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Approximation of holomorphic maps by algebraic morphisms

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Dedicated to Professor Józef Siciak on his seventieth birthday

Abstract. Let X be a nonsingular complex algebraic curve and let Y be a nonsingular rational complex algebraic surface. Given a compact subset K of X, every holomorphic map from a neighborhood of K in X into Y can be approximated by rational maps from X into Y having no poles in K. If Y is a nonsingular projective complex surface with the first Betti number nonzero, then such an approximation is impossible.

1. Introduction. Given two topological spaces S and T, we denote by $\mathcal{C}(S,T)$ the space of all continuous maps from S into T, endowed with the compact-open topology.

Throughout this paper we call quasiprojective irreducible complex algebraic varieties simply *algebraic varieties*. Algebraic morphisms will be called *regular maps*. Unless explicitly stated otherwise, we consider algebraic varieties endowed with the topology induced by the usual metric topology on $\mathbb{C}$.

Let X and Y be algebraic varieties and let S be a subset of X. We say that a continuous map $f : S \to Y$ can be *approximated by regular* (resp. *rational*) *maps from X into Y* if for every neighborhood $\mathcal{N}$ of f in $\mathcal{C}(S,Y)$ there exists a regular map $g : X \to Y$ (resp. there exist a Zariski neighborhood X_0 of S in X and a regular map $g : X_0 \to Y$) such that the restriction $g|S$ belongs to $\mathcal{N}$. The approximation problem becomes interesting if f is holomorphic, that is, extends to a holomorphic map on a neighborhood of S. Results in this direction include the classical Runge approximation theorem [8], Runge-type approximation theorems for maps into Grassmannians [5],

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and, only loosely related, theorems on approximation of holomorphic maps by Nash maps [2, 6].

In the present paper we prove the following.

THEOREM 1.1. *Let X be a nonsingular algebraic curve and let Y be a nonsingular rational algebraic surface. Given a compact subset K of X, every holomorphic map from K into Y can be approximated by rational maps from X into Y.*

Recall that an algebraic surface is said to be *rational* if it is birationally equivalent to $\mathbb{C}^2$, that is, if it contains a nonempty Zariski open subvariety isomorphic to a Zariski open subvariety of $\mathbb{C}^2$; cf. [1] for the theory and examples of rational algebraic surfaces.

Clearly, if $X = \mathbb{C}$ and $Y = \mathbb{C}^2$, Theorem 1.1 is equivalent to the classical Runge approximation theorem.

If Y is projective, then every rational map from X into Y is automatically regular and therefore Theorem 1.1 implies the following.

COROLLARY 1.2. *Let X be a nonsingular algebraic curve and let Y be a nonsingular projective rational surface. Given an open subset U of X, every holomorphic map from U into Y can be approximated by regular maps from X into Y.*

We do not know to what extent the assumption of rationality of Y can be relaxed. Our next result shows that it cannot be relaxed too much.

THEOREM 1.3. *Let X be a nonsingular algebraic curve and let Y be a nonsingular projective algebraic variety. Let D be an open subset of X biholomorphic to the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$. If every holomorphic map from D into Y can be approximated by regular maps from X into Y, then the first Betti number of Y is equal to* 0.

2. Proofs. As usual, $\mathbb{P}^n$ will denote the complex projective n-space. The unit disc in $\mathbb{C}$ will be denoted by $\mathbb{D}$,

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\},$$

and the unit ball in $\mathbb{C}^n$ by $\mathbb{B}^n$,

$$\mathbb{B}^n = \{w \in \mathbb{C}^n \mid \|w\| < 1\}.$$

We let $\pi_n : \widetilde{\mathbb{B}}^n \to \mathbb{B}^n$ denote the monoidal transformation of $\mathbb{B}^n$ at 0,

$$\widetilde{\mathbb{B}}^n = \{(w, \ell) \in \mathbb{B}^n \times \mathbb{P}^{n-1} \mid w \in \ell\}, \qquad \pi_n(w, \ell) = w,$$

where we regard $\mathbb{P}^{n-1}$ as the variety of 1-dimensional vector subspaces of $\mathbb{C}^n$.

Given a holomorphic map $f : M \to N$ between complex analytic manifolds, a point x in M, and a nonnegative integer r, we denote by $j^r f(x)$ the r-jet of f at x.

