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Robin functions and extremal functions

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Abstract. Given a compact set $K \subset \mathbb{C}^N$, for each positive integer n, let

$$V^{(n)}(z) = V_K^{(n)}(z) := \sup\left\{\frac{1}{\deg p} V_{p(K)}(p(z)) : p \text{ holomorphic polynomial}, 1 \le \deg p \le n\right\}.$$

These "extremal-like" functions $V_K^{(n)}$ are essentially one-variable in nature and always increase to the "true" several-variable (Siciak) extremal function,

$$V_K(z) := \max\left[0, \sup\left\{\frac{1}{\deg p}\log|p(z)| : p \text{ holomorphic polynomial}, \|p\|_K \le 1\right\}\right].$$

Our main result is that if K is regular, then all of the functions $V_K^{(n)}$ are continuous; and their associated Robin functions

$$\varrho_{V_K^{(n)}}(z) := \limsup_{|\lambda|\to\infty}[V_K^{(n)}(\lambda z) - \log(|\lambda|)]$$

increase to $\varrho_K := \varrho_{V_K}$ for all z outside a pluripolar set.

0. Introduction. Let

$$L := \{u \text{ plurisubharmonic (psh) in } \mathbb{C}^N : u(z) \le \log^+|z| + C\}$$

denote the class of psh functions of logarithmic growth on $\mathbb{C}^N$ (here $|z| = (\sum_{i=1}^N |z_i|^2)^{1/2}$; $\log^+|z| = \max(0, \log|z|)$; and the constant C can depend on u). We also consider the class

$$L^+ := \{u \in L : \log^+|z| + C_1 \le u(z) \le \log^+|z| + C_2 \text{ for some } C_1, C_2\}.$$

These classes arise naturally in complex potential theory in $\mathbb{C}$ and in pluripotential theory in $\mathbb{C}^N$. For a bounded Borel set E in $\mathbb{C}^N$, define

$$V_E(z) := \sup\{u(z) : u \in L,\ u \le 0 \text{ on } E\}. \tag{0.1}$$

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The upper semicontinuous (usc) regularization $V_E^*(z) := \limsup_{\zeta\to z} V_E(\zeta)$ is called the (*Siciak*) *extremal function of* E. It is well known that $V_E^* \in L^+$ if and only if E is nonpluripolar; i.e., if u psh is $-\infty$ on E, then $u \equiv -\infty$. If K is a compact set in $\mathbb{C}^N$, then the extremal function in (0.1) can be obtained via the formula

$$(0.2)\quad V_K(z) := \max\left[0, \sup\left\{\frac{1}{\deg p}\log|p(z)| : p \text{ holomorphic polynomial, } \|p\|_K \le 1\right\}\right]$$

([K, Theorem 5.1.7]). Here, $\|p\|_K := \sup_{z\in K}|p(z)|$ denotes the uniform norm on K.

To study the asymptotic behavior of such functions, we recall the notion of the Robin function associated to a function $u \in L$. First of all, suppose that $K \subset \mathbb{C}^N$ is compact and regular, i.e., $V_K = V_K^*$ (equivalently, V_K is continuous). The *Robin function* of K is $\varrho_K : \mathbb{C}^N \to \mathbb{R}\cup\{-\infty\}$ defined by

$$\varrho_K(z) := \limsup_{|\lambda|\to\infty}[V_K(\lambda z) - \log(|\lambda|)].$$

More generally, for $u : \mathbb{C}^N \to \mathbb{R}$ in L we define the *Robin function* of u to be

$$(0.3)\qquad \varrho_u(z) := \limsup_{|\lambda|\to\infty}[u(\lambda z) - \log(|\lambda|)]$$

(hence $\varrho_K = \varrho_{V_K}$). Note that for $\lambda \in \mathbb{C}$, $\varrho_u(\lambda z) = \log|\lambda| + \varrho_u(z)$ (logarithmic homogeneity; cf. Section 5). It is known [Bl] that for $u \in L$, the Robin function $\varrho_u(z)$ is plurisubharmonic in $\mathbb{C}^N$; indeed, either $\varrho_u \in L$ or $\varrho_u \equiv -\infty$.

Our aim in this note is two-fold: first, we discuss the Robin function $\varrho_u(z)$—more precisely, the *Robin constant*—associated to a function $u \in L$ in one complex variable. Using these results, we then analyze the Robin function associated to certain "extremal-like" functions associated to a compact set $K \subset \mathbb{C}^N$, $N > 1$. For each positive integer n, let

$$V^{(n)}(z) = V_K^{(n)}(z) := \sup\left\{\frac{1}{\deg p}V_{p(K)}(p(z)) : p \text{ holomorphic polynomial, } 1 \le \deg p \le n\right\}.$$

These functions $V_K^{(n)}$ (discussed in [BCL]) are essentially one-variable in nature and always increase to the "true" extremal function, V_K. Our main result is that if K is regular, then all of the functions $V_K^{(n)}$ are continuous. Concerning their associated Robin functions $\varrho_{V_K^{(n)}}$, we show that $\varrho_{V_K^{(1)}}$ is also continuous and, in this case, the lim sup in (0.3) can be replaced by limit; i.e., the limit exists. Moreover,

$$\lim_{n\to\infty} \varrho_{V^{(n)}}(z) = \varrho_K(z) \tag{0.4}$$

for q.e. $z \in \mathbb{C}^N$ (i.e., all z outside a pluripolar set). We mention that (0.4) is not an immediate consequence of the monotone convergence of the functions $V_K^{(n)}$ to the function V_K; indeed, a necessary and sufficient condition for (0.4) to hold involves the Monge–Ampère measures of these functions (cf. [BT]); this condition is usually difficult to verify. We end with some open questions related to the notions in this and the [BCL] paper.

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1. Subharmonic functions in $\mathbb{C}$. In this section, we work exclusively in $\mathbb{C}$. The major question we want to address is the following: for which functions $u \in L$ does the *limit*

$$\lim_{|t|\to\infty} [u(t) - \log|t|] \tag{1.0}$$

exist? We first discuss some known results about subharmonic functions in $\mathbb{C}$. Let μ be a nonnegative Borel measure on $\mathbb{C}$ of finite total mass. Under what conditions is μ the Laplacian of a function in L?

PROPOSITION 1.1. *Suppose that* $\int_{|t|\le 1} \log|t|\, d\mu(t) > -\infty$. *Then* $\mu(1) := \int_{t\in\mathbb{C}} d\mu(t) \le 1$ *if and only if*

$$u(z) := \int_{t\in\mathbb{C}} [\log|z-t| - \log|t|]\, d\mu(t) \tag{1.1}$$

belongs to L.

REMARK. From Brelot's theorem (cf. [R]), it follows that if $\mu(1) < \infty$ and

$$\int_{|t|\le 1} \log|t|\, d\mu(t) > -\infty,$$

then

$$u(z) := \int_{t\in\mathbb{C}} [\log|z-t| - \log|t|]\, d\mu(t)$$

is a subharmonic (shm) function in $\mathbb{C}$.

Proof of Proposition 1.1. Introduce the notation $n(r) := \int_{|t|\le r} d\mu(t)$. We first recall Jensen's formula: let u be shm in the disk $\{z : |z| < R\}$ and harmonic in a neighborhood of the origin. Then for any $r < R$,

$$M_u(r) := \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta = u(0) + \int_0^r \frac{n(t)}{t}\,dt \tag{1.2}$$
$$= u(0) + \int_0^r [\log r - \log|t|]\,dn(t).$$

Suppose u as in (1.1) is in L but $\mu(1) = \alpha > 1$. Without loss of generality we may assume $u(0) = 0$. Since $u \in L$,

$$\lim_{r\to\infty} [M_u(r) - \log r] < \infty. \tag{1.3}$$

Fix $0 < \beta < 1 - 1/\alpha$. Then, since $\lim_{t\to\infty} n(t) = \alpha$, there exists $r_0 > 1$ such that for all $r > r_0$,

$$(1-\beta)n(r^\beta) =: 1 + \delta > 1.$$

But

$$\begin{aligned} M_u(r) - \log r &= \int_0^r [\log r - \log|t|]\,dn(t) - \log r \\ &\ge \int_0^{r^\beta} [\log r - \log|t|]\,dn(t) - \log r \\ &\ge (1-\beta)(\log r)n(r^\beta) - \log r = \delta \log r, \end{aligned}$$

which contradicts (1.3).

For the converse, we may assume $\mu(1) = 1$. We want to find a constant C such that

$$u(z) := \int_{t\in\mathbb{C}} [\log|z-t| - \log|t|]\,d\mu(t) \le C + \log|z|$$

for all $|z| \ge 1$. Fix such a z and write

$$\begin{aligned} u(z) = &\int_{|t|\le 1} \log\left|1 - \frac{z}{t}\right| d\mu(t) \\ &+ \int_{|t|>1,\,|z|<|t|} \log\left|1 - \frac{z}{t}\right| d\mu(t) + \int_{|t|>1,\,|z|\ge|t|} \log\left|1 - \frac{z}{t}\right| d\mu(t) \\ &\le [n(1)\log|z| + c_1] + [(1-n(1))\log 2] + \int_{|t|>1,\,|z|\ge|t|} \log\frac{2|z|}{|t|}\,d\mu(t) \\ &\le [n(1)\log|z| + c_1] + [(1-n(1))\log 2] + \int_{|t|>1} \log 2|z|\,d\mu(t) \\ &= [n(1)\log|z| + c_1] + [(1-n(1))\log 2] + [1-n(1)]\log 2|z| \\ &= \log|z| + C. \ \blacksquare \end{aligned}$$

It follows from Proposition 1.1 that if $u \in L$ and $u(0) = 0$ then u can be written as in (1.1) with

$$d\mu(t) = \frac{-1}{4\pi i}\, \Delta u(t) dt \wedge d\bar{t} \quad \text{and} \quad \mu(1) \le 1.$$

The fact that $\mu(1) \le 1$ for $u \in L$ follows easily from Jensen's formula (1.2): if $\mu(1) > 1$, then there exist $\delta > 0$ and r_0 such that for all $r > r_0$, we have $n(r) \ge 1 + \delta$. Fixing such an r, we obtain

$$M_u(r) - M_u(r_0) = \int_{r_0}^{r} \frac{n(t)}{t}\, dt \ge (1+\delta) \log \frac{r}{r_0}.$$

The left-hand side of this inequality is dominated by $\log(1+r)$ plus a constant—for all r—yielding a contradiction.

To show that the problem described in (1.0) is nontrivial, we begin with an explicit example of a *continuous* function $u \in L^+(\mathbb{C})$ for which the limit (1.0) does not exist.

