ANNALES
POLONICI MATHEMATICI
80 (2003)

A generalization of Radó's theorem

by E. M. Chirka (Moscow)

Dedicated to Prof. J. Siciak on the occasion of his 70th birthday

Abstract. If Σ is a compact subset of a domain $\Omega \subset \mathbb{C}$ and the cluster values on $\partial\Sigma$ of a holomorphic function f in $\Omega \setminus \Sigma$, $f' \not\equiv 0$, are contained in a compact null-set for the holomorphic Dirichlet class, then f extends holomorphically onto the whole domain Ω.

1. The classical Radó theorem says that a continuous function f in a domain $\Omega \subset \mathbb{C}$ which is holomorphic outside its zero-set $f^{-1}(0)$ is holomorphic everywhere in the domain Ω. It is known that $f^{-1}(0)$ can be replaced by $f^{-1}(E)$ where E is a compact subset of zero capacity in $\mathbb{C}$ (see [3]). There is a more general result in [3]: if f is holomorphic and not locally constant outside a closed subset $\Sigma \subset \Omega$, and the cluster values of f on the boundary of Σ are contained in a compact polar set $E \subset \mathbb{C}$, then f extends holomorphically onto Ω. In this paper we enlarge the class of "images" E still admitting such a continuation and show that this larger class is essentially maximal possible.

We say that a closed set $E \subset \widehat{\mathbb{C}} = \mathbb{C} \cup \infty$ is a *null-set for the holomorphic Dirichlet class* (briefly, $E \in N_{\mathcal{D}}$) if every holomorphic function h in $\widehat{\mathbb{C}} \setminus E$ with finite Dirichlet integral $\int_{\mathbb{C}\setminus E} |h'|^2\, dS$ is constant. Geometrically, the Dirichlet integral equals the area of the image $h(\widehat{\mathbb{C}} \setminus E)$ counted with multiplicities, so it is invariant with respect to conformal transformations of $\widehat{\mathbb{C}} \setminus E$.

This invariant class of sets was introduced and investigated by Ahlfors and Beurling [1]. It follows easily from the definition that every set $E \in N_{\mathcal{D}}$ is *totally disconnected* and, for an arbitrary domain $G \subset \widehat{\mathbb{C}}$, every holomorphic function h in $G \setminus E$ with finite Dirichlet integral $\int_{G\setminus E} |h'|^2\, dS$ extends holomorphically onto G (see [1]).

2000 *Mathematics Subject Classification*: 30C99, 32B15.
Key words and phrases: Radó theorem, holomorphic extension.
Research supported by the Russian FBR, grant 02-01-01291.

THEOREM. *Let Σ be a relatively closed subset of a domain $\Omega \subset \widehat{\mathbb{C}}$ such that all the connected components of Σ are compact. Let $f : \Omega \setminus \Sigma \to \widehat{\mathbb{C}}$ be a meromorphic function such that all the cluster values of f at the boundary points of Σ are contained in a closed set $E \in N_{\mathcal{D}}$. Assume also that f is not constant in some connected component of $\Omega \setminus \Sigma$. Then f extends to a function meromorphic in Ω.*

For f holomorphic in $\Omega \setminus \Sigma$ and for E compact, the extended meromorphic function in Ω is locally bounded, hence holomorphic.

The compactness of the components of Σ is essentially used in the proof below. The opposite case, when each component of Σ has cluster points on $\partial\Omega$, is of quite different nature related to the boundary properties of holomorphic functions. Nevertheless, the author does not know whether the theorem is false without the compactness assumption.

The present formulation of the theorem is due to Prof. Don Marshall who attracted the attention of the author to the paper [1] and pointed out that the proof given originally for singularities E of zero analytic capacity is valid almost literally for singularities of the essentially larger class $N_{\mathcal{D}}$.

The class $N_{\mathcal{D}}$ contains all closed polar sets, all sets of zero analytic capacity, but also many other sets. The metric (Hausdorff) dimension of polar sets equals zero, for sets of zero analytic capacity it does not exceed one, while the metric dimension of sets in $N_{\mathcal{D}}$ can be two. (Simple examples are the direct products of Cantor sets of zero length on the coordinate axes, see [1].)