Let X be a nonsingular algebraic curve and let Y be a nonsingular algebraic variety. Let y_0 be a point in Y and let $\pi : \widetilde{Y} \to Y$ be the monoidal transformation of Y at y_0, that is, the blowing up of Y at y_0. Let $f : U \to Y$ be a holomorphic map defined on an open subset U of X such that $f^{-1}(y_0)$ is a discrete subset of U. Then there exists a unique holomorphic map $\widetilde{f} : U \to \widetilde{Y}$ satisfying $\pi \circ \widetilde{f} = f$.

LEMMA 2.1. *With the notation as above, let $\widetilde{\mathcal{N}}$ be a neighborhood of $\widetilde{f}$ in $\mathcal{C}(U, \widetilde{Y})$, let A be a finite subset of U, and let s be a nonnegative integer. Then there exist a neighborhood $\mathcal{N}$ of f in $\mathcal{C}(U, Y)$, a finite subset B of U, and a nonnegative integer r such that for every holomorphic map $g : U \to Y$ in $\mathcal{N}$, with the set $g^{-1}(y_0)$ discrete in U and $j^r g(x) = j^r f(x)$ for all x in $A \cup B$, the following conditions are satisfied*:

(i) *$\widetilde{g}$ is in $\widetilde{\mathcal{N}}$,*

(ii) *$j^s\widetilde{g}(a) = j^s\widetilde{f}(a)$ for all a in A.*

Proof. The conclusion readily follows from the construction of $\widetilde{f}$, which we recall below for the convenience of the reader.

Choose a neighborhood V of y_0 in Y and a biholomorphic map $\psi : V \to \mathbb{B}^n$ such that $\psi(y_0) = 0$. There exists a biholomorphic map $\widetilde{\psi} : \pi^{-1}(V) \to \widetilde{\mathbb{B}}^n$ satisfying $\pi_n \circ \widetilde{\psi} = \psi \circ (\pi|\pi^{-1}(V))$. Let x_0 be a point in $f^{-1}(y_0)$. Choose a neighborhood U_0 of x_0 in U and a biholomorphic map $\varphi : U_0 \to \mathbb{D}$ such that $\varphi(x_0) = 0$ and $U_0 \cap f^{-1}(y_0) = \{x_0\}$. The holomorphic map

$$\psi \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{B}^n, \qquad \psi \circ f \circ \varphi^{-1} = (f_1, \ldots, f_n),$$

satisfies

$$(\psi \circ f \circ \varphi^{-1})(0) = 0, \qquad (\psi \circ f \circ \varphi^{-1})(\mathbb{D} \setminus \{0\}) \subseteq \mathbb{B}^n \setminus \{0\},$$

and hence

$$(\psi \circ f \circ \varphi^{-1})(z) = z^p(\lambda_1(z), \ldots, \lambda_n(z))$$

for z in $\mathbb{D}$, where p is a positive integer, $\lambda_j : \mathbb{D} \to \mathbb{C}$ is a holomorphic function for $1 \le j \le n$, and $(\lambda_1(z), \ldots, \lambda_n(z))$ belongs to $\mathbb{C}^n \setminus \{0\}$ for all z in $\mathbb{D}$. Then $\widetilde{f} : U_0 \to \widetilde{Y}$ is given by

$$\widetilde{f}(x) = \widetilde{\psi}^{-1}((f_1(z), \ldots, f_n(z)), (\lambda_1(z) : \ldots : \lambda_n(z)))$$

for x in U_0 with $\varphi(x) = z$. ∎

LEMMA 2.2. *Let $f : X \to \mathbb{P}^n$ be a holomorphic map defined on a 1-dimensional complex analytic manifold X. If X has no compact connected component, then there exists a holomorphic map*

$$F = (F_0, \ldots, F_n) : X \to \mathbb{C}^{n+1} \setminus \{0\}$$

such that

$$f(x) = (F_0(x) : \ldots : F_n(x)) \quad \textit{for all } x \textit{ in } X.$$

Proof. Let γ_n be the universal line bundle on $\mathbb{P}^n$. Recall that

$$\{(\ell, w) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid w \in \ell\}$$

is the total space of γ_n. The pullback line bundle $f^*\gamma_n$ on X is a holomorphic subbundle of the trivial vector bundle on X with total space $X \times \mathbb{C}^{n+1}$. Since X has no compact connected component, $f^*\gamma_n$ is holomorphically trivial and hence there exists a holomorphic section $u : X \to f^*\gamma_n$ such that $u(x) \neq 0$ for all x in X (cf. [3, Theorem 30.3]). Note that u is of the form $u(x) = (x, F(x))$ for all x in X, where $F : X \to \mathbb{C}^{n+1} \setminus \{0\}$ is a holomorphic map. The conclusion follows. ∎

Let X be a nonsingular algebraic curve. We say that a nonsingular algebraic variety Y has *property* (X) if for every holomorphic map $f : U \to Y$ defined on an open subset U of X, every neighborhood $\mathcal{N}$ of f in $\mathcal{C}(U, Y)$, every finite subset A of U, and every nonnegative integer s, there exists a regular map $g : X \to Y$ such that $g|U$ belongs to $\mathcal{N}$ and $j^s g(a) = j^s f(a)$ for all a in A. In particular, if Y has property (X), then given an open subset U of X, every holomorphic map from U into Y can be approximated by regular maps from X into Y.