PROPOSITION 1.2. *There exists $u \in L^+(\mathbb{C}) \cap C(\mathbb{C})$ for which $u(t) - \log|t|$ does not have a limit as $|t| \to \infty$.*

Proof. The idea is to construct a sequence of continuous subharmonic functions

$$u_j(t) := \log^+ \frac{|t - t_j|}{r_j} + \log r_j$$

in $L^+(\mathbb{C})$ having Laplacians supported on circles $|t - t_j| = r_j$ of smaller and smaller radii r_j with centers t_j marching to infinity in such a way that an infinite sum

$$u(t) := \sum_j \varepsilon_j u_j(t)$$

gives us the desired function. To make this precise, we first choose a sequence $\{t_j\}$ of positive numbers with $t_1 > 2$ and $t_j \uparrow \infty$ and with

$$2t_j \le t_{j+1} \le 4t_j \tag{1.4}$$

for all j. Next, choose a sequence $\{\varepsilon_j\}$ of positive numbers with

$$\varepsilon_j \downarrow 0, \quad \sum_j \varepsilon_j = 1, \quad \sum_j \varepsilon_j \log t_j < \infty. \tag{1.5}$$

Finally, define the sequence $\{r_j\}$ of positive numbers by

$$r_k := 1 \Big/ \Big[\prod_{j=1}^{k-1} t_j^{\varepsilon_j/\varepsilon_k} \Big]. \tag{1.6}$$

Note that by the choice of r_k in (1.6), we have

$$\varepsilon_k \log \frac{t_k}{r_k} = \sum_{j=1}^{k} \varepsilon_j \log t_j. \tag{1.7}$$

Now let

$$u_j(t) := \log^+ \frac{|t-t_j|}{r_j} + \log r_j \quad \text{and} \quad u(t) := \sum_j \varepsilon_j u_j(t)$$

as above. We show this u satisfies the conditions stated in the proposition.

(i) *u is in $L^+(\mathbb{C})$.* If $t_k \le |t| \le t_{k+1}$, then

$$|t-t_j| \le |t| + t_j \le \begin{cases} 2|t| & \text{for } j \le k, \\ 2t_j & \text{for } j > k. \end{cases}$$

Hence

$$u(t) \le \sum_{j\le k} \varepsilon_j \log 2|t| + \sum_{j>k} \varepsilon_j \log 2t_j \le \Big[\log 2 + \sum_j \varepsilon_j \log t_j\Big] + \log|t|$$

by (1.5). In the other direction, if we write

$$u(t) = \sum_{j\ne k,k+1} \varepsilon_j \log|t| + \sum_{j\ne k,k+1} \varepsilon_j \log \frac{|t-t_j|}{|t|} + \varepsilon_k u_k(t) + \varepsilon_{k+1} u_{k+1}(t)$$

and we use the estimates

$$\frac{|t-t_j|}{|t|} > \frac{|t|-t_j}{|t|} > \frac{1}{2} \quad \text{for } j<k \quad \text{since } |t| \ge t_k > 2t_j,$$

$$\frac{|t-t_j|}{|t|} > \frac{t_j-|t|}{|t|} > \frac{1}{2} \quad \text{for } j>k+1 \quad \text{since } t_j \ge 2t_{k+1} > 2|t|;$$

we obtain

$$\begin{aligned} u(t) &\ge \sum_{j\ne k,k+1} \varepsilon_j \log|t| + \log 1/2 + \varepsilon_k u_k(t) + \varepsilon_{k+1} u_{k+1}(t) \\ &\ge \log|t| + \log\frac{1}{2} + \varepsilon_k \log\frac{r_k}{|t|} + \varepsilon_{k+1}\log\frac{r_{k+1}}{|t|} \\ &\ge \log|t| + \log\frac{1}{2} + \varepsilon_k \log\frac{r_k}{4t_k} + \varepsilon_{k+1}\log\frac{r_{k+1}}{t_{k+1}} \qquad \text{(from (1.4))} \\ &\ge \log|t| + \log\frac{1}{2} - \varepsilon_k \log 4 - 2\varepsilon_{k+1}\log\frac{t_{k+1}}{r_{k+1}} \\ &\ge \log|t| - \log 8 - 2\sum_{j=1}^{k+1} \varepsilon_j \log t_j \\ &\ge \log|t| - \log 8 - 2\sum_{j=1}^{\infty} \varepsilon_j \log t_j = \log|t| + c_1, \end{aligned}$$

where the last two lines follow from (1.5) and (1.7).

(ii) *There exists* $\delta > 0$ *with* $\liminf_{k\to\infty}[u(-t_k) - u(t_k)] \geq \delta$. For

$$u(-t_k) - u(t_k) > \varepsilon_k[u_k(-t_k) - u_k(t_k)] = \varepsilon_k \log \frac{2t_k}{r_k}$$
$$> \varepsilon_k \log \frac{t_k}{r_k} = \sum_{j=1}^{k} \varepsilon_j \log t_j \quad \text{(from (1.7))}$$

and the result follows from convergence of $\sum_{j=1}^{\infty} \varepsilon_j \log t_j$ (see (1.5)).

(iii) u *is continuous on* $\mathbb{C}$. It suffices to show that the series $\sum_j \varepsilon_j u_j$ converges uniformly on compact sets in $\mathbb{C}$ since each u_j is continuous on $\mathbb{C}$. Fix $t \in \mathbb{C}$. If there exists $r > 0$ so that the disk $\Delta(t,r) := \{z : |z - t| < r\}$ avoids each of the disks $\overline{\Delta}(t_j, r_j)$, then clearly the series $\sum_j \varepsilon_j u_j$ converges uniformly to u on $\overline{\Delta}(t, r/2)$. Otherwise we can choose $r > 0$ sufficiently small so that the disk $\Delta(t,r) := \{z : |z - t| < r\}$ meets at most one of the disks $\overline{\Delta}(t_j, r_j)$, say $\Delta(t,r) \cap \Delta(t_k, r_k) \neq \emptyset$. Then for $z \in \Delta(t,r)$,

$$u(z) = \varepsilon_k \log r_k + \sum_{j=1,\ j\neq k}^{\infty} \varepsilon_j \log|z - t_j| \quad \text{if } z \in \Delta(t,r) \cap \Delta(t_k, r_k),$$

while

$$u(z) = \sum_{j=1}^{\infty} \varepsilon_j \log|z - t_j| \quad \text{if } z \in \Delta(t,r) \setminus \Delta(t_k, r_k).$$

Note that $|t - t_j| > 1$ for $j > k$; thus, for $z \in \Delta(t,r)$, we also have $|z - t_j| > 1$; hence if $N > k$ we obtain the estimate

$$\Big|u(z) - \sum_{j=1}^{N} \varepsilon_j u_j(z)\Big| = \sum_{j=N+1}^{\infty} \varepsilon_j \log|z - t_j| \leq \sum_{j=N+1}^{\infty} \varepsilon_j \log(r + |t - t_j|)$$
$$\leq \sum_{j=N+1}^{\infty} \varepsilon_j \log M|t - t_j| = \log M \sum_{j=N+1}^{\infty} \varepsilon_j + \sum_{j=N+1}^{\infty} \varepsilon_j \log|t - t_j|$$

where $M = M(r)$. Thus given $\varepsilon > 0$, we choose $N > k$ sufficiently large so that

$$\sum_{j=N+1}^{\infty} \varepsilon_j < \frac{\varepsilon}{2\log M} \quad \text{and} \quad \sum_{j=N+1}^{\infty} \varepsilon_j \log|t - t_j| < \frac{\varepsilon}{2}.$$

This yields

$$\Big|u(z) - \sum_{j=1}^{N} \varepsilon_j u_j(z)\Big| < \varepsilon$$

for all $z \in \Delta(t,r)$; i.e., the partial sums $u_N(z) := \sum_{j=1}^{N} \varepsilon_j u_j(z)$ converge uniformly to $u(z)$ on $\Delta(t,r)$. ∎

Remark. Siciak has pointed out how to construct lots of examples using facts from complex potential theory: start with a compact, nonpolar, polynomially convex set $K \subset \mathbb{C}$ such that $0 \in K$ is the only irregular point of K. Then the extremal function V_K^* belongs to $L^+(\mathbb{C})$ and is continuous on $\mathbb{C} \setminus \{0\}$. The function $u(z) := V_K^*(1/z) + \log|z|$ for $z \neq 0$ extends continuously to $z = 0$ upon setting $u(0) := \lim_{z\to 0,\, z\neq 0} u(z) = -\log \operatorname{cap} K$, where $\operatorname{cap} K$ denotes the logarithmic capacity of K, and this u provides another example of a function satisfying the criteria of Proposition 1.2. As a concrete example of such a set K, take $K := \{0\} \cup \bigcup_{k=1}^{\infty} [e^{-2\cdot 3^k}, e^{-3^k}]$.

We still recover a one-sided estimate for general functions $u \in L^+(\mathbb{C})$. We claim that we may write u as the sum of the logarithmic potential of its Laplacian plus a constant:

$$u(t) := \int \log|t-s|\, d\mu(s) + \Big[u(0) - \int \log|s|\, d\mu(s)\Big] \tag{1.8}$$

where

$$d\mu(t) = \frac{-1}{4\pi i}\, \Delta u(t) dt \wedge d\bar{t}$$

is the probability measure associated to the Laplacian $\Delta u(t)$. Recall that we defined

$$n(r) := \int_{|t|\le r} d\mu(t)$$

for $r > 0$. Since we are only concerned with asymptotic behavior of u, we may assume there exists $\delta > 0$ with $n(r) = 0$ for $r \le \delta$. The following facts follow from arguments similar to those used in Proposition 1.1:

(i) $\int \log|s|\, d\mu(s) = \int \log r\, dn(r)$ is finite;
(ii) $\lim_{r\to\infty} n(r) = 1$;
(iii) $\lim_{r\to\infty} \int_1^r (1-n(t))t^{-1}\, dt$ exists (and is finite).

