Approximation of holomorphic maps
by algebraic morphisms

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Dedicated to Professor Józef Siciak on his seventieth birthday

Abstract. Let $X$ be a nonsingular complex algebraic curve and let $Y$ be a nonsingular rational complex algebraic surface. Given a compact subset $K$ of $X$, every holomorphic map from a neighborhood of $K$ in $X$ into $Y$ can be approximated by rational maps from $X$ into $Y$ having no poles in $K$. If $Y$ is a nonsingular projective complex surface with the first Betti number nonzero, then such an approximation is impossible.

1. Introduction. Given two topological spaces $S$ and $T$, we denote by $C(S,T)$ the space of all continuous maps from $S$ into $T$, endowed with the compact-open topology.

Throughout this paper we call quasiprojective irreducible complex algebraic varieties simply algebraic varieties. Algebraic morphisms will be called regular maps. Unless explicitly stated otherwise, we consider algebraic varieties endowed with the topology induced by the usual metric topology on $\mathbb{C}$.

Let $X$ and $Y$ be algebraic varieties and let $S$ be a subset of $X$. We say that a continuous map $f : S \to Y$ can be approximated by regular (resp. rational) maps from $X$ into $Y$ if for every neighborhood $\mathcal{N}$ of $f$ in $C(S,Y)$ there exists a regular map $g : X \to Y$ (resp. there exist a Zariski neighborhood $X_0$ of $S$ in $X$ and a regular map $g : X_0 \to Y$) such that the restriction $g|S$ belongs to $\mathcal{N}$. The approximation problem becomes interesting if $f$ is holomorphic, that is, extends to a holomorphic map on a neighborhood of $S$. Results in this direction include the classical Runge approximation theorem [8], Runge-type approximation theorems for maps into Grassmannians [5],

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and, only loosely related, theorems on approximation of holomorphic maps by Nash maps [2, 6].

In the present paper we prove the following.

**Theorem 1.1.** Let $X$ be a nonsingular algebraic curve and let $Y$ be a nonsingular rational algebraic surface. Given a compact subset $K$ of $X$, every holomorphic map from $K$ into $Y$ can be approximated by rational maps from $X$ into $Y$.

Recall that an algebraic surface is said to be *rational* if it is birationally equivalent to $\mathbb{C}^2$, that is, if it contains a nonempty Zariski open subvariety isomorphic to a Zariski open subvariety of $\mathbb{C}^2$; cf. [1] for the theory and examples of rational algebraic surfaces.

Clearly, if $X = \mathbb{C}$ and $Y = \mathbb{C}^2$, Theorem 1.1 is equivalent to the classical Runge approximation theorem.

If $Y$ is projective, then every rational map from $X$ into $Y$ is automatically regular and therefore Theorem 1.1 implies the following.

**Corollary 1.2.** Let $X$ be a nonsingular algebraic curve and let $Y$ be a nonsingular projective rational surface. Given an open subset $U$ of $X$, every holomorphic map from $U$ into $Y$ can be approximated by regular maps from $X$ into $Y$.

We do not know to what extent the assumption of rationality of $Y$ can be relaxed. Our next result shows that it cannot be relaxed too much.

**Theorem 1.3.** Let $X$ be a nonsingular algebraic curve and let $Y$ be a nonsingular projective algebraic variety. Let $D$ be an open subset of $X$ biholomorphic to the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$. If every holomorphic map from $D$ into $Y$ can be approximated by regular maps from $X$ into $Y$, then the first Betti number of $Y$ is equal to 0.

2. **Proofs.** As usual, $\mathbb{P}^n$ will denote the complex projective $n$-space. The unit disc in $\mathbb{C}$ will be denoted by $D$,

$$D = \{z \in \mathbb{C} \mid |z| < 1\},$$

and the unit ball in $\mathbb{C}^n$ by $B^n$,

$$B^n = \{w \in \mathbb{C}^n \mid \|w\| < 1\}.$$

We let $\pi_n : \tilde{B}^n \to B^n$ denote the monoidal transformation of $B^n$ at 0,

$$\tilde{B}^n = \{(w, \ell) \in B^n \times \mathbb{P}^{n-1} \mid w \in \ell\}, \quad \pi_n(w, \ell) = w,$$

where we regard $\mathbb{P}^{n-1}$ as the variety of 1-dimensional vector subspaces of $\mathbb{C}^n$.

