of $\mathbb{C}^n$ is defined by the formula

$$\Phi_E(z) = \sup\{|p(z)|^{1/\deg p};\ p : \mathbb{C}^n \to \mathbb{C} \text{ is a nonconstant polynomial with } \sup |p|(E) \le 1\}, \quad z \in \mathbb{C}^n.$$

It was introduced more than forty years ago by Józef Siciak who used it to prove a multivariate version of the Bernstein–Walsh theorem on polynomial approximation of germs of holomorphic functions on compact subsets of $\mathbb{C}^n$ (Siciak 1962). It is known (Zakharyuta 1976a, Siciak 1981) that $\log \Phi_E(z) = V_E(z)$, where

$$V_E(z) := \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^n),\ u \le 0 \text{ on } E\},$$

where $\mathcal{L}(\mathbb{C}^n) = \{u \in \mathrm{PSH}(\mathbb{C}^n) : u(z) - \log|z| \le O(1) \text{ as } |z| \to \infty\}$ is the *Lelong class* of plurisubharmonic functions with logarithmic growth at infinity. If E is nonpluripolar, then the plurisubharmonic function

$$V_E^*(z) = \limsup_{w \to z} V_E(w)$$

is the unique function in the class $\mathcal{L}(\mathbb{C}^n)$ which vanishes on E except possibly on a pluripolar subset and satisfies the *complex Monge–Ampère equation*

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$(\mathrm{dd}^c V_E^*)^n = 0$ in $\mathbb{C}^n \setminus E$ (Bedford and Taylor 1982). If $n = 1$, the Monge–Ampère equation reduces to the classical *Laplace equation.* For this reason, the function V_E^* is a natural counterpart of the classical Green function with pole at infinity and it is called the *plurisubharmonic* or *pluricomplex Green function* for *E with pole at infinity.* Thus the Siciak extremal function establishes an important link between polynomial approximation in several complex variables and pluripotential theory founded by Bedford and Taylor. This suggests its numerous applications in complex and real analysis. I am not in a position to cite all of them here. However, following a comprehensive list drawn up by Maciej Klimek in his well-known monograph "Pluripotential Theory", published over ten years ago (Klimek 1991), let me cite some articles (updated by papers which have appeared during the last decade):

• *separately analytic functions* (Siciak 1969a, 1969b, 1981, 2001, Zakharyuta 1976b, Nguyen Thanh Van and Zeriahi 1991, 1995, Öktem 1998, Alehyane and Zeriahi 2001, Jarnicki and Pflug 2001, 2003a, 2003b);

• *isomorphisms of spaces of analytic functions* (Zakharyuta 1974a, 1974b, Zeriahi 1990);

• *transfinite diameter, Chebyshev constant, and Chebyshev and orthogonal polynomials in* $\mathbb{C}^n$ (Zakharyuta 1975, 1976a, Molzon and Shiffman 1982, Nguyen Thah Van 1980, Nguyen Thanh Van and Zeriahi 1983, Zeriahi 1985, Klimek 1986, Jędrzejowski 1992, Bloom 1997, 1998, Siciak 1997a, Bloom and Calvi 1999, Bloom and Levenberg 2001);

• *estimates of growth of entire functions* (Winiarski 1970, 1973, Zeriahi 1987);

• *quasianalytic functions of several variables* (Pleśniak 1971, 1972, 1973, 1977, 1981);

• *lacunary power series in* $\mathbb{C}^n$ *and sets of convergence of series of homogeneous polynomials* (Siciak 1981);

• *Leja's Polynomial Condition, uniformly bounded families of polynomials and determining measures* (Nguyen Thanh Van 1975, 1985, Siciak 1971, 1981, 1988, Cegrell 1980, Pleśniak 1980b, 1980c, 1995, Klimek 1982, Nguyen Than Van and Zeriahi 1983, Nguyen Thanh Van and Pleśniak 1984, Levenberg 1985);

• *distribution of the zeros of polynomials of best approximation* (Pleśniak 1980a, Gilewicz and Pleśniak 1993, Wójcik 1988, Bloom and Szczepański 1999, Hécart 1999, 2000).

The above list is a testimony of the power of Siciak's extremal function in complex analysis. The purpose of my lecture is to supplement it by some applications in Constructive Function Theory. My presentation does not aim at being complete and/or objective. Instead, I would like to concentrate on problems that have been investigated by myself and/or by my co-workers.