The representation (1.8) follows. For simplicity we assume

$$u(0) - \int \log|s|\, d\mu(s) = 0. \tag{1.9}$$

Lemma 1.3. *Under the hypothesis* (1.9) *on* u,

$$\limsup_{|t|\to\infty} \int \log \frac{|t-s|}{|t|}\, d\mu(s) \le 0.$$

Proof. Fix $t \in \mathbb{C}$ with $|t| > 1$ and a positive integer k. We split the integral into two parts:

$$\text{if } |s| \le |t|/k, \text{ then } \frac{|t-s|}{|t|} \le \frac{|t|+|s|}{|t|} \le \frac{k+1}{k};$$

$$\text{if } |s| \ge |t|/k, \text{ then } \frac{|t-s|}{|t|} \le \frac{|t|+|s|}{|t|} \le \frac{(k+1)|s|}{|t|}.$$

Then for the first part we have

$$\int_{|s|\le |t|/k} \log \frac{|t-s|}{|t|}\, d\mu(s) \le \log\left(\frac{k+1}{k}\right) \cdot n(|t|/k) \le \log\left(\frac{k+1}{k}\right);$$

while for the second part,

$$\begin{aligned}
&\int_{|s|\ge |t|/k} \log \frac{|t-s|}{|t|}\, d\mu(s) \\
&\quad \le \int_{|s|\ge |t|/k} \log\,[(k+1)|s|]\, d\mu(s) - \int_{|s|\ge |t|/k} \log|t|\, d\mu(s) \\
&\quad = \log(k+1)\cdot[1-n(|t|/k)] + \int_{|s|\ge |t|/k} \log|s|\, d\mu(s) - \log|t|\cdot[1-n(|t|/k)] \\
&\quad \le \log(k+1)\cdot[1-n(|t|/k)] + \int_{|s|\ge |t|/k} \log|s|\, d\mu(s).
\end{aligned}$$

Using (i) and (ii) we obtain

$$\limsup_{|t|\to\infty} \int \log \frac{|t-s|}{|t|}\, d\mu(s) \le \log\left(\frac{k+1}{k}\right)$$

and the result follows. ∎

We now show that by suitably averaging the function u, we will get existence of the limit above with u replaced by this averaged version. Precisely, fix $r > 0$ and define

$$u^r(t) := \frac{1}{2\pi}\int_0^{2\pi} u(t + re^{i\theta})\, d\theta.$$

Then $u^r \in L^+(\mathbb{C}) \cap C(\mathbb{C})$ and u^r satisfies (1.9) if $r < \delta$ with

$$d\mu^r(t) = \frac{-1}{4\pi i}\, \Delta u^r(t)dt \wedge d\bar{t};$$

thus

$$\limsup_{|t|\to\infty} \int \log \frac{|t-s|}{|t|}\, d\mu^r(s) \le 0.$$

LEMMA 1.4. $\lim_{|t|\to\infty}[u^r(t) - \log|t|] = 0.$

Proof. It suffices to show

$$\liminf_{|t|\to\infty}[u^r(t) - \log|t|] \ge 0.$$

Fix $t \in \mathbb{C}$ with $|t| > 1$. For simplicity, take $r = 1$. By (ii), (iii) and Fubini's theorem, we can write

$$u^1(t) - \log|t| = \int \log^+ |t-s|\, d\mu(s) - \int \log|t|\, d\mu(s) \tag{1.10}$$
$$= \int_{|t-s|\geq 1} \log \frac{|t-s|}{|t|}\, d\mu(s) - \int_{|t-s|\leq 1} \log|t|\, d\mu(s).$$

The second term is equal to $n(t;1)\log|t|$ where $n(t;1) := \int_{|t-s|\leq 1} d\mu(s)$ is the mass of the measure μ in the disk of radius 1 centered at t; clearly

$$n(t;1) \leq n(|t|+1) - n(|t|-1)$$

so that

$$n(t;1)\log|t| \leq C \int_{|t|-1}^{|t|+1} \log r\, dn(r)$$

for some constant C which is independent of t. By (i), we see that

$$\lim_{|t|\to\infty} n(t;1)\log|t| = 0.$$

Let

$$G(t) := \int_{|t-s|\geq 1} \log \frac{|t-s|}{|t|}\, d\mu(s).$$

From (1.10), we must show $\liminf_{|t|\to\infty} G(t) \geq 0$. Clearly we need only consider the nonpositive part

$$G^-(t) := \int_{1\leq|t-s|\leq|t|} \log \frac{|t-s|}{|t|}\, d\mu(s)$$

and show that $\liminf_{|t|\to\infty} G^-(t) \geq 0$.

To this end, fix $\varepsilon > 0$ and split up $G^-(t)$ into two parts:

$$G^-(t) := \int_{1\leq|t-s|\leq|t|,\, |s|\leq\varepsilon|t|} \log \frac{|t-s|}{|t|}\, d\mu(s) \tag{1.11}$$
$$+ \int_{1\leq|t-s|\leq|t|,\, |s|\geq\varepsilon|t|} \log \frac{|t-s|}{|t|}\, d\mu(s).$$

In the first integral, we have

$$\frac{|t-s|}{|t|} \geq \frac{|t|-|s|}{|t|} \geq 1-\varepsilon$$

so that $\log(|t-s|)/|t| \geq \log(1-\varepsilon) = O(\varepsilon)$. We split up the second integral in (1.11) into two parts: one with $|s| \geq |t|$ and one with $|s| \leq |t|$. Defining

$$U(\varepsilon, t) := \{s : 1 \leq |t-s| \leq |t| \text{ and } \varepsilon|t| \leq |s| \leq |t|\},$$

for $s \in U(\varepsilon, t)$ we have

$$\frac{|t-s|}{|t|} \geq \frac{1}{|t|} \geq \frac{\varepsilon}{|s|}.$$

Thus

$$\int\limits_{U(\varepsilon,t)} \log \frac{|t-s|}{|t|}\, d\mu(s) \geq \int\limits_{U(\varepsilon,t)} \log \varepsilon\, d\mu(s) - \int\limits_{U(\varepsilon,t)} \log |s|\, d\mu(s)$$

$$\geq \log \varepsilon \cdot [n(|t|) - n(\varepsilon|t|)] - \int\limits_{\varepsilon|t|}^{|t|} \log r\, dn(r).$$

We may assume $\varepsilon|t| > 1$ since we are interested (fixing $\varepsilon > 0$) in the behavior of $G^-(t)$ for $|t|$ large. For s satisfying $|s| \geq |t|$ and $1 \leq |t-s| \leq |t|$,

$$\frac{|t-s|}{|t|} \geq \frac{1}{|t|} \geq \frac{1}{|s|}.$$

Hence

$$\int\limits_{1\leq|t-s|\leq|t|,\, |s|\geq|t|} \log \frac{|t-s|}{|t|}\, d\mu(s) \geq - \int\limits_{|s|\geq|t|} \log |s|\, d\mu(s) = - \int\limits_{r\geq|t|} \log r\, dn(r).$$

Altogether, we obtain the estimate

$$\liminf_{|t|\to\infty} G^-(t) \geq \liminf_{|t|\to\infty} \Big\{ O(\varepsilon) n(\varepsilon|t|) + \log \varepsilon \cdot [n(|t|) - n(\varepsilon|t|)]$$

$$- \int\limits_{\varepsilon|t|}^{|t|} \log r\, dn(r) - \int\limits_{r\geq|t|} \log r\, dn(r) \Big\}.$$

Again using (i) and (ii), we have

$$\liminf_{|t|\to\infty} G^-(t) \geq \liminf_{|t|\to\infty} O(\varepsilon) n(\varepsilon|t|) = O(\varepsilon)$$

and the result follows. ∎

We will use these results in Section 4 when we discuss the existence of directional limits for Robin functions ϱ_u associated to functions $u \in L$ in $\mathbb{C}^N$, $N > 1$.

2. Computing V_K using one-variable methods. This section is essentially contained in [BCL]. It contains the primary motivation for our results; for omitted proofs we refer the reader to [BCL]. Let $K \subset \mathbb{C}^N$ be compact. We recall that K is nonpluripolar as a subset of $\mathbb{C}^N$ if and only if $V_K^* \in L$ (equivalently, $V_K^* \not\equiv \infty$) and that K is regular if and only if $V_K^* = V_K$ (equivalently, V_K is continuous on $\mathbb{C}^N$). Moreover, if we let

$$\widehat{K} := \{z \in \mathbb{C}^N : |p(z_1, \ldots, z_N)| \leq \|p\|_K \text{ for all polynomials } p\}$$

denote the polynomial hull of K, then $\widehat{K} = \{z \in \mathbb{C}^N : V_K(z) = 0\}$ and $V_{\widehat{K}} = V_K$.

We recall from [BCL] how to relate the notions of regularity and (pluri-) polarity in one and several variables for K and $p(K)$ when p is a nonconstant polynomial.

LEMMA 2.1 [BCL]. *Suppose that $E \subset \mathbb{C}^N$ is a bounded Borel set and that $p : \mathbb{C}^N \to \mathbb{C}$ is a nonconstant polynomial. Then* (a) *if E is nonpluripolar, $p(E)$ is nonpolar, and* (b) *if E is a regular compact set, then $p(E)$ is regular.*

If K is compact and regular and p_d is a polynomial of degree d, then

$$\frac{1}{d} V_{p_d(K)}(p_d(z)) \le V_K(z);$$

conversely, if $\|p_d\|_K \le 1$, then $p_d(K) \subset U$, the unit disk in $\mathbb{C}$, so that $V_{p_d(K)}(w) \ge V_U(w) = \log^+(|w|)$ for all $w \in \mathbb{C}$, so $V_{p_d(K)}(p_d(z)) \ge \log^+(|p_d(z)|)$, from which it follows that

$$V_K(z) \le \sup_{p_d} \frac{1}{d} V_{p_d(K)}(p_d(z)).$$

Thus

$$V_K(z) = \sup_{p_d} \frac{1}{d} V_{p_d(K)}(p_d(z)). \tag{2.1}$$

If $d = 1$, this implies that for any complex affine function $\ell(z)$, we have

$$V_{\ell(K)}(\ell(z)) \le V_K(z).$$

Define

$$V^{(1)}(z) := \sup\{V_{\ell(K)}(\ell(z)) : \ell \in (\mathbb{C}^N)^*,\ \ell \ne 0\} \tag{2.2}$$

where $(\mathbb{C}^N)^*$ is the class of all complex-linear functionals on $\mathbb{C}^N$. Note that if we replace ℓ by a scalar multiple $t\ell$, then $V_{t\ell(K)} \circ t\ell = V_{\ell(K)} \circ \ell$. Thus considering upper envelopes over all complex-linear functionals or simply, e.g., over all linear functionals normalized to have norm 1, yields the same function $V^{(1)}$; similarly, if $\ell \in (\mathbb{C}^N)^*$ and $a \in \mathbb{C}$ is constant, then we have $V_{(\ell+a)(K)}((\ell + a)(z)) = V_{\ell(K)}(\ell(z))$. If $E \subset \mathbb{C}^N$ is a bounded Borel set, we define

$$V^{(1)}(z) := \sup\{V^*_{\ell(E)}(\ell(z)) : \ell \in (\mathbb{C}^N)^*,\ \ell \ne 0\}$$

and by [K, Corollary 5.2.5] it follows that $V^{(1)*} \le V^*_E$.

Returning to the case where K is compact and regular, note that $V^{(1)}$ is lower semicontinuous as the upper envelope of a family of continuous functions. Since we will show (Proposition 3.5) that in this setting, $V^{(1)}$ is actually continuous, it is natural to ask for the most general situation under which we have the equality $V^{(1)} = V_K$. A necessary condition is given in [BCL].

PROPOSITION 2.2 [BCL]. *Let $N > 1$. Suppose $K \subset \mathbb{C}^N$ is compact, regular, and polynomially convex* ($K = \widehat{K}$). *Define $V^{(1)}(z)$ using* (2.2). *If $V^{(1)}(z) = V_K(z)$ in $\mathbb{C}^N$, then K is lineally convex*; *i.e., the complement of K is the union of complex hyperplanes.*

For each positive integer n, we can define

$$V^{(n)}(z) = V_K^{(n)}(z) := \sup\left\{\frac{1}{\deg p} V_{p(K)}(p(z)) : 1 \le \deg p \le n\right\}.$$

Equation (2.1) shows that, for any regular compact set K, the functions $V^{(n)}$ increase monotonically to V_K; i.e.,

$$V^{(n)} \le V^{(n+1)}, \quad n = 1, 2, \ldots, \quad \text{and} \quad \lim_{n\to\infty} V^{(n)}(z) = V_K(z), \quad z \in \mathbb{C}^N.$$

We study the functions $V^{(n)}$ in the rest of the paper.