Given a holomorphic map $f : M \to N$ between complex analytic manifolds, a point $x$ in $M$, and a nonnegative integer $r$, we denote by $j^r f(x)$ the $r$-jet of $f$ at $x$. 
Let $X$ be a nonsingular algebraic curve and let $Y$ be a nonsingular algebraic variety. Let $y_0$ be a point in $Y$ and let $\pi : \tilde{Y} \to Y$ be the monoidal transformation of $Y$ at $y_0$, that is, the blowing up of $Y$ at $y_0$. Let $f : U \to Y$ be a holomorphic map defined on an open subset $U$ of $X$ such that $f^{-1}(y_0)$ is a discrete subset of $U$. Then there exists a unique holomorphic map $\tilde{f} : U \to \tilde{Y}$ satisfying $\pi \circ \tilde{f} = f$.

**Lemma 2.1.** With the notation as above, let $\tilde{N}$ be a neighborhood of $\tilde{f}$ in $\mathcal{C}(U, \tilde{Y})$, let $A$ be a finite subset of $U$, and let $s$ be a nonnegative integer. Then there exist a neighborhood $N$ of $f$ in $\mathcal{C}(U, Y)$, a finite subset $B$ of $U$, and a nonnegative integer $r$ such that for every holomorphic map $g : U \to Y$ in $N$, with the set $g^{-1}(y_0)$ discrete in $U$ and $j^r g(x) = j^r f(x)$ for all $x$ in $A \cup B$, the following conditions are satisfied:

(i) $\tilde{g}$ is in $\tilde{N}$;

(ii) $j^s \tilde{g}(a) = j^s \tilde{f}(a)$ for all $a$ in $A$.

**Proof.** The conclusion readily follows from the construction of $\tilde{f}$, which we recall below for the convenience of the reader.

Choose a neighborhood $V$ of $y_0$ in $Y$ and a biholomorphic map $\psi : V \to \mathbb{B}^n$ such that $\psi(y_0) = 0$. There exists a biholomorphic map $\tilde{\psi} : \pi^{-1}(V) \to \mathbb{B}^n$ satisfying $\pi_n \circ \tilde{\psi} = \psi \circ (\pi|\pi^{-1}(V))$. Let $x_0$ be a point in $f^{-1}(y_0)$. Choose a neighborhood $U_0$ of $x_0$ in $U$ and a biholomorphic map $\varphi : U_0 \to \mathbb{D}$ such that $\varphi(x_0) = 0$ and $U_0 \cap f^{-1}(y_0) = \{x_0\}$. The holomorphic map

$$\psi \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{B}^n,$$

satisfies

$$(\psi \circ f \circ \varphi^{-1})(0) = 0, \quad (\psi \circ f \circ \varphi^{-1})(\mathbb{D} \setminus \{0\}) \subseteq \mathbb{B}^n \setminus \{0\},$$

and hence

$$(\psi \circ f \circ \varphi^{-1})(z) = z^p(\lambda_1(z), \ldots, \lambda_n(z))$$

for $z$ in $\mathbb{D}$, where $p$ is a positive integer, $\lambda_j : \mathbb{D} \to \mathbb{C}$ is a holomorphic function for $1 \leq j \leq n$, and $(\lambda_1(z), \ldots, \lambda_n(z))$ belongs to $\mathbb{C}^n \setminus \{0\}$ for all $z$ in $\mathbb{D}$. Then $\tilde{f} : U_0 \to \tilde{Y}$ is given by

$$\tilde{f}(x) = \tilde{\psi}^{-1}((f_1(z), \ldots, f_n(z)), (\lambda_1(z) : \ldots : \lambda_n(z)))$$

for $x$ in $U_0$ with $\varphi(x) = z$.