LEMMA 2.3. *Let Y be a nonsingular algebraic variety and let $\pi : \widetilde{Y} \to Y$ be the monoidal transformation of Y at a point y_0 in Y. Let X be a nonsingular algebraic curve. Then Y has property (X) if and only if $\widetilde{Y}$ has property (X).*

Proof. Assume that Y has property (X). Let $\varphi : U \to \widetilde{Y}$ be a holomorphic map defined on an open subset U of X, let $\widetilde{\mathcal{N}}$ be a neighborhood of φ in $\mathcal{C}(U, \widetilde{Y})$, let A be a finite subset of U, and let s be a nonnegative integer. We assume below that U has no compact connected component, since otherwise $U = X$, X is projective, and φ is a regular map.

Choose a compact subset K of U and a neighborhood $\widetilde{\mathcal{N}}_K$ of $\varphi|K$ in $\mathcal{C}(K, \widetilde{Y})$ such that K contains A and every holomorphic map $\varphi' : U \to \widetilde{Y}$ belongs to $\widetilde{\mathcal{N}}$, provided that $\varphi'|K$ is in $\widetilde{\mathcal{N}}_K$.

We assert that there exists a holomorphic map $\psi : V \to \widetilde{Y}$, defined on a neighborhood V of K in U, such that $\psi|K$ belongs to $\widetilde{\mathcal{N}}_K$, $j^s\psi(a) = j^s\varphi(a)$ for all a in A, and the set $\psi^{-1}(\pi^{-1}(y_0))$ is finite. It suffices to prove this assertion for U connected and φ satisfying $\varphi(U) \subseteq \pi^{-1}(y_0)$, and therefore we assume that these conditions are satisfied. Choose a neighborhood M of y_0 in Y biholomorphic to $\mathbb{B}^n$, where $n = \dim Y$. Then there exists a biholomorphic map $\sigma : \pi^{-1}(M) \to \widetilde{\mathbb{B}}^n$ with $\sigma(\pi^{-1}(y_0)) = \{0\} \times \mathbb{P}^{n-1}$. By Lemma 2.2, there exists a holomorphic map

$$H = (H_1, \ldots, H_n) : U \to \mathbb{C}^n \setminus \{0\}$$

such that

$$(\sigma \circ \varphi)(x) = (0, (H_1(x) : \ldots : H_n(x))) \quad \text{for all } x \text{ in } U.$$

Choose a holomorphic function $\lambda : U \to \mathbb{C}$ with $\lambda^{-1}(0) = A$ and $j^s\lambda(a) = 0$ for all a in A (cf. [3, Theorem 26.5]). If $\varepsilon > 0$ is sufficiently small, then one can find a neighborhood V of K in U such that $\varepsilon\lambda(x)H(x)$ is in $\mathbb{B}^n$ for all x in V. If $\varepsilon > 0$ is small enough, then the holomorphic map

$$\psi : V \to \widetilde{Y}, \qquad \psi(x) = \sigma^{-1}(\varepsilon\lambda(x)H(x), (H_1(x) : \ldots : H_n(x)))$$

satisfies all the requirements, and hence the assertion is proved.

Put $f = \pi \circ \psi$. Clearly, the set $f^{-1}(y_0)$ is finite and, in the notation of Lemma 2.1, $\widetilde{f} = \psi$. Let $\mathcal{N}$ be a neighborhood of f in $\mathcal{C}(V, Y)$, let B be a finite subset of V, and let r be a nonnegative integer. Since Y has property (X), there exists a regular map $g : X \to Y$ such that $g|V$ is in $\mathcal{N}$ and $j^r g(x) = j^r f(x)$ for all x in $A \cup B$. Shrinking $\mathcal{N}$, if necessary, we see that the set $g^{-1}(y_0)$ is finite. Furthermore, in view of Lemma 2.1, a suitable choice of $\mathcal{N}$, B, and r ensures that $\widetilde{g}|K$ is in $\widetilde{\mathcal{N}}_K$ and $j^s\widetilde{g}(a) = j^s\varphi(a)$ for all a in A. By construction, $\widetilde{g}|U$ is in $\widetilde{\mathcal{N}}$. Since $\widetilde{g}$ is a regular map, it follows that $\widetilde{Y}$ has property (X).