Note that if K is nonpluripolar, then $V^* := V^{(1)*}$ (and hence $V^{(n)*}$ for each $n = 1, 2, \ldots$) is in the class L^+ where

$$L^+ := \{u \in L : \log^+|z| + C_1 \le u(z) \le \log^+|z| + C_2 \text{ for some } C_1, C_2\}.$$

For it is well known that $V_K^* \in L^+$ if K is nonpluripolar; letting $\ell_j(z) = z_j$, $j = 1, \ldots, N$, we have

$$V_K^*(z) \ge V_K(z) \ge V^{(1)}(z) \ge \max_{j=1,\ldots,N} V_{\ell_j(K)}(\ell_j(z)).$$

But $\max_{j=1,\ldots,N} V_{\ell_j(K)}(\ell_j(z)) = V_{\ell_1(K)\times\ldots\times\ell_N(K)}(z)$ and $V^*_{\ell_1(K)\times\ldots\times\ell_N(K)} \in L^+$ since each $\ell_j(K)$ is nonpolar by Lemma 2.1.

Finally, we note that if $N = 1$, then $V^{(1)} = V_K$ for all compact sets K.

3. Continuity of $V^{(n)}$. In this section, *we will always assume that $K \subset \mathbb{C}^N$ is compact and regular*; moreover, we may assume $K \subset B$, the unit ball. Our main task in this section is to show that each of the functions $V^{(n)} = V_K^{(n)}$, $n = 1, 2, \ldots$, is continuous. We first work with $V^{(1)}$ and see which results generalize. Recall that we may assume our linear functionals ℓ are normalized to have norm 1; in the case of $V^{(n)}$ for $n > 1$, since $V_{tp(K)} \circ tp = V_{p(K)} \circ p$, we are again free to normalize in an appropriate fashion. For example, writing $p = H_n + H_{n-1} + \ldots + H_0$ where H_k is a homogeneous polynomial of degree k, we may require that $\|H_n\|_B = 1$. We begin by stating a lemma which will be useful in the next section in proving continuity of $\varrho_{V^{(1)}}$.

LEMMA 3.0. *Fix a positive integer n. If $K \subset \mathbb{C}^N$ is compact and regular, then*

$$\inf_p \operatorname{cap}(p(K)) > 0$$

where the infimum is taken over all nonconstant polynomials $p = H_n + H_{n-1} + \ldots + H_0$ of degree at most n with $\|H_n\|_B = 1$.

Proof. We know from the previous section that $V^{(n)} \in L^+$; in particular, there exists a constant C so that for $|z| > 1$, we have $V^{(n)}(z) \le C + \log|z|$. Thus for any p,

$$\frac{1}{n} V_{p(K)}(p(z)) \le C + \log|z|, \qquad |z| > 1.$$

For motivational purposes, we first give a proof for the case $n = 1$ (linear case) using this normalization: for a linear functional $\ell(z) = a_1 z_1 + \ldots + a_N z_N$, we suppose $|a_1|^2 + \ldots + |a_N|^2 = 1$. Given $t \in \mathbb{C}$, setting $z_1 = t\overline{a}_1, \ldots, z_N = t\overline{a}_N$ yields a point $z \in \mathbb{C}^N$ with $\ell(z) = t$ and $|z| = |t|$. Thus for such z and t with $|z| = |t| > 1$, we have

$$V_{\ell(K)}(t) = V_{\ell(K)}(\ell(z)) \le C + \log|t| + \log\frac{|z|}{|t|} = C + \log|t|.$$

Letting $|t| \to \infty$, we conclude that $\varrho_{\ell(K)} \le C$ so that $\mathrm{cap}(\ell(K)) > e^{-C}$.

For the general case, write $p(z) := t$, $t \in \mathbb{C}$. Then if $|z| > 1$ and $t \ne 0$,

$$V_{p(K)}(t) = V_{p(K)}(p(z)) \le nC + \log|t| + n\log\frac{|z|}{|t|^{1/n}}.$$

Now since $\|H_n\|_B = 1$, for any $R \ge 1$, $\|H_n\|_{B(R)} = R^n$ and hence

$$\|p\|_{B(R)} \ge R^n, \qquad R \ge 1.$$

Choose a sequence $\{R_k\}$ of radii each larger than R_0 and increasing to ∞, and choose corresponding points $\{z_k\}$ with $|z_k| = R_k$ such that $|p(z_k)| =: |t_k| = \|p\|_{B(R_k)} \ge R_k^n$. Then $|t_k| \uparrow \infty$ and, since

$$\frac{|z_k|}{|t_k|^{1/n}} \le \frac{R_k}{R_k} = 1,$$

for the points t_k we have

$$V_{p(K)}(t_k) \le nC + \log|t_k|.$$

Letting $k \to \infty$, we have $\varrho_{p(K)} \le nC$ so that $\mathrm{cap}(p(K)) > e^{-nC}$. Note we are using the fact that for planar (nonpolar) compact sets, such as $E = p(K)$, the limit

$$\lim_{t\to\infty}[V_E(t) - \log|t|] = \varrho_E$$

exists. ∎

In the next few results, we use the fact that for regular compact sets E, F in $\mathbb{C}^N$ (even $N = 1$),

$$\|V_E - V_F\|_{\mathbb{C}^N} = \|V_E - V_F\|_{E\cup F} = \max[\|V_E\|_F, \|V_F\|_E]. \tag{3.1}$$

For $K \subset \mathbb{C}^N$ and $\delta > 0$, we define

$$K^\delta := \{z \in \mathbb{C}^N : \mathrm{dist}(z, K) \le \delta\}.$$

Lemma 3.1. *Let $K \subset \mathbb{C}^N$ be compact and regular. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if K' is compact and regular and*

$$K \subset (K')^\delta, \quad K' \subset K^\delta, \tag{3.2}$$

then $\|V_K - V_{K'}\|_{\mathbb{C}^N} \le \varepsilon$.

Proof. Since $K^\delta, (K')^\delta$ decrease to K, K' as δ decreases to 0, we can choose δ so that

$$K^\delta \subset \{z \in \mathbb{C}^N : V_K(z) < \varepsilon\} \quad \text{and} \quad (K')^\delta \subset \{z \in \mathbb{C}^N : V_{K'}(z) < \varepsilon\}.$$

Then for all $z \in \mathbb{C}^N$,

$$V_K(z) - \varepsilon \le V_{K^\delta}(z) \quad \text{and} \quad V_{K'}(z) - \varepsilon \le V_{(K')^\delta}(z).$$

By (3.2), $V_{(K')^\delta} \le V_K$ and $V_{K^\delta} \le V_{K'}$; combining with the above equation, we obtain

$$V_K(z) - \varepsilon \le V_{K^\delta}(z) \le V_{K'}(z) \quad \text{and} \quad V_{K'}(z) - \varepsilon \le V_{(K')^\delta}(z) \le V_K(z);$$

i.e., $\|V_K - V_{K'}\|_{\mathbb{C}^N} \le \varepsilon$. ■

Corollary 3.2. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $T : \mathbb{C}^N \to \mathbb{C}^N$ is an invertible linear transformation with $\|T - I\|, \|T^{-1} - I\| < \delta$, then*

$$\|V_K - V_{T(K)}\|_{\mathbb{C}^N} < \varepsilon.$$

Proof. We know that given $\varepsilon > 0$, there exists $\delta > 0$ such that if K' is compact and regular and

$$K \subset (K')^\delta, \quad K' \subset K^\delta,$$

then $\|V_K - V_{K'}\|_{\mathbb{C}^N} \le \varepsilon$. If $\|T - I\|, \|T^{-1} - I\| < \delta$, since $K \subset B$, for $z \in T(K)$ we have

$$\operatorname{dist}(z, K) \le |T^{-1}(z) - z| < \delta.$$

This says that $T(K) \subset K^\delta$. Similarly, for $z \in K$ we have

$$\operatorname{dist}(z, T(K)) \le |T(z) - z| < \delta.$$

This says that $K \subset (T(K))^\delta$ and the result follows. ■

Lemma 3.3. *Let $T : \mathbb{C}^N \to \mathbb{C}^N$ be an invertible linear transformation. For any $\mathbb{C}$-linear $\ell : \mathbb{C}^N \to \mathbb{C}$ with $\ell \not\equiv 0$,*

$$\|V_{\ell(K)} - V_{\ell(T(K))}\|_{\mathbb{C}} \le \|V_K - V_{T(K)}\|_{\mathbb{C}^N}.$$

Proof. From (3.1), we need only estimate $|V_{\ell(K)}(w) - V_{\ell(T(K))}(w)|$ at points $w \in \ell(K) \cup \ell(T(K))$. Fix $w \in \ell(T(K))$. Then $V_{\ell(T(K))}(w) = 0$, and, writing $w = \ell(T(z))$ for some $z \in K$, we have

$$\begin{aligned} V_{\ell(K)}(w) &= V_{\ell(K)}(\ell(T(z))) = [V_{\ell(K)} \circ \ell](T(z)) \\ &\le V_K^{(1)}(T(z)) \le V_K(T(z)) \le \|V_K\|_{T(K)}. \end{aligned}$$

Similarly, if $w \in \ell(K)$, we obtain the inequality $V_{\ell(T(K))}(w) \le \|V_{T(K)}\|_K$. The result follows from (3.1). ■

We will need the following linear algebra lemma.

LEMMA 3.4. *Fix* $z \in \mathbb{C}^N \setminus \{0\}$ *and* $0 < \delta < 1/2$. *For each* $z' \in B(z, \delta|z|) := \{z' : |z - z'| < \delta|z|\}$, *there exists an invertible linear transformation* $T : \mathbb{C}^N \to \mathbb{C}^N$ *with* $T(z) = z'$ *and* $\|T - I\|, \|T^{-1} - I\| < 2\delta$.

Proof. For simplicity, we take $z = (z_1, 0, \ldots, 0)$. Define T on the standard basis vectors $e_j := (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in the jth slot) by

$$T(e_1) = T(z)/z_1 := z'/z_1, \qquad T(e_j) = e_j, \quad j = 2, \ldots, n.$$

For a vector $w = (w_1, \ldots, w_N) \in \mathbb{C}^N$, we have $|w - T(w)| = |1 - z'_1/z_1|\,|w_1|$ so that

$$\|T - I\| \le |1 - z'_1/z_1| = \left|\frac{z_1 - z'_1}{z_1}\right| \le |z - z'|/|z| < \delta|z|/|z| = \delta.$$

Since $\|T^{-1} - I\| \le \delta/(1 - \delta) < 2\delta$, the result follows. ■

PROPOSITION 3.5. *For* K *regular*, $V_K^{(1)}$ *is continuous on* $\mathbb{C}^N$.