**Lemma 2.2.** Let $f : X \to \mathbb{P}^n$ be a holomorphic map defined on a 1-dimensional complex analytic manifold $X$. If $X$ has no compact connected component, then there exists a holomorphic map

$$F = (F_0, \ldots, F_n) : X \to \mathbb{C}^{n+1} \setminus \{0\}$$

such that

$$f(x) = (F_0(x) : \ldots : F_n(x)) \quad \text{for all } x \in X.$$
Proof. Let $\gamma_n$ be the universal line bundle on $\mathbb{P}^n$. Recall that
$$\{(\ell, w) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid w \in \ell\}$$
is the total space of $\gamma_n$. The pullback line bundle $f^*\gamma_n$ on $X$ is a holomorphic subbundle of the trivial vector bundle on $X$ with total space $X \times \mathbb{C}^{n+1}$. Since $X$ has no compact connected component, $f^*\gamma_n$ is holomorphically trivial and hence there exists a holomorphic section $u : X \to f^*\gamma_n$ such that $u(x) \neq 0$ for all $x$ in $X$ (cf. [3, Theorem 30.3]). Note that $u$ is of the form $u(x) = (x, F(x))$ for all $x$ in $X$, where $F : X \to \mathbb{C}^{n+1} \setminus \{0\}$ is a holomorphic map. The conclusion follows.

Let $X$ be a nonsingular algebraic curve. We say that a nonsingular algebraic variety $Y$ has property $(X)$ if for every holomorphic map $f : U \to Y$ defined on an open subset $U$ of $X$, every neighborhood $\mathcal{N}$ of $f$ in $\mathcal{C}(U, Y)$, every finite subset $A$ of $U$, and every nonnegative integer $s$, there exists a regular map $g : X \to Y$ such that $g|U$ belongs to $\mathcal{N}$ and $j^s g(a) = j^s f(a)$ for all $a$ in $A$. In particular, if $Y$ has property $(X)$, then given an open subset $U$ of $X$, every holomorphic map from $U$ into $Y$ can be approximated by regular maps from $X$ into $Y$.

**Lemma 2.3.** Let $Y$ be a nonsingular algebraic variety and let $\pi : \tilde{Y} \to Y$ be the monoidal transformation of $Y$ at a point $y_0$ in $Y$. Let $X$ be a nonsingular algebraic curve. Then $Y$ has property $(X)$ if and only if $\tilde{Y}$ has property $(X)$.

**Proof.** Assume that $Y$ has property $(X)$. Let $\varphi : U \to \tilde{Y}$ be a holomorphic map defined on an open subset $U$ of $X$, let $\tilde{N}$ be a neighborhood of $\varphi$ in $\mathcal{C}(U, \tilde{Y})$, let $A$ be a finite subset of $U$, and let $s$ be a nonnegative integer. We assume below that $U$ has no compact connected component, since otherwise $U = X$, $X$ is projective, and $\varphi$ is a regular map.

Choose a compact subset $K$ of $U$ and a neighborhood $\tilde{N}_K$ of $\varphi|K$ in $\mathcal{C}(K, \tilde{Y})$ such that $K$ contains $A$ and every holomorphic map $\varphi' : U \to \tilde{Y}$ belongs to $\tilde{N}$, provided that $\varphi'|K$ is in $\tilde{N}_K$.

We assert that there exists a holomorphic map $\psi : V \to \tilde{Y}$, defined on a neighborhood $V$ of $K$ in $U$, such that $\psi|K$ belongs to $\tilde{N}_K$, $j^s \psi(a) = j^s \varphi(a)$ for all $a$ in $A$, and the set $\psi^{-1}(\pi^{-1}(y_0))$ is finite. It suffices to prove this assertion for $U$ connected and $\varphi$ satisfying $\varphi(U) \subseteq \pi^{-1}(y_0)$, and therefore we assume that these conditions are satisfied. Choose a neighborhood $M$ of $y_0$ in $Y$ biholomorphic to $\mathbb{B}^n$, where $n = \dim Y$. Then there exists a biholomorphic map $\sigma : \pi^{-1}(M) \to \mathbb{B}^n$ with $\sigma(\pi^{-1}(y_0)) = \{0\} \times \mathbb{P}^{n-1}$. By Lemma 2.2, there exists a holomorphic map
$$H = (H_1, \ldots, H_n) : U \to \mathbb{C}^n \setminus \{0\}$$
such that
\[(\sigma \circ \varphi)(x) = (0, (H_1(x) : \ldots : H_n(x)))\] for all \(x\) in \(U\).