The converse is obvious: if $\widetilde{Y}$ has property (X), then so does Y. Indeed, it suffices to note that any holomorphic map $\alpha : U \to Y$ is of the form $\alpha = \pi \circ \beta$ for some holomorphic map $\beta : U \to \widetilde{Y}$. ■

Given a topological space T, a continuous function $f : T \to \mathbb{C}$, and a compact subset K of T, we set

$$\|f\|_K = \sup\{|f(t)| \mid t \in K\}.$$

LEMMA 2.4. *Let X be a nonsingular algebraic curve. Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open subset U of X. Let K be a compact subset of U, let A be a finite subset of U, and let s be a nonnegative integer. Given $\varepsilon > 0$, there exists a rational function $g : X \to \mathbb{C}$, without poles on U, such that*

$$\|g - f\|_K < \varepsilon, \qquad j^s g(a) = j^s f(a) \quad \text{for all } a \text{ in } A.$$

Proof. If $U = X$ and X is projective, then f is a constant function and the conclusion is obvious. Otherwise, by removing finitely many points from $X \setminus U$, we may assume that X is an affine algebraic curve and $X \setminus U$ has no compact connected component. Then every holomorphic function on U can be uniformly approximated on compact subsets of U by holomorphic functions on X [3, Theorem 25.5], and hence also by regular functions on X. The last assertion is easy to justify. Indeed, regard X as an algebraic subset of $\mathbb{C}^n$ for some n. Since X is a Stein submanifold of $\mathbb{C}^n$, every holomorphic function on X extends to a holomorphic function on $\mathbb{C}^n$ [4, p. 245, Theorem

18], and the latter can be uniformly approximated by polynomial functions on $\mathbb{C}^n$. Thus the assertion follows.

By the Riemann–Roch theorem, there exists a regular function $\mu : X \to \mathbb{C}$ with $\mu^{-1}(0) = B$, where B is a finite set containing A. Set $\lambda = \mu^s$, and choose a positive integer r such that $j^r\lambda(a) \neq 0$ for all a in B (in particular, $r > s$). Choose a regular function $\varphi : X \to \mathbb{C}$ with $j^r\varphi(a) = j^r f(a)$ for all a in B. Then $(f - \varphi)/\lambda$ is a holomorphic function on U, and therefore we can find a regular function $\psi : X \to \mathbb{C}$ satisfying

$$\|\lambda(\psi - (f - \varphi)/\lambda)\|_K < \varepsilon.$$

Obviously, $g = \varphi + \lambda\psi$ is a regular function on X. Moreover,

$$g - f = \varphi + \lambda\psi - f = \lambda(\psi - (f - \varphi)/\lambda),$$

and hence $\|g - f\|_K < \varepsilon$. By construction,

$$j^s g(a) = j^s(\varphi + \lambda\psi) = j^s\varphi(a) = j^s f(a)$$

for all a in B. ∎

LEMMA 2.5. *Projective n-space $\mathbb{P}^n$ has property (X) for every nonsingular algebraic curve X.*

Proof. The conclusion follows from Lemmas 2.2 and 2.4, and the fact that every rational map from X into $\mathbb{P}^n$ is regular. ∎

Proof of Theorem 1.1. It is well known that Y can be regarded as a Zariski open subvariety of a nonsingular projective surface. Hence we may assume without loss of generality that Y itself is projective. Then every birational equivalence $\sigma : Y \to \mathbb{P}^2$ is the composition of finitely many algebraic isomorphisms, monoidal transformations and their inverses [1, Theorem II.7]. We complete the proof by applying Lemmas 2.3 and 2.5. ∎

Let us record the following direct consequence of Lemmas 2.3 and 2.5.

REMARK 2.6. Let $Y = Y_k \to Y_{k-1} \to \ldots \to Y_0 = \mathbb{P}^n$ be a sequence of monoidal transformations. Let X be a nonsingular algebraic curve. Given an open subset U of X, every holomorphic map from U into Y can be approximated by regular maps from X into Y.