Proof. Given $\varepsilon > 0$, choose $\delta > 0$ as in Corollary 3.2. Then for $T : \mathbb{C}^N \to \mathbb{C}^N$ an invertible linear transformation with $\|T - I\|, \|T^{-1} - I\| < \delta$, we obtain

$$\|V_K - V_{T(K)}\|_{\mathbb{C}^N} < \varepsilon.$$

For such a T, by Lemma 3.3, if $\ell : \mathbb{C}^N \to \mathbb{C}$ is linear with $\ell \not\equiv 0$,

$$\|V_{\ell(K)} - V_{\ell(T(K))}\|_{\mathbb{C}} \le \|V_K - V_{T(K)}\|_{\mathbb{C}^N} < \varepsilon. \tag{3.3}$$

We claim that (3.3) implies that

$$\|V_K^{(1)} - V_{T(K)}^{(1)}\|_{\mathbb{C}^N} < \varepsilon. \tag{3.4}$$

For, given any $z \in \mathbb{C}^N$, (3.3) gives

$$|V_{\ell(K)}(\ell(z)) - V_{\ell(T(K))}(\ell(z))| < \varepsilon.$$

Thus

$$V_{\ell(K)}(\ell(z)) \le \varepsilon + V_{\ell(T(K))}(\ell(z)) \le \varepsilon + V_{T(K)}^{(1)}(z).$$

As this holds for all $\ell : \mathbb{C}^N \to \mathbb{C}$ with $\ell \not\equiv 0$,

$$V_K^{(1)}(z) \le \varepsilon + V_{T(K)}^{(1)}(z).$$

Reversing the roles of K and $T(K)$ together with the above inequality yields (3.4).

However,

$$V_{T(K)}^{(1)}(z) = \sup\{V_{\ell(T(K))}(\ell(z)) : \ell \not\equiv 0\}$$
$$= \sup\{V_{\ell(T(K))}(\ell \circ T(T^{-1}(z))) : \ell \not\equiv 0\} = V_K^{(1)}(T^{-1}(z)).$$

Combining with (3.4) gives

$$|V_K^{(1)}(z) - V_K^{(1)}(T^{-1}(z))| < \varepsilon \tag{3.5}$$

for all $z \in \mathbb{C}^N$. Fixing $z \in \mathbb{C}^N \setminus \{0\}$, and setting $\delta' = \delta'(z, \varepsilon) := \delta|z|/2$, by Lemma 3.4, for each z' with $|z - z'| < \delta' = \delta|z|/2$, we can find T as above with $T^{-1}(z) = z'$. Thus, applying (3.5), we have shown that $|z - z'| < \delta'$ implies that $|V_K^{(1)}(z) - V_K^{(1)}(z')| < \varepsilon$; i.e., $V_K^{(1)}$ is continuous at z. For $z = 0$, we observe that for any $a \in \mathbb{C}^N$, $V_K^{(1)}(z) = V_{K+a}^{(1)}(z + a)$; hence continuity of $V_K^{(1)}$ at 0 follows from continuity of $V_{K+a}^{(1)}$ at a. ■

Note that this argument generalizes to show that $V_K^{(n)}$ is continuous for $n = 1, 2, \ldots$ Lemmas 3.1, 3.4 and Corollary 3.2 are general facts about linear transformations and (usual) extremal functions. Lemma 3.3 remains valid upon replacing ℓ by a nonconstant polynomial p_d; hence the argument of Proposition 3.5 can be repeated virtually line-by-line to obtain continuity of $V_K^{(n)}$ (note that T, T^{-1} invertible, p_d a nonconstant polynomial implies $p_d \circ T, p_d \circ T^{-1}$ are nonconstant polynomials of the same degree as p_d). Thus we may state the following.

PROPOSITION 3.5′. *For K regular, $V_K^{(n)}$ is continuous on $\mathbb{C}^N$, $n = 1, 2, \ldots$*

4. Existence of directional limits and continuity of $\varrho_{V^{(1)}}$. We begin with a general fact about extremal functions in one variable. Let Δ denote the unit disk in $\mathbb{C}$.

LEMMA 4.1. *Let $X \subset \Delta$ be nonpolar. For $|\eta| \le 1$, $|\xi| \gg 1$,*

$$|V_X^*(\xi + \eta) - V_X^*(\xi)| = |\eta|/|\xi| + O(|\eta|/|\xi|^2)$$

where $O(|\eta|/|\xi|^2)$ is independent of X.

Proof. Consider

$$|V_X^*(\xi + \eta) - V_X^*(\xi)| = \left| \int_X \log \frac{|\xi + \eta - t|}{|\xi - t|}\, d\mu_X(t) \right|$$
$$= \left| \log \frac{|\xi + \eta|}{|\xi|} + \int_X \log \frac{|\xi + \eta - t|\,|\xi|}{|\xi - t|\,|\xi + \eta|}\, d\mu_X(t) \right|$$
$$=: \left| \log \left| 1 + \frac{|\eta|}{|\xi|} \right| + R(\xi, \eta) \right|$$

where

$$R(\xi,\eta) := \int_X \log \frac{|\xi+\eta-t|\,|\xi|}{|\xi-t|\,|\xi+\eta|}\,d\mu_X(t).$$

Now

$$\begin{aligned}\frac{|\xi+\eta-t|\,|\xi|}{|\xi-t|\,|\xi+\eta|} &= \frac{|\xi^2+\xi(\eta-t)|}{|\xi^2+\xi(\eta-t)-t\eta|}\\ &\le \frac{|\xi^2+\xi(\eta-t)|}{|\xi^2+\xi(\eta-t)|-|t\eta|} = \frac{1}{1-|t\eta|/|\xi^2+\xi(\eta-t)|}\\ &= 1+O(|\eta|/|\xi|^2)\end{aligned}$$

(note that $|\eta|,|t|\le 1$). Similarly,

$$\frac{|\xi+\eta-t|\,|\xi|}{|\xi-t|\,|\xi+\eta|} \ge \frac{1}{1+|t\eta|/|\xi^2+\xi(\eta-t)|} = 1-O(|\eta|/|\xi|^2).$$

Thus, since $\mu_X(X)=1$,

$$|R(\xi,\eta)| = \log(1+O(|\eta|/|\xi|^2)) = O(|\eta|/|\xi|^2).$$

Finally,

$$\left|\log\left|1+\frac{|\eta|}{|\xi|}\right|\right| \le \frac{|\eta|}{|\xi|} + O(|\eta|^2/|\xi|^2) \le \frac{|\eta|}{|\xi|} + O(|\eta|/|\xi|^2)$$

and the result follows. We only use $\mu_X(X)=1$ and X nonpolar so $O(|\eta|/|\xi|^2)$ is independent of X. ■

We write $V := V_K^{(1)}$ below for simplicity.

Corollary 4.2. *Let $K \subset B \subset \mathbb{C}^N$ be a regular compact set. There exists $M \ge 1$ such that given any $\varepsilon > 0$, there exists $R = R(K,\varepsilon) > 1$ with*

$$|V(z+\eta)-V(z)| \le M|\eta|/|z| + \varepsilon \quad \textit{for all } |\eta|\le 1,\ |z|>R.$$

Proof. Note that for ℓ normalized so that $\|\ell\|=1$, each set $\ell(K)$ is a compact, nonpolar subset of Δ. Thus, by the lemma,

$$|V_{\ell(K)}(t+s)-V_{\ell(K)}(t)| \le |s|/|t| + O(|s|/|t|^2), \quad |s|\le 1,\ |t|\gg 1.$$

Since $O(|s|/|t|^2)$ is independent of ℓ, we can choose $R' \gg 1$ so that

$$|V_{\ell(K)}(t+s)-V_{\ell(K)}(t)| \le 2|s|/|t|, \quad |s|\le 1,\ |t|>R',$$

for all ℓ.

Since K is nonpluripolar, $V \in L^+$ (cf. Section 2); thus, there exist C_1, C_2 with

$$\log^+|z| + C_1 \le V(z) \le \log^+|z| + C_2 \tag{4.1}$$

in all of $\mathbb{C}^N$. By Lemma 3.0, $\operatorname{cap}(\ell(K)) \geq a$ for some $a > 0$ if $\|\ell\| = 1$. Thus, there exists $c' = c'(a) > 0$ such that

$$\log^+ |w| \leq V_{\ell(K)}(w) \leq \log^+ |w| + c' \tag{4.2}$$

for all $\|\ell\| = 1$ and all $w \in \mathbb{C}$.

Next, choose $R > R'$ so that $\log R \gg \max[|C_1|, |C_2|, \varepsilon, c']$. Given $|z| > R$, choose $\ell = \ell_z$ such that

$$V(z) \geq V_{\ell(K)}(\ell(z)) \geq V(z) - \varepsilon. \tag{4.3}$$

Combining (4.1), (4.2) and (4.3) for $\ell = \ell_z$ at the point z we obtain

$$\log^+ |\ell(z)| + (c' - C_1) \geq \log^+ |z| - \varepsilon.$$

By the choice of R, we have

$$|\ell(z)| \geq \frac{|z|}{e^{\varepsilon + c' - C_1}} \geq b_1 |z|$$

where $b_1 = b_1(a) := 1/e^{1+c'-C_1}$ (we may assume $\varepsilon < 1$). Note that as long as $\varepsilon < 1$, the constant b_1 depends only on K (from (4.1)) and hence a (from (4.2)). Thus we are free to take $R = R(K, \varepsilon)$ sufficiently large so that, e.g., $R > 4/b_1$. This we do.

Now given $|\eta| \leq 1$, $|z| > R$, choose $\ell = \ell_z$ so that (4.3) holds and ℓ_η so that

$$V(z+\eta) \geq V_{\ell_\eta(K)}(\ell_\eta(z+\eta)) \geq V(z+\eta) - \varepsilon \tag{4.4}$$

and $|\ell_\eta(z+\eta)| \geq b_1|z+\eta|$ so that

$$|\ell_\eta(z)| \geq b_1|z| - b_1|\eta| - 1 \geq b_1|z| - (b_1 + 1) \geq b|z|$$

where $b = b(b_1)$ (since we may assume $R > 4/b_1$ (so $|z| > 4/b_1$) and take $b = b_1/2$). Using (4.3), (4.4) and the fact that

$$V_{\ell_\eta(K)}(\ell_\eta(z)) \leq V(z), \qquad V_{\ell(K)}(\ell(z+\eta)) \leq V(z+\eta),$$

we obtain

$$\begin{aligned} |V(z+\eta) - V(z)| &\leq \max\left[\frac{|\ell(\eta)|}{|\ell(z)|} + O\left(\frac{|\ell(\eta)|}{|\ell(z)|^2}\right), \frac{|\ell_\eta(\eta)|}{|\ell_\eta(z)|} + O\left(\frac{|\ell_\eta(\eta)|}{|\ell_\eta(z)|^2}\right)\right] + \varepsilon \\ &\leq \frac{1}{b}\left[\frac{|\eta|}{|z|} + O\left(\frac{|\eta|}{|z|^2}\right)\right] + \varepsilon \end{aligned}$$

where we have used the facts that $|\ell(z)|, |\ell_\eta(z)| \geq b|z|$ and $|\ell(\eta)|, |\ell_\eta(\eta)| \leq |\eta|$. Since $O(|\eta|/|z|^2) \leq O(|\eta|/|z|)$ and this quantity is independent of ℓ, ℓ_η, the result follows. ■

Remark. Note that the constant M in the corollary is independent of ε.