Choose a holomorphic function \(\lambda : U \to \mathbb{C}\) with \(\lambda^{-1}(0) = A\) and \(j^s\lambda(a) = 0\) for all \(a\) in \(A\) (cf. [3, Theorem 26.5]). If \(\varepsilon > 0\) is sufficiently small, then one can find a neighborhood \(V\) of \(K\) in \(U\) such that \(\varepsilon \lambda(x)H(x)\) is in \(\mathbb{B}^n\) for all \(x\) in \(V\). If \(\varepsilon > 0\) is small enough, then the holomorphic map
\[\psi : V \to \tilde{Y}, \quad \psi(x) = \sigma^{-1}(\varepsilon \lambda(x)H(x), (H_1(x) : \ldots : H_n(x)))\]
satisfies all the requirements, and hence the assertion is proved.

Put \(f = \pi \circ \psi\). Clearly, the set \(f^{-1}(y_0)\) is finite and, in the notation of Lemma 2.1, \(\tilde{f} = \psi\). Let \(N\) be a neighborhood of \(f\) in \(C(V, Y)\), let \(B\) be a finite subset of \(V\), and let \(r\) be a nonnegative integer. Since \(Y\) has property \((X)\), there exists a regular map \(g : X \to Y\) such that \(g|V\) is in \(N\) and \(j^rg(x) = j^r f(x)\) for all \(x\) in \(A \cup B\). Shrinking \(N\), if necessary, we see that the set \(g^{-1}(y_0)\) is finite. Furthermore, in view of Lemma 2.1, a suitable choice of \(N, B, r\) ensures that \(\tilde{g}|K\) is in \(\tilde{N}_K\) and \(j^s\tilde{g}(a) = j^s\varphi(a)\) for all \(a\) in \(A\). By construction, \(\tilde{g}|U\) is in \(\tilde{N}\). Since \(\tilde{g}\) is a regular map, it follows that \(\tilde{Y}\) has property \((X)\).

The converse is obvious: if \(\tilde{Y}\) has property \((X)\), then so does \(Y\). Indeed, it suffices to note that any holomorphic map \(\alpha : U \to Y\) is of the form \(\alpha = \pi \circ \beta\) for some holomorphic map \(\beta : U \to \tilde{Y}\).

Given a topological space \(T\), a continuous function \(f : T \to \mathbb{C}\), and a compact subset \(K\) of \(T\), we set
\[\|f\|_K = \sup\{|f(t)| \mid t \in K\}\]

**Lemma 2.4.** Let \(X\) be a nonsingular algebraic curve. Let \(f : U \to \mathbb{C}\) be a holomorphic function defined on an open subset \(U\) of \(X\). Let \(K\) be a compact subset of \(U\), let \(A\) be a finite subset of \(U\), and let \(s\) be a nonnegative integer. Given \(\varepsilon > 0\), there exists a rational function \(g : X \to \mathbb{C}\), without poles on \(U\), such that
\[\|g - f\|_K < \varepsilon, \quad j^sg(a) = j^sf(a)\] for all \(a\) in \(A\).

**Proof.** If \(U = X\) and \(X\) is projective, then \(f\) is a constant function and the conclusion is obvious. Otherwise, by removing finitely many points from \(X \setminus U\), we may assume that \(X\) is an affine algebraic curve and \(X \setminus U\) has no compact connected component. Then every holomorphic function on \(U\) can be uniformly approximated on compact subsets of \(U\) by holomorphic functions on \(X\) [3, Theorem 25.5], and hence also by regular functions on \(X\). The last assertion is easy to justify. Indeed, regard \(X\) as an algebraic subset of \(\mathbb{C}^n\) for some \(n\). Since \(X\) is a Stein submanifold of \(\mathbb{C}^n\), every holomorphic function on \(X\) extends to a holomorphic function on \(\mathbb{C}^n\) [4, p. 245, Theorem
and the latter can be uniformly approximated by polynomial functions on $\mathbb{C}^n$. Thus the assertion follows.