That the Siciak extremal function is especially useful in problems connected with polynomial approximation in $\mathbb{C}^n$ can be attributed to the fact that the continuity of Φ_E in $\mathbb{C}^n$ is equivalent to the *Bernstein–Walsh type inequality*:

For each $b > 1$ *there exists a neighbourhood* U *of* E *and a constant* $M > 0$ *such that*

$$\text{(BWI)} \qquad \sup |p|(U) \le M b^{\deg p} \sup |p|(E)$$

for each polynomial $p \in \mathbb{C}[z]$ (Zakharyuta 1976a, Siciak 1981).

Moreover if E is determining for germs of holomorphic functions on E, then the continuity of Φ_E is equivalent to the following *analytic extension property*:

If $f : E \to \mathbb{C}$ *and*

$$\limsup_{n\to\infty} (\inf\{\sup |f - p|(E) : p \text{ is a polynomial of degree } \le n\})^{1/n} < 1,$$

then f *extends to a holomorphic function defined on a neighbourhood of* E (Baouendi–Goulaouic 1974, the case where $E \subset \mathbb{R}^n$; Nguyen Thanh Van and Siciak 1974, the general case).

As mentioned above, the crucial point for applications is to establish the continuity of the function Φ_E. In such a case the set E is said to be *L-regular*. The space $\mathcal{R}$ of all compact, polynomially convex L-regular subsets of $\mathbb{C}^n$ is a complete metric space if the distance between two sets $E, F \in \mathcal{R}$ is defined by

$$\Gamma(E,F) := \max\{\sup V_E(F), \sup V_F(E)\} = \sup |V_E - V_F|(\mathbb{C}^n)$$

(Klimek 1995). Thanks to the product formula

$$\Phi_{E\times F}(z,w) = \max\{\Phi_E(z), \Phi_F(w)\} \quad \text{for } (z,w) \in \mathbb{C}^n \times \mathbb{C}^m,$$

where $E \subset \mathbb{C}^n$ and $F \subset \mathbb{C}^m$ (Siciak 1962), and due to the simple fact that $\Phi_E \le \Phi_F$ if $F \subset E \subset \mathbb{C}^n$, one can infer some continuity criteria from the classical one-dimensional case. Let me also recall here another useful result usually called the *analytic accessibility criterion*:

Given $a \in E$, *suppose there exists an analytic map* $h : [0,1] \to E$ *such that* $h(0) = a$. *If for each* $t \in (0,1]$ *the function* Φ_E *is continuous at* $h(t)$ *then* Φ_E *is also continuous at* a (Sadullaev 1980, Pleśniak 1980e, 1984a, Cegrell 1985).

L-regularity is invariant under nondegenerate holomorphic maps. More precisely:

If $h : \mathbb{C}^n \supset E \to \mathbb{C}^m$ $(m \le n)$ *is a germ of a nondegenerate holomorphic map*, *where* E *is compact*, *polynomially convex and* L*-regular*, *then the function* $\Phi_{h(E)}$ *is continuous in* $\mathbb{C}^m$ (Pleśniak 1978, Klimek 1981, 1982, 2001, Sadullaev 1981, Nguyen Thanh Van and Pleśniak 1984).

This invariance property gives rise to the question of whether the inverse image of an L-regular compact set E under a "good" holomorphic map h is also L-regular. Klimek (1982) proved that this is the case if both E and $h(E)$ are of the same dimension. In particular, he obtained the L-regularity of analytic polyhedrons of type $P = \{z \in \mathbb{C}^n : |h_j(z)| \le 1,\ j = 1, \ldots, n\}$. In the general case where $j = 1, \ldots, m$ (with any m), such a result was established by Pleśniak (1984b) with the aid of the analytic accessibility criterion and techniques of subanalytic geometry à la Hironaka–Łojasiewicz. More exactly:

If E *is a set in* $\mathbb{C}^n$ *which is subanalytic as a subset of* $\mathbb{R}^{2n}$, *and if* $\operatorname{int} E$ *is dense in* E, *then* E *is* (*locally*) L*-regular at every point* $a \in \overline{E}$ (Pleśniak 1984b).

Let us add that the last result can also be obtained in a more general setting of *polynomially bounded o-minimal structures* which are essential generalizations of subanalytic geometry (Pleśniak 2001).

