Multidimensional analogue of the
van der Corput–Visser inequality
and its application to the estimation of the Bohr radius

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Abstract. We present a multidimensional analogue of an inequality by van der Corput–Visser concerning the coefficients of a real trigonometric polynomial. As an application, we obtain an improved estimate from below of the Bohr radius for the hypercone $\mathcal{D}_1^n = \{z \in \mathbb{C}^n : |z_1| + \ldots + |z_n| < 1\}$ when $3 \leq n \leq 10$.

1. Bohr radius. The following classical result can be found in [9]: If the power series

$$\sum_{k=0}^{\infty} c_k z_1^k$$

converges in the unit disk $\{z_1 \in \mathbb{C} : |z_1| < 1\}$ and the modulus of its sum is smaller than 1, then

$$\sum_{k=0}^{\infty} |c_k z_1^k| < 1$$

in the disk $\{z_1 \in \mathbb{C} : |z_1| < 1/3\}$ and the constant $1/3$ cannot be improved. We now turn to the formulation of multidimensional analogues of this result. Let $\mathcal{D}$ be a complete Reinhardt domain. We denote by $R(\mathcal{D})$ the largest nonnegative number r with the following property: if the power series

$$\sum_{|\alpha| \geq 0} c_\alpha z^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, all α_j are nonnegative integers, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, $|\alpha| = |\alpha_1| + \ldots + |\alpha_n|$, converges in $\mathcal{D}$ and the modulus of its sum is smaller

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than 1, then

$$\sum_{|\alpha| \geq 0} |c_\alpha z^\alpha| < 1$$

in the homothety $r\mathcal{D}$. The number $R(\mathcal{D})$ is called the *Bohr radius*. We do not know of any bounded domain $\mathcal{D} \subset \mathbb{C}^n$ for which the precise value of the Bohr radius is known. However, there are a number of cases where estimates for the Bohr radius from above and from below have been obtained. Such is the case of the estimates in [8]. There, for the unit polydisk

$$U^n = \{z \in \mathbb{C}^n : |z_j| < 1, \, j = 1, \ldots, n\},$$

the following inequalities are proved for $n > 1$:

$$\frac{1}{3\sqrt{n}} < R(U^n) < \frac{2\sqrt{\log n}}{\sqrt{n}}.$$

This means that $R(U^n)$ tends to 0 as $n \to \infty$.

In [2] the following estimates for the Bohr radius of the hypercone

$$\mathcal{D}_1^n = \{z \in \mathbb{C}^n : |z_1| + \ldots + |z_n| < 1\}$$

are obtained:

(1.1) $$\frac{1}{3\sqrt[3]{e}} < R(\mathcal{D}_1^n) \leq \frac{1}{3}.$$

We point out that unlike the case of the polydisk, this Bohr radius does not tend to 0 as $n \to \infty$.

Further estimates of the Bohr radius for the domain

$$\mathcal{D}_p^n = \{z \in \mathbb{C}^n : |z_1|^p + \ldots + |z_n|^p < 1\},$$

where $1 \leq p < \infty$, can be found in [7]. Other multidimensional analogues of the Bohr theorem can be found in [3]–[6], [11].

The estimate (1.1) was significantly improved in [1] for the case $n = 2$. The main ingredient there was the following result due to J. G. van der Corput and C. Visser ([10]): Let

$$F(t) = \sum_{j=-N}^{N} a_j e^{ijt}$$

be a real trigonometric polynomial. Then

(1.2) $$|a_0| + |a_k| + |a_{-k}| \leq \max_{0 \leq t \leq 2\pi} |F(t)|$$

whenever $k > N/2$.

We remark here that the function $e^{iNt} F(t)$ is a holomorphic polynomial in e^{it}. This implies that the inequality (1.2) is significantly sharper than the Cauchy estimates.

In the present article we obtain a multidimensional analogue for the inequality (1.2), which, in turn, improves the estimate (1.1) of the Bohr radius from below for small dimensions n.