By the Riemann–Roch theorem, there exists a regular function $\mu : X \to \mathbb{C}$ with $\mu^{-1}(0) = B$, where $B$ is a finite set containing $A$. Set $\lambda = \mu^s$, and choose a positive integer $r$ such that $j^r \lambda(a) \neq 0$ for all $a$ in $B$ (in particular, $r > s$). Choose a regular function $\varphi : X \to \mathbb{C}$ with $j^r \varphi(a) = j^r f(a)$ for all $a$ in $B$. Then $(f - \varphi)/\lambda$ is a holomorphic function on $U$, and therefore we can find a regular function $\psi : X \to \mathbb{C}$ satisfying

$$\|\lambda \psi - (f - \varphi)/\lambda\|_K < \varepsilon.$$ 

Obviously, $g = \varphi + \lambda \psi$ is a regular function on $X$. Moreover,

$$g - f = \varphi + \lambda \psi - f = \lambda \psi - (f - \varphi)/\lambda,$$

and hence $\|g - f\|_K < \varepsilon$. By construction,

$$j^s g(a) = j^s (\varphi + \lambda \psi) = j^s \varphi(a) = j^s f(a)$$

for all $a$ in $B$.

**Lemma 2.5.** Projective $n$-space $\mathbb{P}^n$ has property $(X)$ for every nonsingular algebraic curve $X$.

**Proof.** The conclusion follows from Lemmas 2.2 and 2.4, and the fact that every rational map from $X$ into $\mathbb{P}^n$ is regular.

**Proof of Theorem 1.1.** It is well known that $Y$ can be regarded as a Zariski open subvariety of a nonsingular projective surface. Hence we may assume without loss of generality that $Y$ itself is projective. Then every birational equivalence $\sigma : Y \to \mathbb{P}^2$ is the composition of finitely many algebraic isomorphisms, monoidal transformations and their inverses [1, Theorem II.7]. We complete the proof by applying Lemmas 2.3 and 2.5.

Let us record the following direct consequence of Lemmas 2.3 and 2.5.

**Remark 2.6.** Let $Y = Y_k \to Y_{k-1} \to \ldots \to Y_0 = \mathbb{P}^n$ be a sequence of monoidal transformations. Let $X$ be a nonsingular algebraic curve. Given an open subset $U$ of $X$, every holomorphic map from $U$ into $Y$ can be approximated by regular maps from $X$ into $Y$.

We shall now begin the preparation for the proof of Theorem 1.3. Given a set $X$, an Abelian group $A$, a positive integer $n$, and a map $f : X \to A$, we let $f^{(n)}$ denote the map from the Cartesian $n$-fold product $X^n$ into $A$ defined by

$$f^{(n)}(x_1, \ldots, x_n) = f(x_1) + \ldots + f(x_n)$$

for all $(x_1, \ldots, x_n)$ in $X^n$.

**Lemma 2.7.** Let $X$ be a nonsingular algebraic curve, let $U$ be an open subset of $X$, and let $x_0$ be a point in $U$. Let $f : X \to A$ be a regular map
satisfying \( f(x_0) = 0 \), where \( A \) is a simple Abelian variety. Then there exists a positive integer \( n_{X,U} \) such that for every integer \( n \geq n_{X,U} \), one has either \( f^{(n)}(U^n) = \{0\} \) or \( f^{(n)}(U^n) = A \).

**Proof.** Without loss of generality, we may assume that \( X \) is projective. Let \( g \) be the genus of \( X \).

If \( g = 0 \), then \( f(X) = \{0\} \) [1, p. 84], and hence the conclusion follows with \( n_{X,U} = 1 \).