The class $N_{\mathcal{D}}$ in our considerations is maximal possible. Indeed, by [1], a compact set $E \subset \mathbb{C}$ is not contained in $N_{\mathcal{D}}$ if and only if there is a conformal map g of $\widehat{\mathbb{C}} \setminus E$ into $\widehat{\mathbb{C}}$, $g(\infty) = \infty$, which is not fractional-linear. Take $\Omega = \widehat{\mathbb{C}}$, $\Sigma = \widehat{\mathbb{C}} \setminus g(\widehat{\mathbb{C}} \setminus E)$ and $f = g^{-1}$. Then the cluster values of f on $\partial\Sigma$ are contained in E. Assume that f extends meromorphically onto Ω; then it is rational. As E is compact in $\mathbb{C}$, this rational function has only one simple pole at infinity, hence, it is linear. Thus, g is fractional-linear, contrary to $E \notin N_{\mathcal{D}}$. This shows that f has no meromorphic extension onto Ω.

I would like to express my gratitude to Don Marshall, Lee Stout and Alex Volberg for fruitful discussions.

2. Here we prove the theorem.

STEP 1. *Reduction to a proper map.* If the function f is constant in a component Ω' of $\Omega \setminus \Sigma$ then $\partial\Omega' \subset \Sigma \cup \partial\Omega$ intersects Σ (otherwise $\partial\Omega' \subset \partial\Omega$, hence $\Omega' = \Omega$ because Ω is connected, hence $f \equiv \text{const}$); thus, this constant value is contained in E. It follows that, for $a \in \mathbb{C} \setminus E$, the zero-set of the function $f - a$ in $\Omega \setminus \Sigma$ is discrete, and its closure does not intersect Σ. Thus, substituting f by $1/(f-a)$ and removing from Ω a closed neighbourhood of this discrete set which does not intersect Σ either, we can assume further that f is holomorphic in $\Omega \setminus \Sigma$ and $E \subset \mathbb{C}$ is compact.

Next, we can assume that $\infty \in \Omega \setminus \Sigma$ and f is not constant in the unbounded component of $\Omega \setminus \Sigma$. It follows from the definition of components (of Σ) that there is an increasing sequence of subdomains $\Omega_\nu \subset \Omega$ such that $\bigcup \Omega_\nu = \Omega$ and $\Sigma \cap \Omega_\nu$ is compact for each ν. Thus, we can assume further that Σ is compact in $\mathbb{C}$.

Denote by $\widetilde{\Omega}$ the union of Ω and the polynomially convex hull $\widehat{\Sigma}$ of Σ. Then $\widetilde{\Omega}$ is a domain in $\widehat{\mathbb{C}}$ and $\widetilde{\Omega} \setminus \widehat{\Sigma} \subset \Omega \setminus \Sigma$. If we could prove the theorem for $\widetilde{\Omega}, \widehat{\Sigma}$ and $f|\widetilde{\Omega} \setminus \widehat{\Sigma}$ (instead of Ω, Σ and f) then it would follow also that $\widehat{\Sigma}$ is totally disconnected (being contained in the preimage of the totally disconnected set E under the meromorphic map extending f), hence, $\widehat{\Sigma} = \Sigma$ and $\widetilde{\Omega} = \Omega$. Thus, we reduce the proof to the case when Σ is a polynomially convex compact set in $\mathbb{C}$. Then $\Omega \setminus \Sigma$ is connected (has only one component) and f is not constant in this domain.

As E is totally disconnected, the set $f^{-1}(E) \equiv \{z \in \Omega \setminus \Sigma : f(z) \in E\}$ is also totally disconnected. Hence, there is a subdomain $\Omega_0 \Subset \Omega$ containing Σ and such that $\partial\Omega_0$ does not intersect $f^{-1}(E)$ and $\Omega_0 \cap (\Sigma \cup f^{-1}(E))$ is a (polynomially convex) compact set. Then $f(\partial\Omega_0)$ is a compact set disjoint from E, hence there is a neighbourhood $U_0 \supset E$ such that $U_0 \cap f(\partial\Omega_0)$ is empty. Set $V_0 = (\Omega_0 \cap f^{-1}(U_0 \setminus E)) \cup \Sigma$. Then V_0 is a relatively compact open subset of Ω containing Σ and

$$f : V_0 \setminus \Sigma \to U_0 \setminus E$$

is a proper holomorphic mapping. It is enough to show that f extends holomorphically in V_0, i.e., in each connected component of V_0. If V is such a component then $f(V \setminus \Sigma)$ is connected and $f(\partial V) \subset \partial U_0$. Hence, there is a component U of U_0 such that $f(V \setminus \Sigma)$ is contained in U, the map $f : V \setminus \Sigma \to U \setminus E$ is proper, and it is enough to show that f extends holomorphically onto V.