2. Multivariate van der Corput–Visser inequality. Consider the trigonometric polynomial

$$(2.1) \qquad F(t) = F(t_1, \ldots, t_n) = \sum_{|\alpha| \leq N} a_\alpha e^{i(\alpha_1 t_1 + \ldots + \alpha_n t_n)}.$$

Denote by $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ the unit vector in $\mathbb{R}^n$ with all entries zero except the jth one which is 1. Now, we state the following

THEOREM 2.1. *If the trigonometric polynomial* (2.1) *is real, then*

$$(2.2) \qquad |a_0| + \sum_{j=1}^{n}(|a_{Ne_j}| + |a_{-Ne_j}|) \leq \max_{\substack{0 \leq t_j \leq 2\pi \\ j=1,\ldots,n}} |F(t)|.$$

Proof. The proof is by induction. For $n = 1$ this is the inequality (1.2). Assume that (2.2) holds for $n - 1$. Then we write the polynomial (2.1) as

$$F(t) = \sum_{\alpha_n = -N}^{N} e^{i\alpha_n t} \sum_{|\alpha_1| + \ldots + |\alpha_{n-1}| \leq N - |\alpha_n|} a_\alpha e^{i(\alpha_1 t_1 + \ldots + \alpha_{n-1} t_{n-1})}.$$

Applying to $F(t)$ the one-dimensional result, we obtain

$$|a_{Ne_n}| + |a_{-Ne_n}| + \Big| \sum_{|\alpha_1| + \ldots + |\alpha_{n-1}| \leq N} a_{\alpha_1, \ldots, \alpha_{n-1}, 0} e^{i(\alpha_1 t_1 + \ldots + \alpha_{n-1} t_{n-1})} \Big|$$
$$\leq \max |F(t)|.$$

Since the trigonometric polynomial

$$\sum_{|\alpha_1| + \ldots + |\alpha_{n-1}| \leq N} a_{\alpha_1, \ldots, \alpha_{n-1}, 0} e^{i(\alpha_1 t_1 + \ldots + \alpha_{n-1} t_{n-1})}$$

is real and its modulus does not exceed

$$\max |F(t)| - |a_{Ne_n}| - |a_{-Ne_n}|,$$

we apply the result for dimension $n - 1$ and complete the proof. ∎

3. An estimate of the Bohr radius. We are now going to apply the inequality (2.2) to the estimation from below of the Bohr radius for the hypercone $\mathcal{D}_1^n$.

Consider the homogeneous polynomial of degree k in n complex variables

$$P_k(z) = \sum_{|\alpha| = k} c_\alpha z^\alpha, \qquad k = 1, 2, \ldots,$$

satisfying

(3.1) $$|P_k(z)| \leq 1, \quad z \in \overline{\mathcal{D}_1^n}.$$

For such homogeneous polynomials, define

$$A_k^n = \sup \sum_{|\alpha|=k} |c_\alpha| \, |z^\alpha|,$$

where the sup is taken not only over $z \in \mathcal{D}_1^n$ but also over all the homogeneous polynomials of degree k which satisfy (3.1).

THEOREM 3.1. *We have*

(3.2) $$A_k^n \leq \sqrt{\binom{n+k-1}{k} - n\sqrt{1 - n^{-2k}(3n-2)} + n^{1-k}}, \quad k = 1,2,\dots$$

Proof. From the assumption (3.1) it follows that

$$\left| P_k\left(\frac{1}{n}, \frac{1}{n} e^{it_2}, \dots, \frac{1}{n} e^{it_n}\right) \right| \leq 1.$$

Therefore

(3.3) $$n^{2k} \left| P_k\left(\frac{1}{n}, \frac{1}{n} e^{it_2}, \dots, \frac{1}{n} e^{it_n}\right) \right|^2 \leq n^{2k}.$$

The left-hand side of (3.3) is a trigonometric polynomial in $n-1$ variables. This polynomial is obviously real and is equal to

$$\sum_{|\alpha|=k} |c_\alpha|^2 + \sum_{j=2}^{n} (\bar{c}_{ke_1} c_{ke_j} e^{ikt_j} + c_{ke_1} \bar{c}_{ke_j} e^{-ikt_j})$$
$$+ \text{ terms with at least two nonzero indices.}$$