PROPOSITION 4.3. *Let $K \subset B$ be regular. For each $\alpha \in \partial B$, the directional limit*

$$\lim_{|\lambda|\to\infty} [V(\lambda\alpha) - \log|\lambda|] = \varrho_V(\alpha)$$

exists.

Before proving the proposition, we recall what information we already know from the results in Section 1. Given a function $u \in L^+(\mathbb{C})$ (one variable), we assume that

$$u(t) := \int \log|t-s|\, d\mu(s);$$

i.e., using (1.8) and (1.9), we assume $u(0) = \int \log|s|\, d\mu(s)$, where

$$d\mu(t) = \frac{-1}{4\pi i}\, \Delta u(t) dt \wedge d\bar{t}.$$

Then we showed in Section 1 that

(1) $\limsup_{|z|\to\infty} [u(z) - \log|z|] \le 0$, and
(2) for $r > 0$, $\lim_{|z|\to\infty} [u^r(z) - \log|z|] = 0$ where

$$u^r(z) := \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})\, dt.$$

Thus, to verify for a given $u \in L^+(\mathbb{C})$ that the limit $\lim_{|z|\to\infty}[u(z) - \log|z|]$ exists (recall that by Proposition 1.2, this is NOT always the case, even if u is continuous), it suffices to verify that

$$\lim_{|z|\to\infty} [u^1(z) - u(z)] = 0.$$

Moreover, since u is subharmonic, $u^1(z) \ge u(z)$ for all z so that

$$\liminf_{|z|\to\infty} [u^1(z) - u(z)] \ge 0.$$

Thus we must show:

$$(*) \qquad \limsup_{|z|\to\infty} [u^1(z) - u(z)] \le 0.$$

Proof of Proposition 4.3. Fix $\alpha \in \partial B$ and consider $u(\lambda) := V(\lambda\alpha)$. Given $\varepsilon > 0$, from the corollary we have

$$u^1(\lambda) - u(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} [u(\lambda + e^{it}) - u(\lambda)]\, dt \le \frac{M}{|\lambda|} + \varepsilon$$

for $|\lambda| > R = R(K, \varepsilon)$. Letting $|\lambda| \to \infty$, we obtain $(*)$; i.e.,

$$\limsup_{|\lambda|\to\infty} [u^1(\lambda) - u(\lambda)] \le \varepsilon$$

valid for all $\varepsilon > 0$. ∎

COROLLARY 4.4. *Let $K \subset B$ be regular. Then ϱ_V is continuous. Moreover, we have uniformity in the limits defining the Robin function: given $\varepsilon > 0$, there exists R depending only on ε such that for all $\alpha \in \partial B$,*

$$\varrho_V(\alpha) - [V(\lambda\alpha) - \log|\lambda|] < \varepsilon$$

for $|\lambda| > R$.

Proof. We first prove the continuity. Fix $\alpha \in \partial B$. Given $\varepsilon > 0$, choose $\delta > 0$ as in Corollary 3.2; then for $T : \mathbb{C}^N \to \mathbb{C}^N$ an invertible linear transformation with $\|T - I\|, \|T^{-1} - I\| < \delta$, we obtain

$$|V(z) - V(T^{-1}(z))| < \varepsilon. \tag{3.5}$$

In particular, for $\alpha' \in \partial B$ with $|\alpha - \alpha'| < \delta$, choose T unitary with $T(\alpha') = \alpha$ and $\|T - I\|, \|T^{-1} - I\| = |\alpha - \alpha'| < \delta$. Then if we choose $R = R(\varepsilon, \alpha, \alpha')$ so that for all $|\lambda| > R$,

$$|V(\lambda\alpha) - \log|\lambda| - \varrho_V(\alpha)| < \varepsilon \quad \text{and} \quad |V(\lambda\alpha') - \log|\lambda| - \varrho_V(\alpha')| < \varepsilon,$$

then

$$|\varrho_V(\alpha) - \varrho_V(\alpha')| \le |V(\lambda\alpha) - V(\lambda\alpha')| + 2\varepsilon.$$

Since $T(\lambda\alpha') = \lambda\alpha$ and $\|T - I\|, \|T^{-1} - I\| < \delta$, from (3.5) we obtain

$$|V(\lambda\alpha) - V(\lambda\alpha')| < \varepsilon.$$

Thus, given $\varepsilon > 0$, choosing $\delta > 0$ as in Corollary 3.2 gives

$$|\varrho_V(\alpha) - \varrho_V(\alpha')| < 3\varepsilon$$

provided $|\alpha - \alpha'| < \delta$.

For the uniformity in the limits defining the Robin function, we first note that ϱ_V is uniformly continuous on ∂B; hence, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\alpha', \alpha'' \in \partial B$ with $|\alpha' - \alpha''| < \delta$ implies $|\varrho_V(\alpha') - \varrho_V(\alpha'')| < \varepsilon$. By compactness of ∂B, we can choose finitely many points $\alpha_1, \ldots, \alpha_m \in \partial B$ with

$$\partial B \subset \bigcup_{i=1}^{m} \{\alpha \in \partial B : |\alpha - \alpha_i| < \delta\}.$$

Then

$$\begin{aligned} \varrho_V(\alpha_i) &= \liminf_{|\lambda|\to\infty} [V(\lambda\alpha_i) - \log|\lambda|] \\ &= \lim_{R\to\infty} \big[\inf_{|\lambda|>R} \{V(\lambda\alpha_i) - \log|\lambda|\}\big], \qquad i = 1, \ldots, m, \end{aligned}$$

so that there exist R_i, $i = 1, \ldots, m$, such that

$$V(\lambda\alpha_i) - \log|\lambda| > \varrho_V(\alpha_i) - \varepsilon, \qquad |\lambda| > R_i.$$

Set $R := \max[R_1, \ldots, R_m]$.

Now fix $\alpha \in \partial B$ and choose $i \in \{1, \ldots, m\}$ with $|\alpha - \alpha_i| < \delta$. Choose a unitary map T with $T(\alpha) = \alpha_i$ and $\|T - I\|, \|T^{-1} - I\| < \delta$. Again, as in the proof of Proposition 3.5, we obtain

$$|V(z) - V(T^{-1}(z))| < \varepsilon \tag{3.5}$$

for all $z \in \mathbb{C}^N$. In particular, since $T(\lambda\alpha) = \lambda\alpha_i$, we have

$$|V(\lambda\alpha) - V(\lambda\alpha_i)| < \varepsilon$$

for all $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}\{\varrho_V(\alpha) - [V(\lambda\alpha) - \log|\lambda|]\} - \{\varrho_V(\alpha_i) - [V(\lambda\alpha_i) - \log|\lambda|]\} \\ = \varrho_V(\alpha) - \varrho_V(\alpha_i) + V(\lambda\alpha_i) - V(\lambda\alpha) < 2\varepsilon,\end{aligned}$$

which gives $\varrho_V(\alpha) - [V(\lambda\alpha) - \log|\lambda|] < 3\varepsilon$ for $|\lambda| > R$. ∎

For completeness, we give a proof of the analogous (known) result for the Robin function $\varrho_K := \varrho_{V_K}$ associated to the extremal function V_K of a regular compact set. We begin with a lemma in the spirit of Corollary 4.2. We assume $K \subset B$.

Lemma 4.5. *Let $z_0 \in \mathbb{C}^N$ with $|z_0| > 1$. For any $\eta \in \mathbb{C}^N$ with $|\eta| \le 1$,*

$$|V_K(z_0 + \eta) - V_K(z_0)| \le \omega(|\eta|/|z_0|)$$

where $\omega = \omega(\delta)$ is the modulus of continuity of V_K on $\{z : |z| \le 2\}$.

Proof. First of all, following the idea in the proof of Lemma 3.4, we can find an invertible linear transformation T with $T(z_0) = z_0 + \eta$ and $\|T - I\| \le |\eta|/|z_0|$. We may assume that $V_K(z_0 + \eta) \ge V_K(z_0)$. Define

$$v(z) := (V_K \circ T)(z) - \omega(|\eta|/|z_0|).$$

Then $v \in L$ and if $|z| \le 1$, we have $|T(z)| \le 2$ so that

$$|(V_K \circ T)(z) - V_K(z)| \le \omega(|\eta|/|z_0|) \quad \text{for } |z| \le 1.$$

Since $K \subset B$, this implies $v(z) \le V_K(z)$ for all $z \in \mathbb{C}^N$ and setting $z = z_0$ gives the result. ∎

Corollary 4.6. *Let $K \subset \mathbb{C}^N$ be regular. For each $\alpha \in \partial B$, the directional limit*

$$\lim_{|\lambda| \to \infty} [V_K(\lambda\alpha) - \log|\lambda|] = \varrho_K(\alpha)$$

exists. In addition, ϱ_K is continuous on $\mathbb{C}^N$.

Proof. The existence of the directional limit follows as in the proof of Proposition 4.3 with Lemma 4.5 in place of Corollary 4.2. The continuity of ϱ_K is then shown as in the proof of Corollary 4.4 with (3.5) replaced by $\|V_K - V_{T(K)}\|_{\mathbb{C}^N} < \varepsilon$. ∎

REMARK. Corollary 4.6 also follows from [S1] by using the formula $V_K(z) = \widetilde{V}_K(1, z)$ (see [S1]).

The previous result generalizes to the case of a weighted extremal function and a locally L-regular set K. Let $K \subset \mathbb{C}^N$ be a compact set and let w be an admissible weight function on K; i.e., w is usc and $\{z \in K : w(z) > 0\}$ is not pluripolar. Let $Q := -\log w$ and define the *weighted* extremal function

$$V_{K,Q}(z) := \sup\{u(z) : u \in L,\, u \le Q \text{ on } K\}.$$

Next, a set $E \subset \mathbb{C}^N$ is said to be *locally L-regular at a point $a \in \overline{E}$* if for each $r > 0$, $V_{E\cap\overline{B}(a,r)}$ is continuous at a where

$$\overline{B}(a, r) = \{z \in \mathbb{C}^N : |z - a| \le r\}.$$

The set E is *locally L-regular* if it is locally L-regular at each point $a \in \overline{E}$ (cf. [S2]). Clearly if E is a locally L-regular compact set then E is regular.

COROLLARY 4.6′. *Let K be a locally L-regular compact set and let $w \ge 0$ be a continuous weight function on K. Then $V_{K,Q}$ and $\varrho_{K,Q} = \varrho_{V_{K,Q}}$ are continuous. Moreover, for each $\alpha \in \partial B$, the directional limit*

$$\lim_{|\lambda|\to\infty} [V_{K,Q}(\lambda\alpha) - \log|\lambda|] = \varrho_{K,Q}(\alpha)$$

exists.