Suppose now that \( g \geq 1 \). Let \( J \) be the Jacobian variety of \( X \) and let \( \mu : X \to J \) be the canonical regular map corresponding to \( x_0 \) (in particular, \( \mu(x_0) = 0 \)). It follows from the universal property of \((J, \mu)\) that there exists a morphism of Abelian varieties \( \varphi : J \to A \) satisfying \( f = \varphi \circ \mu \) (cf. [1, pp. 82, 84]). Since \( A \) is simple, one has either \( \varphi(J) = \{0\} \) or \( \varphi(J) = A \).

It is well known that \( \mu^{(g)}(X^g) = J \) (cf. [3, Theorem 21.9]), and hence \( \mu^{(g)}(U^g) \) contains a nonempty open subset \( G \) of \( J \). Since \( J \) is a complex torus, there exists a positive integer \( k \) such that the sum \( G + \ldots + G \), with \( k \) terms, is equal to \( J \). Set \( n_{X,U} = kg \) and let \( n \) be an integer, \( n \geq n_{X,U} \). By construction, \( \mu^{(n)}(U^n) = J \). The conclusion follows since \( f^{(n)} = \varphi \circ \mu^{(n)} \).

**Proof of Theorem 1.3.** Suppose that the first Betti number of \( Y \) is nonzero. Then the Albanese variety \( \text{Alb}(Y) \) of \( Y \) is nontrivial. Choose a point \( y_0 \) in \( Y \) and let \( \alpha : Y \to \text{Alb}(Y) \) be the corresponding Albanese map (in particular, \( \alpha(y_0) = 0 \)). There exists an isogeny \( \lambda : \text{Alb}(Y) \to A_1 \times \ldots \times A_r \), where the \( A_i \) are simple Abelian varieties [7, p. 122]. Let \( \pi : A_1 \times \ldots \times A_r \to A_1 \) be the canonical projection and let \( \varphi = \pi \circ \lambda \circ \alpha \). Put \( A = A_1 \). Since the set \( \alpha(Y) \) generates \( \text{Alb}(Y) \) as a group [1, p. 84], it follows that the regular map \( \varphi : Y \to A \) is nonconstant. Of course, \( \varphi(y_0) = 0 \).

Fix an integer \( n \) with \( n \geq n_{X,D} \). Choose a small neighborhood \( V \) of \( y_0 \) in \( Y \) such that the set \( A \setminus \varphi^{(n)}(V^n) \) has a nonempty interior. By shrinking \( V \) if necessary, we may assume that there exists a biholomorphic map \( \sigma : V \to \mathbb{B}^m \) with \( \sigma(y_0) = 0 \), where \( m = \dim Y \). Since \( V \) is Zariski dense in \( Y \) and \( \varphi \) is continuous in the Zariski topology, it follows that \( \varphi \) is nonconstant on \( V \). Choose a point \( y_1 \) in \( V \) with \( \varphi(y_1) \neq 0 \). Let \( L \) be the vector subspace of \( \mathbb{C}^m \) generated by the vector \( \sigma(y_1) \) and let \( M = \sigma^{-1}(L \cap \mathbb{B}^m) \). Then \( M \) is a complex analytic submanifold of \( V \) containing \( y_0 \) and biholomorphic to \( D \) such that \( \varphi \) is nonconstant on \( M \). Choose a holomorphic map \( f : D \to Y \) such that \( f \) maps biholomorphically \( D \) onto \( M \). We claim that \( f \) cannot be approximated by regular maps from \( X \) into \( Y \). Indeed, by construction, \( A \setminus (\varphi \circ f)^{(n)}(D^n) \) has a nonempty interior and \( (\varphi \circ f)^{(n)} \) is nonconstant on \( D^n \). On the other hand, by Lemma 2.7, if \( g : X \to Y \) is a regular map, then either \((\varphi \circ g)^{(n)}(D^n) = \{0\}\) or \((\varphi \circ g)^{(n)}(D^n) = A \). Obviously, \( g|D \) cannot be arbitrarily close to \( f \) in the compact-open topology. ■
References


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