Applying (2.2) we deduce that

(3.4) $$\sum_{|\alpha|=k} |c_\alpha|^2 + 2|c_{ke_1}|(|c_{ke_2}| + \dots + |c_{ke_n}|) \leq n^{2k}.$$

Similarly to (3.4) one gets the other $n-1$ inequalities, in which the factor next to 2 is of the type $|c_{ke_j}|$ (i.e. k is at place j, $j = 2, \dots, n-1$). Taking the arithmetic mean of those n inequalities we obtain

(3.5) $$\sum_{\substack{|\alpha|=k \\ \alpha \in I}} |c_\alpha|^2 + \frac{n-2}{n} (|c_{ke_1}|^2 + \dots + |c_{ke_n}|^2)$$
$$+ \frac{2}{n} (|c_{ke_1}| + \dots + |c_{ke_n}|)^2 \leq n^{2k},$$

where $I = \{\alpha \in \mathbb{N}^n : \alpha_j \neq k, \, j = 1, \dots, n\}$. The inequality (3.5) and a

classical inequality imply that

$$(3.6) \qquad \sum_{\substack{|\alpha|=k \\ \alpha \in I}} |c_\alpha|^2 + \frac{3n-2}{n^2} \left(|c_{ke_1}| + \ldots + |c_{ke_n}| \right)^2 \le n^{2k}.$$

Note that we pass from (3.5) to (3.6) in order to make the subsequent calculations easier. Proceeding with (3.5) cannot improve the final result.

Furthermore, one observes that the modulus of the polynomial $P_k(z)$ does not exceed 1 on the sections of the hypercone by coordinate hyperplanes $\{ z \in \mathbb{C}^n : z_l = 0, \, l = 1, \ldots, j-1, j+1, \ldots, n \}$, $j = 1, \ldots, n$. That is, in the disks $\{ z_j \in \mathbb{C} : |z_j| < 1 \}$ one has $|P_k(z)| \le 1$. Therefore

$$(3.7) \qquad |c_{ke_j}| \le 1, \qquad j = 1, \ldots, n.$$

Now, we deduce that

$$A_k^n \le B_k^n = \sup \sum_{|\alpha|=k} |c_\alpha| x^\alpha,$$

where the sup is taken over all $(x_1, \ldots, x_n)$ with $x_j \ge 0$, $j = 1, \ldots, n$, and $x_1 + \ldots + x_n = 1$ and over all $|c_\alpha|$ satisfying (3.6) and (3.7) at the same time.

Since B_k^n is homogeneous with respect to $|c_\alpha|$, we may consider

$$\sum_{\substack{|\alpha|=k \\ \alpha \in I}} |c_\alpha|^2 + \frac{3n-2}{n^2} A^2 = n^{2k}$$

instead of (3.6), where $A = \sum_{j=1}^n |c_{ke_j}|$. Ignoring (3.7) for a moment, we arrive at a conditional extremum problem with respect to the variables $|c_\alpha|$, $\alpha \in I$, and x. It can be easily solved by using the Lagrange multiplier method. First, the maximum is achieved for $x_1 = \ldots = x_n = 1/n$, and thus $x^\alpha = n^{-k}$, and, secondly, all $|c_\alpha|$, $\alpha \in I$, are equal. Since the number of these $|c_\alpha|$ is $\binom{n+k-1}{k} - n$ (see, e.g., [12, Ch. IV, §2]), we obtain

$$B_k^n = n^{-k} \sqrt{\binom{n+k-1}{k} - n} \sqrt{n^{2k} - \frac{3n-2}{n^2} A^2} + n^{-k} A.$$

It is a routine matter to show that B_k^n increases in A. Recalling (3.7), we take $A = n$ and readily arrive at (3.2). This completes the proof of the theorem. ∎

4. Applications. In the proof of Theorem 9 in [2] it was shown that

$$(4.1) \qquad A_k^n \le k^k / k!.$$

The advantage of the estimate (4.1) is that it is independent of the number n of independent variables. At the same time, (3.2) for small n sometimes gives a better bound than (4.1), for example, for $n = 2$ and $k = 2, 3, 4, 5$, even

though there are cases where the converse is true: if $n = 4$ and $k = 2$, then (4.1) gives the bound of 2 while (3.2) gives the bound of 2.6511. Therefore the following corollary might be of importance.