Proof. The continuity of $V_{K,Q}$ follows from [S2, Proposition 2.16]. Next, let

$$Z = Z(K) := \{z \in \mathbb{C}^N : V_{K,Q}(z) \le M = M(K) := \|V_{K,Q}\|_K\}.$$

Then $V_{K,Q}(z) = V_Z(z) + M$ for $z \in \mathbb{C}^N \setminus Z$ since both functions are *maximal* outside Z and agree on ∂Z (cf. [K]). Thus the function $u(z) := \max[0, V_{K,Q}(z) - M]$ belongs to L and is equal to 0 at all points of Z; hence $u = V_Z$ on all of $\mathbb{C}^N$. In particular, V_Z is continuous; this implies the continuity of $\varrho_{K,Q} = \varrho_Z + M$ ([S1, Proposition 2.3(ii)]) and the existence of the directional limits. ∎

We would like to adapt the arguments used to prove Proposition 4.3 and Corollary 4.4 to study $\varrho_{V^{(n)}} = \varrho_{V_K^{(n)}}$ for $n = 2, 3, \ldots$ To this end, we need a modified version of Corollary 4.2, which in turn requires a generalization of Lemma 4.1.

LEMMA 4.7. *Fix a positive integer $n \ge 2$ and $m > 1$. There exist constants C_1, C_2 and R depending on n and C_3 depending on n and m such that for each nonpolar set $X \subset \Delta_m := \{t \in \mathbb{C} : |t| < m\}$ and all nonconstant polynomials $p : \mathbb{C}^N \to \mathbb{C}$ of degree at most n,*

$$|V_X^*(p(z+\eta)) - V_X^*(p(z))| \le C_1|\eta|\,\|p\|_{\overline{B}}/\log|z| + C_2|\eta|^2\|p\|_{\overline{B}}^2/(\log|z|)^2 + C_3|\eta|\,\|p\|_{\overline{B}}/(|z|^{n-1}(\log|z|)^2)$$

for all $|\eta| \le 1$ and all $|z| \ge R$ with $|p(z)| \ge \log|z| \cdot |z|^{n-1}$.

Proof. Let $p : \mathbb{C}^N \to \mathbb{C}$ be a polynomial of degree n. We have the following Cauchy estimate for p:

$$|D^\alpha p(z)| \le \frac{\alpha!\|p\|_{P(z,1/n)}}{(1/n)^{|\alpha|}}$$

where $P(z,1/n)$ is the polydisc $\Delta(z_1,1/n)\times\ldots\times\Delta(z_N,1/n)$.

If $z\in\overline{B}:=\overline{B}(0,1)$ and $w\in P(z,1/n)$ then $w\in B(0,1+\sqrt{N}/n)$, hence, by the Bernstein–Walsh inequality (cf. [S2]),

$$\|p\|_{P(z,1/n)} \le \|p\|_{\overline{B}}(1+\sqrt{N}/n)^n.$$

Putting these estimates together for an arbitrary $z\in\overline{B}$, and using the fact that $\alpha!\le n^{|\alpha|}$, we get

$$\|D^\alpha p\|_{\overline{B}} \le n^{2|\alpha|}(1+\sqrt{N}/n)^n\|p\|_{\overline{B}}$$

for all multi-indices α; hence, again by the Bernstein–Walsh inequality,

$$|D^\alpha p(z)| \le n^{2|\alpha|}(1+\sqrt{N}/n)^n|z|^{n-|\alpha|}\|p\|_{\overline{B}}, \quad \text{where } z\in\mathbb{C}^N \text{ with } |z|\ge 1.$$

Now fix a polynomial p with $\|p\|_{\overline{B}}=1$ and write

$$p(z+\eta)=p(z)+\eta\nabla p(z)+O(|\eta|^2)p_2(z)$$

where $p_2(z)$ involves at least second-order partial derivatives of p. From the above inequality, we have

$$|p(z+\eta)-p(z)|\le[n^2N|\eta||z|^{n-1}+c_2|\eta|^2|z|^{n-2}+\ldots+c_n|\eta|^n](1+\sqrt{N}/n)^n\|p\|_{\overline{B}}$$

for each such p where $c_2=c_2(n,N),\ldots,c_n=c_n(n,N)$ are independent of p. Thus, for $|z|>R=R(n)$ and $|\eta|\le 1$,

$$|p(z+\eta)-p(z)|\le An^2|\eta|\,|z|^{n-1}\|p\|_{\overline{B}} \tag{4.5}$$

where $A=A(n,N)$. Hence, if we write $p(z+\eta)=p(z)+q_\eta(z):=p(z)+q(z)$, then (4.5) can be written as

$$|q(z)|\le An^2|\eta|\,|z|^{n-1}\|p\|_{\overline{B}}. \tag{4.6}$$

To estimate

$$|V_X^*(p(z+\eta))-V_X^*(p(z))|=\left|\int_X \log\frac{|p(z+\eta)-t|}{|p(z)-t|}\,d\mu_X(t)\right|,$$

we estimate

$$\begin{aligned}\log\frac{|p(z+\eta)-t|}{|p(z)-t|} &= \log\frac{|p(z)+q(z)-t|}{|p(z)-t|}\\ &= \log\frac{|p(z)+q(z)|}{|p(z)|}+\log\frac{|p(z)+q(z)-t|\,|p(z)|}{|p(z)-t|\,|p(z)+q(z)|}.\end{aligned}$$

First, for $|z| \ge R$ with $|p(z)| \ge \log|z| \cdot |z|^{n-1}$ and $|\eta| \le 1$, using (4.6) we have

$$\frac{|q(z)|}{|p(z)|} \le \frac{An^2|\eta|\,\|p\|_{\overline{B}}}{\log|z|}.$$

Thus

$$\log\left|1+\frac{q(z)}{p(z)}\right| \le \frac{An^2|\eta|\,\|p\|_{\overline{B}}}{\log|z|} + O\biggl(\frac{|\eta|^2\|p\|^2_{\overline{B}}}{(\log|z|)^2}\biggr)$$

for such z, η. Then

$$\begin{aligned}\frac{|p(z)+q(z)-t|\,|p(z)|}{|p(z)-t|\,|p(z)+q(z)|} &\le \frac{|p(z)^2+p(z)q(z)-tp(z)|}{|p(z)^2+p(z)q(z)-tp(z)|-|tq(z)|}\\ &= 1+O\biggl(\frac{|tq(z)|}{|p(z)^2+p(z)q(z)-tp(z)|}\biggr)\\ &= 1+O\biggl(\frac{|tq(z)|}{|p(z)|^2}\biggr).\end{aligned}$$

Hence

$$\log\biggl(\frac{|p(z)+q(z)-t|\,|p(z)|}{|p(z)-t|\,|p(z)+q(z)|}\biggr) = O\biggl(\frac{|tq(z)|}{|p(z)|^2}\biggr) = O\biggl(\frac{m|\eta|\,\|p\|_{\overline{B}}}{|z|^{n-1}(\log|z|)^2}\biggr).$$

The "big-O" terms in these estimates depend on $\|p\|_{\overline{B}}$; note, however, that the points z for which these estimates hold depend on $p(z)$ (since we require $|p(z)| \ge \log|z| \cdot |z|^{n-1}$). ∎

Now we modify Corollary 4.2. For $m > 1$, we define

$$\begin{aligned} u_m(z) &= u^{(n)}_{m,K}(z)\\ &:= \sup\biggl\{\frac{1}{n}\,V_{p(K)}(p(z)) : p = H_n + H_{n-1} + \ldots,\ \|H_n\|_{\overline{B}} = 1,\ \|p\|_{\overline{B}} \le m\biggr\}.\end{aligned}$$

Note that $\{u_m\}_{m=2,3,\ldots}$ increases pointwise to $V := V^{(n)}$ on all of $\mathbb{C}^N$; moreover, if each u_m is continuous, then by Dini's theorem, $u_m \to V$ uniformly on compact subsets of $\mathbb{C}^N$.

COROLLARY 4.8. *Fix a positive integer $n \ge 2$ and $m > 1$. Let $K \subset B$ be a regular compact set. There exists $C \ge 1$ depending on n, m and K such that for any $\varepsilon > 0$, there exists $R = R(m, n, K, \varepsilon) > 1$ with*

$$|u_m(z+\eta) - u_m(z)| \le C|\eta|/\log|z| + \varepsilon \quad \textit{for all } |\eta| \le 1,\ |z| > R.$$

Proof. We first make a remark on the use of Lemma 4.7 for p of degree n with $\|p\|_{\overline{B}} \le m$. By Lemma 4.7, since C_1, C_2 and C_3 are independent of p, and $\|p\|_{\overline{B}} \le m$, we can choose $R' = R'(n, m)$ sufficiently large and $C = C(n, m)$ so that

$$(+) \qquad |V_{p(K)}(p(z+\eta)) - V_{p(K)}(p(z))| \le C|\eta|/\log|z|$$

for all such p if $|\eta| \le 1$, $|z| \ge R'$, and $|p(z)| \ge \log|z| \cdot |z|^{n-1}$.

Since K is nonpluripolar, $V \in L^+$ (cf. Section 2) and hence $u_m \in L^+$; thus, there exist C_1', C_2' with

$$\log^+ |z| + C_1' \le u_m(z) \le \log^+ |z| + C_2' \tag{4.7}$$

in all of $\mathbb{C}^N$; indeed, we can take $C_1' = 0$ since $K \subset B$. Now, to begin the actual proof of Corollary 4.8, given $\varepsilon > 0$, we choose $R > R'$ so that $\log R \gg \max[|C_1'|, |C_2'|, \varepsilon]$. Given $|z| > R$, choose $p = p_z = H_n + H_{n-1} + \ldots$ with $\|H_n\|_{\overline{B}} = 1$ and $\|p\|_{\overline{B}} \le m$ such that

$$u_m(z) \ge \frac{1}{n} V_{p(K)}(p(z)) \ge u_m(z) - \varepsilon. \tag{4.8}$$

Using Lemma 3.0, for some $a > 0$ we have $\operatorname{cap}(p(K)) \ge a$ if $\|H_n\|_{\overline{B}} = 1$. Thus there exists $c' = c'(a) > 0$ such that

$$\log^+ |w| \le V_{p(K)}(w) \le \log^+ |w| + c' \tag{4.9}$$

for all p with $\|H_n\|_{\overline{B}} = 1$ and all $w \in \mathbb{C}$; combining (4.7), (4.8) and (4.9) for $p = p_z$ at the point z we obtain

$$\log^+ |p(z)| + (c' - nC_1') \ge n \log^+ |z| - n\varepsilon.$$

By the choice of R, we have

$$|p(z)| \ge \frac{|z|^n}{e^{n(\varepsilon - C_1') + c'}} \ge b_1 |z|^n$$

where $b_1 = b_1(a) := 1/e^{n(1-C_1')+c'}$ (we may assume $\varepsilon < 1$). Note $c' > 0$ and recall we can take $C_1' = 0$ since $K \subset B$; hence $0 < b_1 = e^{-(n+c')} < 1$.