STEP 2. *Extension of the graph.* As $f|_{V\setminus\Sigma}$ is proper and its cluster values on $\partial\Sigma$ are contained in E, the graph $A = \{(z, w) : z \in V \setminus \Sigma,\ w = f(z)\}$ of the function f over $V \setminus \Sigma$ is closed in $(V \times U) \setminus (\Sigma \times E)$. Let m be the multiplicity (degree) of the proper map $f|_{V\setminus\Sigma}$. Then A is an m-sheeted analytic cover over $U \setminus E$, hence, it is represented as the zero-set of a Weierstrass polynomial $P(z, w)$,

$$A: \quad P(z, w) \equiv z^m + a_1(w)z^{m-1} + \ldots + a_m(w) = 0, \qquad w \in U \setminus E,$$

with coefficients a_j holomorphic in $U \setminus E$ (see, e.g., [2]). The construction of this P is the following. If the value w_0 is not critical for f then the roots of the equation $f(z) = w$ with respect to z are holomorphic functions $\alpha_\nu(w)$ in a neighbourhood of w_0. By the Vieta formulas, the coefficients $a_j(w)$ are homogeneous polynomials of these roots with constant integer coefficients. As $\overline{V}$ is compact, it follows that a_j are uniformly bounded, hence extend holomorphically onto the discrete set of critical values of f in $U \setminus E$.

We show that a_j extend holomorphically onto U. By the Newton formulas, each a_j is a polynomial with rational coefficients of $b_1, \ldots, b_j$ where $b_s := \sum_{\nu=1}^m \alpha_\nu^s$. As $E \in N_{\mathcal{D}}$, it is enough to show that $b_s(w)$ have finite Dirichlet integrals in $U \setminus E$. As $b_s' = s \sum \alpha_j^{s-1} \alpha_j'$ and $\alpha_j(w)$ are uniformly bounded in $U \setminus E$, we have

$$\int_{U\setminus E} |b_s'|^2 \, dS_w \le C \int_{U\setminus E} \sum_{j=1}^m |\alpha_j'|^2 \, dS_w = C \int_{V\setminus \Sigma} dS_z = C \operatorname{mes}_2(V \setminus \Sigma) < \infty$$

because $\alpha_j'(f(z)) = 1/f'(z)$, being the derivative of the inverse map.

Let $\widetilde{a}_j$ be holomorphic extensions of a_j on U, $\widetilde{P}(z,w) := z^m + \widetilde{a}_1(w) z^{m-1} + \ldots + \widetilde{a}_m(w)$ and $\widetilde{A} = \{(z,w) : \widetilde{P}(z,w) = 0,\ w \in U\}$. Then $\widetilde{A}$ is a purely one-dimensional complex analytic subset of $\mathbb{C} \times U$ containing A.

Step 3. *Extension of f.* As E is nowhere dense in $\mathbb{C}$ and the projection of $\widetilde{A}$ onto U is proper, the set $\widetilde{A} \cap (\mathbb{C} \times E)$ is nowhere dense in $\widetilde{A}$. As $\widetilde{P} = P$ in $U \setminus E$, one has $\widetilde{A} \setminus (\mathbb{C} \times E) = A \setminus (\mathbb{C} \times E)$, hence the set $\widetilde{A}$ is the closure of A in $\mathbb{C} \times U$. Thus, $\widetilde{A} \subset A \cup (\Sigma \times E)$. It follows that the projection of $\widetilde{A}$ into the z-plane is proper, hence it is an analytic covering over V. As the number of sheets over $V \setminus \Sigma$ equals one (A is a graph), it follows that it equals one everywhere in V, hence $\widetilde{A}$ is a graph $w = \widetilde{f}(z)$ over V, $\widetilde{f} = f$ in $V \setminus \Sigma$. As $\widetilde{A}$ is complex-analytic, the function $w - \widetilde{f}(z)$ is a Weierstrass polynomial (see [2, Th. 2.8]), i.e. the function $\widetilde{f}$ is holomorphic in V. ■

It is not difficult to rewrite the proof of the theorem without this lifting to $\mathbb{C}^2$, analytic sets etc., but we wanted to emphasize the geometry of the elimination of singularities, having in mind some generalizations to Riemann surfaces and complex analytic sets of finite area.

References

[1] L. Ahlfors and A. Beurling, *Conformal invariants and function-theoretical null-sets*, Acta Math. 83 (1950), 101–129.

[2] E. M. Chirka, *Complex Analytic Sets*, Nauka, Moscow, 1985 (in Russian); English transl., Kluwer, Dordrecht, 1989.

[3] E. L. Stout, *A generalization of a theorem of Radó*, Math. Ann. 177 (1968), 339–340.

Steklov Mathematical Institute
Gubkin st. 8
Moscow, Russia
E-mail: chirka@mi.ras.ru

Reçu par la Rédaction le 30.1.2002
Révisé le 8.4.2002 (1388)