We shall now begin the preparation for the proof of Theorem 1.3. Given a set X, an Abelian group A, a positive integer n, and a map $f : X \to A$, we let $f^{(n)}$ denote the map from the Cartesian n-fold product X^n into A defined by

$$f^{(n)}(x_1, \ldots, x_n) = f(x_1) + \ldots + f(x_n)$$

for all $(x_1, \ldots, x_n)$ in X^n.

LEMMA 2.7. *Let X be a nonsingular algebraic curve, let U be an open subset of X, and let x_0 be a point in U. Let $f : X \to A$ be a regular map* *satisfying* $f(x_0) = 0$, *where* A *is a simple Abelian variety. Then there exists a positive integer* $n_{X,U}$ *such that for every integer* $n \geq n_{X,U}$, *one has either* $f^{(n)}(U^n) = \{0\}$ *or* $f^{(n)}(U^n) = A$.

Proof. Without loss of generality, we may assume that X is projective. Let g be the genus of X.

If $g = 0$, then $f(X) = \{0\}$ [1, p. 84], and hence the conclusion follows with $n_{X,U} = 1$.

Suppose now that $g \geq 1$. Let J be the Jacobian variety of X and let $\mu : X \to J$ be the canonical regular map corresponding to x_0 (in particular, $\mu(x_0) = 0$). It follows from the universal property of (J, μ) that there exists a morphism of Abelian varieties $\varphi : J \to A$ satisfying $f = \varphi \circ \mu$ (cf. [1, pp. 82, 84]). Since A is simple, one has either $\varphi(J) = \{0\}$ or $\varphi(J) = A$.

It is well known that $\mu^{(g)}(X^g) = J$ (cf. [3, Theorem 21.9]), and hence $\mu^{(g)}(U^g)$ contains a nonempty open subset G of J. Since J is a complex torus, there exists a positive integer k such that the sum $G + \ldots + G$, with k terms, is equal to J. Set $n_{X,U} = kg$ and let n be an integer, $n \geq n_{X,U}$. By construction, $\mu^{(n)}(U^n) = J$. The conclusion follows since $f^{(n)} = \varphi \circ \mu^{(n)}$. ∎

Proof of Theorem 1.3. Suppose that the first Betti number of Y is nonzero. Then the Albanese variety $\mathrm{Alb}(Y)$ of Y is nontrivial. Choose a point y_0 in Y and let $\alpha : Y \to \mathrm{Alb}(Y)$ be the corresponding Albanese map (in particular, $\alpha(y_0) = 0$). There exists an isogeny $\lambda : \mathrm{Alb}(Y) \to A_1 \times \ldots \times A_r$, where the A_i are simple Abelian varieties [7, p. 122]. Let $\pi : A_1 \times \ldots \times A_r \to A_1$ be the canonical projection and let $\varphi = \pi \circ \lambda \circ \alpha$. Put $A = A_1$. Since the set $\alpha(Y)$ generates $\mathrm{Alb}(Y)$ as a group [1, p. 84], it follows that the regular map $\varphi : Y \to A$ is nonconstant. Of course, $\varphi(y_0) = 0$.

Fix an integer n with $n \geq n_{X,D}$. Choose a small neighborhood V of y_0 in Y such that the set $A \setminus \varphi^{(n)}(V^n)$ has a nonempty interior. By shrinking V if necessary, we may assume that there exists a biholomorphic map $\sigma : V \to \mathbb{B}^m$ with $\sigma(y_0) = 0$, where $m = \dim Y$. Since V is Zariski dense in Y and φ is continuous in the Zariski topology, it follows that φ is nonconstant on V. Choose a point y_1 in V with $\varphi(y_1) \neq 0$. Let L be the vector subspace of $\mathbb{C}^m$ generated by the vector $\sigma(y_1)$ and let $M = \sigma^{-1}(L \cap \mathbb{B}^m)$. Then M is a complex analytic submanifold of V containing y_0 and biholomorphic to D such that φ is nonconstant on M. Choose a holomorphic map $f : D \to Y$ such that f maps biholomorphically D onto M. We claim that f cannot be approximated by regular maps from X into Y. Indeed, by construction, $A \setminus (\varphi \circ f)^{(n)}(D^n)$ has a nonempty interior and $(\varphi \circ f)^{(n)}$ is nonconstant on D^n. On the other hand, by Lemma 2.7, if $g : X \to Y$ is a regular map, then either $(\varphi \circ g)^{(n)}(D^n) = \{0\}$ or $(\varphi \circ g)^{(n)}(D^n) = A$. Obviously, $g|D$ cannot be arbitrarily close to f in the compact-open topology. ∎

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