Now given $|\eta| \le 1$, $|z| > R$, choose $p = p_z$ so that (4.8) holds and $p_\eta = \widetilde{H}_n + \widetilde{H}_{n-1} + \ldots$ with $\|\widetilde{H}_n\|_{\overline{B}} = 1$ and $\|p_\eta\|_{\overline{B}} \le m$ so that

$$u_m(z+\eta) \ge \frac{1}{n} V_{p_\eta(K)}(p_\eta(z+\eta)) \ge u_m(z+\eta) - \varepsilon \tag{4.10}$$

and $|p_\eta(z+\eta)| \ge b_1 |z+\eta|^n \ge b|z|^n$ where $b = b(b_1)$ (as in the proof of Corollary 4.2). Since b_1 (and hence b) depends only on K (from (4.7)) and a (from (4.9)), we can assume from the beginning that $b_1 R \ge \log R$ so that (+) is valid. Using (4.8), (4.10) and the fact that

$$\frac{1}{n} V_{p_\eta(K)}(p_\eta(z)) \le u_m(z), \qquad \frac{1}{n} V_{p(K)}(p(z+\eta)) \le u_m(z+\eta),$$

we obtain

$$\begin{aligned} |u_m(z+\eta) - u_m(z)| &\le \frac{1}{n} \max[|V_{p_\eta(K)}(p_\eta(z+\eta)) - V_{p_\eta(K)}(p_\eta(z))|, \\ &\qquad |V_{p(K)}(p(z+\eta)) - V_{p(K)}(p(z))|] + \varepsilon \\ &\le \frac{1}{n} C|\eta|/\log|z| + \varepsilon. \ \blacksquare \end{aligned}$$

Finally, the analogue of Proposition 4.3 follows by applying Corollary 4.8.

COROLLARY 4.9. *For $K \subset B$ a regular compact set, for $n = 2, 3, \ldots$, and $m > 1$, define*

$$u_m(z) := \sup \left\{ \frac{1}{n} V_{p(K)}(p(z)) : p = H_n + H_{n-1} + \ldots,\ \|H_n\|_{\overline{B}} = 1,\ \|p\|_{\overline{B}} \le m \right\}.$$

Then for each $\alpha \in \partial B$, the directional limit

$$\lim_{|\lambda| \to \infty} [u_m(\lambda\alpha) - \log|\lambda|] = \varrho_{u_m}(\alpha)$$

exists.

REMARK. It seems likely that Corollary 4.9 is valid for $V^{(n)}$, $n = 2, 3, \ldots$, but we have been unable to verify this.

5. Final remarks. We give an explicit example of a compact set K in $\mathbb{C}^N$, $N > 1$, such that ϱ_K is not continuous. Note from Corollary 4.6 that K cannot be regular. Indeed, we construct such an example with K circled, i.e., $z \in K$ if and only if $e^{it} z \in K$. Let

$$H := \{u \in L : u(\lambda z) = u(z) + \log|\lambda| \text{ for } \lambda \in \mathbb{C},\ z \in \mathbb{C}^N\}$$

be the *log-homogeneous* psh functions. For K circled,

$$\begin{aligned} V_K(z) &= \max[0, \sup\{u(z) : u \in H,\ u \le 0 \text{ on } K\}] \\ &= \max\left[0, \sup\left\{\frac{1}{\deg p} \log|p(z)| : p \text{ homogeneous polynomial, } \|p\|_K \le 1\right\}\right]; \end{aligned}$$

moreover, we have the following.

LEMMA 5.1. *Let $K \subset \mathbb{C}^N$ be compact, circled, and nonpluripolar. Then $V_K^*(z) = \max[0, \varrho_K(z)]$.*

Proof. This follows from the above formula for V_K and the definition of ϱ_K. If $V_K^*(z) > 0$, then

$$\begin{aligned} \varrho_K(z) &:= \limsup_{|\lambda| \to \infty} [V_K^*(\lambda z) - \log|\lambda|] \\ &= \limsup_{|\lambda| \to \infty} [V_K^*(z) + \log|\lambda| - \log|\lambda|] = V_K^*(z). \end{aligned}$$

Thus $\varrho_K \in H$ and $\varrho_K(z) = V_K^*(z)$ if $V_K^*(z) > 0$; hence $\{z \in \mathbb{C}^N : \varrho_K(z) \le 0\}$ differs from K by at most a pluripolar set and the result follows. ∎

The following example is due to Cegrell [C]; we elaborate on the details. Let $\{a_j\}$ be a countable dense sequence of points in the unit circle and let $\{\alpha_j\}$ be a sequence of positive numbers with $\sum_j \alpha_j < \infty$. We can reorder the $\{a_j\}$ and choose the $\{\alpha_j\}$ accordingly so that, in addition, $\sum_j \alpha_j \log|1 - a_j| > -\infty$. For example, for $n = 1, 2, \ldots$ and $j =$

$2^n+1,\dots,2^{n+1}$, we can take $\alpha_j = 2^{-2n}$; and we take a_j, $j = 2^n+1,\dots,2^{n+1}$, to be the 2^n-roots of unity (omitting 1 and repeating any other). Define

$$g(z_1, z_2) := \exp\Big\{\sum_j \alpha_j \log|z_1 - a_j z_2|\Big\}.$$

Then g is discontinuous at all points (z_1, z_2) with $|z_1| = |z_2|$ and

$$\sum_j \alpha_j \log|z_1 - a_j z_2| > -\infty.$$

Moreover, $\log g \in H(\mathbb{C}^2)$. Next, let

$$h(z_1, z_2) := g(z_1, z_2) + \max(|z_1|, |z_2|).$$

Then

$$\begin{aligned} N(h) &:= \{(z_1, z_2) \in \mathbb{C}^2 : h \text{ is discontinuous at } (z_1, z_2)\} \\ &= \Big\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| \text{ and } \sum_j \alpha_j \log|z_1 - a_j z_2| > -\infty\Big\}. \end{aligned}$$

Note that $N(h) \neq \emptyset$ by the assumption that $\sum_j \alpha_j \log|1 - a_j| > -\infty$; indeed,

$$\{(re^{it}, re^{it}) : 0 < r < 1,\ 0 \le t \le 2\pi\} \subset N(h).$$

Now, in $\mathbb{C}^3$, we define

$$\begin{aligned} W(z_1, z_2, z_3) := {}& \exp\Big\{\sum_j \alpha_j \max[\log|z_1 - a_j z_2|, \log|z_3|]\Big\} \\ &+ \max(|z_1|, |z_2|, |z_3|). \end{aligned}$$

Then $\log W \in H(\mathbb{C}^3)$ and from the discussion on $N(h)$, we see that

$$N(W) := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : W \text{ is discontinuous at } (z_1, z_2, z_3)\} \subset \{z_3 = 0\}$$

and

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 : (z_1, z_2) \in N(h),\ z_3 = 0\} \subset N(W)$$

so that $N(W)$ is nonempty and pluripolar. Let

$$D := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : W(z_1, z_2, z_3) < 1\}.$$

Then by a result of Siciak [S3], $K := \overline{D}$ is a compact, circled, and nonpluripolar subset of $\mathbb{C}^3$ with

$$V_K^* = \max[0, \log W] = \max[0, \varrho_K].$$

More generally, the above argument is valid for any $W \in H(\mathbb{C}^N)$ with $W(z) \ge c|z|$, $c > 0$ and $N(W) \neq \emptyset$ to yield a compact set $K = \overline{D}$ such that ϱ_K is not continuous; here, $D = \{z \in \mathbb{C}^N : W(z) < 1\}$.

We end the paper with the following relationship on the asymptotic behavior of $\varrho_{V^{(n)}}$ for $K \subset B$ a regular compact set.

THEOREM 5.2. *For $K \subset B$ a regular compact set,*

$$\lim_{n\to\infty} \varrho_{V^{(n)}}(z) = \varrho_K(z)$$

for q.e. $z \in \mathbb{C}^N$.

Proof. Note for $n = 1, 2, \ldots$ that

$$\begin{aligned} V^{(n)}(z) &= \sup\left\{\frac{1}{\deg p} V_{p(K)}(p(z)) : 1 \le \deg p \le n\right\} \\ &\ge \sup\left\{\frac{1}{\deg p}\log^+ \frac{|p(z)|}{\|p\|_K} : 1 \le \deg p \le n\right\} \\ &\qquad\qquad (\text{since } p(K) \subset \{t \in \mathbb{C} : |t| \le \|p\|_K\}) \\ &\ge \sup\left\{\frac{1}{|\alpha|}\log \frac{|p_\alpha(z)|}{\|p_\alpha\|_K} : 1 \le |\alpha| = \deg p_\alpha \le n\right\} \end{aligned}$$

where $\{p_\alpha\}$ is any sequence of polynomials with $1 \le \deg p_\alpha \le n$. Thus we have

$$\varrho_{V^{(n)}}(z) \ge \sup\left\{\frac{1}{|\alpha|}\log \frac{|\widehat{p}_\alpha(z)|}{\|p_\alpha\|_K} : 1 \le |\alpha| = \deg p_\alpha \le n\right\} \tag{5.1}$$

where $\widehat{p}_\alpha$ denotes the top degree homogeneous part of p_α. On the other hand, if we take a family of Chebyshev-type polynomials $\{Q_\alpha\}$ as in, e.g., [Bl, Theorem 2.3], then

$$\left[\limsup_{|\alpha|\to\infty} \frac{1}{|\alpha|}\log \frac{|\widehat{Q}_\alpha(z)|}{\|Q_\alpha\|_K}\right]^* = \varrho_K(z).$$

Thus

$$\lim_{n\to\infty}\left[\sup \frac{1}{|\alpha|}\log \frac{|\widehat{p}_\alpha(z)|}{\|p_\alpha\|_K} : 1 \le |\alpha| = \deg p_\alpha \le n\right] = \varrho_K(z) \quad \text{q.e.},$$

which, together with (5.1), shows that $\lim_{n\to\infty} \varrho_{V^{(n)}}(z) = \varrho_K(z)$ q.e. ∎

REMARK. It is not always true that if $u_n, u \in L$ and u_n increases pointwise to u, then ϱ_{u_n} increases q.e. to ϱ_u (as a simple example, take $u_n(z) = (1 - 1/n)\log|z|$). A necessary and sufficient condition that this occurs, even with $u_n, u \in L^+$, is given in [BT, Theorem 6.6]; a condition that is admittedly very difficult to verify in practice.

6. Open questions. 1. *Compute $\mu_K^{(n)} := (dd^c V_K^{(n)})^N$ for K regular. Does $\mu_K^{(n)}$ have compact support?*

2. *Let*

$$K = \{(x, y) \in \mathbb{R}^2 : x, y \ge 0,\ x + y \le 1\}.$$

In [BCL], *it was shown that $V_K^{(1)} \ne V_K$. Compute $V_K^{(1)}$ explicitly.*

3. *Compute* $\mu_K^{(n)} := (dd^c V_K^{(n)})^2$ *for the set*

$$K = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0,\ x + y \leq 1\}.$$

Does $\mu_K^{(n)}$ *have compact support?*

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