COROLLARY 4.1.

$$(4.2) \qquad A_k^n \le \min\left(\sqrt{\binom{n+k-1}{k}} - n \sqrt{1 - n^{-2k}(3n-2)} + n^{1-k}, \frac{k^k}{k!} \right).$$

This result allows one in some cases to improve the estimate from below in (1.1), that is,

$$(4.3) \qquad R(\mathcal{D}_1^n) > \frac{1}{3\sqrt[3]{e}}.$$

The right-hand side is approximately equal to 0.238844. We remark here that in [8] the following estimate was obtained for $n > 1$:

$$R(U^n) > \frac{2}{5\sqrt{n}}$$

from which the same estimate holds for any complete Reinhardt domain, in particular, for the hypercone:

$$(4.4) \qquad R(\mathcal{D}_1^n) > \frac{2}{5\sqrt{n}}.$$

The estimate (4.4) is better than (4.3) for $n = 2$, but weaker if $n > 2$. With the help of inequality (1.2), in [1] the estimate (3.2) is given for $n = 2$:

$$R(\mathcal{D}_1^2) > 0.304236,$$

which is sharper than (4.3) and (4.4).

Similarly, we will now use Theorem 3.1 in order to improve, for some cases when $n > 2$, the estimate (4.3) (and therefore (4.4) since (4.3) is better than (4.4) for $n > 2$).

COROLLARY 4.2. *The following estimates hold*:

$$R(\mathcal{D}_1^3) > 0.271114, \qquad R(\mathcal{D}_1^4) > 0.258685, \qquad R(\mathcal{D}_1^5) > 0.251975,$$
$$R(\mathcal{D}_1^6) > 0.247485, \qquad R(\mathcal{D}_1^7) > 0.245426, \qquad R(\mathcal{D}_1^8) > 0.243525,$$
$$R(\mathcal{D}_1^9) > 0.242473, \qquad R(\mathcal{D}_1^{10}) > 0.241522.$$

Proof. Let $f(z) = \sum_{|\alpha| \ge 0} c_\alpha z^\alpha$ be a holomorphic function in the hypercone $\mathcal{D}_1^n$ with $|f(z)| < 1$ for $z \in \mathcal{D}_1^n$. Then, as shown in [1], for $k \ge 1$ we have the Cauchy–Landau type estimate

$$\left| \sum_{|\alpha|=k} c_\alpha z^\alpha \right| < 2(1 - |c_{0,\dots,0}|), \qquad z \in \mathcal{D}_1^n.$$

Furthermore, using Corollary 4.1, one has

$$\sum_{\substack{|\alpha|\geq 0 \\ |z_1|+\ldots+|z_n|\leq x}} |c_\alpha z^\alpha| < |c_{0,\ldots,0}| + 2(1 - |c_{0,\ldots,0}|) \sum_{k=1}^{\infty} C_k^n x^k,$$

where C_k^n denotes the right-hand side in (4.2). Hence it can be readily seen that $R(\mathcal{D}_1^n) \geq x_k$, where x_k is the root of the equation

$$(4.5) \qquad \sum_{k=1}^{\infty} C_k^n x^k = 1/2.$$

After solving numerically the equation (4.5) for $n = 3, \ldots, 10$ by using MAPLE we find the results claimed in the statement of Corollary 4.2. $\blacksquare$

Concluding, we remark that for sufficiently large n the min in (4.2) is equal to the right-hand side of (4.1) and we again return to the earlier estimate (4.3).

REMARK 4.1. The bounds of the Bohr radius from Corollary 4.2 also hold for every Reinhardt domain of the type

$$D = \{z \in \mathbb{C}^n : \Phi(|z_1|, \ldots, |z_n|) < 0\},$$

where Φ is a convex function, that is, D is a union of hypercones

$$\{z \in \mathbb{C}^n : a_1|z_1| + \ldots + a_n|z_n| < 1\}.$$

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