The result about the L-regularity of subanalytic sets can be significantly strengthened with the aid of the Hironaka rectilinearization theorem and Łojasiewicz's inequality:

If U *is a nonvoid bounded subanalytic set in* $\mathbb{R}^n$ *then the extremal function* Φ_E *with* $E = \overline{U}$ *has the following Hölder Continuity Property*:

$$\text{(HCP)} \qquad \Phi_E(z) \le 1 + A\delta^m \quad \text{if } \operatorname{dist}(z, E) \le \delta \le 1,$$

where the positive constants A *and* m *do not depend on* z (Pawłucki and Pleśniak 1986).

Actually, the last statement remains valid for the much larger family of *uniformly polynomially cuspidal* subsets of $\mathbb{R}^n$ (Pawłucki and Pleśniak 1986).

As was observed a long time ago by Siciak (1967), the (HCP) property of Φ_E together with the Cauchy formula yield easily the *Markov property* of E:

For any polynomial $p \in \mathbb{C}[z]$, *one has*

$$\text{(MI)} \qquad |\operatorname{grad} p(z)| \le M(\deg p)^r \sup |p|(E) \quad \text{for } z \in E$$

with some positive constants M *and* r *that do not depend on* p.

The question of whether the converse implication is also true remains open. However, Białas-Cież (1995) and Totik (1995) showed that it holds if $n = 1$ and E is a Cantor-type compact set. Let us add that, in general, L-regularity of E does not ensure that E is Markov (Pleśniak 1990a, Totik 1995, Klimek 2001).

Markov's inequality (MI), together with the Bernstein–Walsh inequality, is one of the most fundamental tools of Constructive Function Theory. Combined with Jackson's theorem, it permits one to characterize C^∞ functions on compact subsets of $\mathbb{R}^n$ (see the Bernstein-type theorem of Pawłucki and Pleśniak (1986)). It turns out that one can also find important applications of (MI) in differential analysis. For example, it yields a relatively simple construction of a continuous linear operator extending Whitney jets of C^∞ functions or jets of ultradifferentiable functions from a compact subset of $\mathbb{R}^n$ that preserves (MI), to the whole space $\mathbb{R}^n$ (Pawłucki and Pleśniak 1988, Pleśniak 1990b, 1994, Pleśniak and Skiba 1990, Zeriahi 1993, Bos and Milman 1995, Beaugendre 2001). For other applications of Markov's inequality in differential analysis we refer the reader to Bos and Milman (1995), and Pleśniak (1998).

We have just seen that subanalytic geometry furnishes nice examples of L-regular and HCP-sets. Another natural domain yielding such examples is complex dynamics. Starting from Klimek's results establishing L-regularity of *filled-in Julia sets* in $\mathbb{C}^n$ arising from iterations of polynomial maps with *Łojasiewicz's exponent* strictly greater than 1 (Klimek 1995, 1999), Marta Kosek (1997, 1998) proved that such sets actually have the (HCP) property. She also obtained corresponding results for polynomial iterations on algebraic sets in $\mathbb{C}^n$ (Kosek 2000). Let us also mention that Siciak (1997b) gave useful sufficient conditions of Wiener type on a compact set $E \subset \mathbb{C}^n$ guaranteeing that the function Φ_E is continuous or Hölder continuous.

The notion of Siciak's extremal function extends naturally to compact subsets of analytic sets in $\mathbb{C}^n$ (see e.g. Sadullaev 1983, Zeriahi 1987, 1991, 1996, 2000). Let us recall here an important result of Sadullaev (1983) furnishing a beautiful characterization of algebraicity of analytic subsets of $\mathbb{C}^n$:

An analytic subset A of $\mathbb{C}^n$ is algebraic if and only if the extremal function Φ_E is locally bounded on A for some (*and hence for each*) *nonpluripolar compact subset E of A.*

This result is crucial for developing the theory of polynomial inequalities (of Bernstein–Walsh, Markov or van der Corput–Schaake type) on algebraic sets (Baran and Pleśniak 2000a). It also plays a fundamental role in characterizing compact pieces of an algebraic variety in $\mathbb{C}^n$ in terms of tangential Markov, Bernstein or van der Corput–Schaake inequalities (Bos, Levenberg and Taylor 1995, Bos, Levenberg, Milman and Taylor 1995, Baran and Pleśniak 1997, 2000b). Let us add that the techniques developed in Baran and Pleśniak (2000b) are based on fine bounds for Siciak's extremal function associated with a ball in $\mathbb{R}^n$ which are due to Baran (1988, 1992, 1998) and which have been inspired by Lundin (1985), who solved an old question asked by Siciak about a formula for the extremal function associated with the Euclidean unit ball in $\mathbb{R}^n$.

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