On the removal of subharmonic singularities of plurisubharmonic functions

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Dedicated to Prof. J. Siciak on the occasion of his 70th birthday

Abstract. It is proved that any subharmonic function in a domain $\Omega \subset \mathbb{C}^n$ which is plurisubharmonic outside of a real hypersurface of class $C^1$ is indeed plurisubharmonic in $\Omega$.

1. Let $E$ be a closed nowhere dense subset of a domain $\Omega$ in $\mathbb{C}^n$ and $u$ be a subharmonic function in $\Omega$, which is plurisubharmonic (psh) in $\Omega \setminus E$. The question is what conditions on $E$ guarantee the plurisubharmonicity of $u$ in the whole domain $\Omega$. For the question to be nontrivial, we consider the sets $E$ which are nonremovable for general bounded psh functions in $\Omega \setminus E$. The simplest class of such singularities is given by smooth hypersurfaces in $\Omega$, so the following theorem can be considered as a first step towards the solution of the general problem.

**Theorem.** Let $\Gamma$ be a $C^1$-hypersurface in a domain $\Omega \subset \mathbb{C}^n$ and $u$ be a subharmonic function in $\Omega$ which is plurisubharmonic in $\Omega \setminus \Gamma$. Then $u$ is plurisubharmonic in $\Omega$.

Note that we do not assume any smoothness of $u$ in $\Omega$. If $\Gamma$ divides $\Omega$ into two components $\Omega_\pm$ and $u \in C^2(\Omega \setminus \Gamma) \cap C^1(\overline{\Omega}_\pm)$ is continuous in $\Omega$, the condition of subharmonicity of $u$ in $\Omega$ means that $\Delta u \geq 0$ in $\Omega \setminus \Gamma$ and $\partial u / \partial n_+ + \partial u / \partial n_- \geq 0$ on $\Gamma$ where $\partial u / \partial n_\pm$ are the (inner) normal derivatives of $u|_{\Omega_\pm}$ at points of $\Gamma$. This case of “classical” smoothness of $u$ was considered by P. Blanchet [1] who proved the theorem under these additional assumptions. The general case needs another technique, even for $C^\infty$-hypersurfaces and $u$ piecewise smooth. On the other hand, our proof

2000 Mathematics Subject Classification: 32U30, 32U40.

Key words and phrases: plurisubharmonic function, removal of singularities.

Research supported by the Russian FBR, grant 02-01-01291.
does not work already in the case of Lipschitz graphs (the $C^1$-smoothness of $\Gamma$ is essential). The following question remains open:

Let $E$ be a closed subset of $\Omega \subset \mathbb{C}^n$ with locally finite Hausdorff $(2n-1)$-measure and let a function $u$ be subharmonic in $\Omega$ and psh in $\Omega \setminus E$. Does it follow that $u$ is plurisubharmonic in $\Omega$?

2. The proof of the theorem is based on the notion of positive currents (see [3, 2]). Recall that $\nu \in \operatorname{psh}(\Omega)$ if and only if the current $dd^c\nu = i \sum v_{jk} dz_j \wedge d\zbar_k$ of bidegree $(1,1)$ in $\Omega$ is positive, that is, $(dd^c\nu, \Phi) \geq 0$ for each positive $(n-1, n-1)$-form $\Phi$ of class $C^\infty_0(\Omega)$. (Here $d^c = i(\bar{\partial} - \partial)$; the class of test forms $\Phi$ can be reduced to $\Phi$ if the type $\varphi \prod_{\nu=1}^{n-1} (i d\nu \wedge d\bar{\nu})$ where $\varphi \in C^\infty_0(\Omega)$, $\varphi \geq 0$ and $d\nu$ are $\mathbb{C}$-linear functions in $\mathbb{C}^n$, see [2].) As is well known (see [3]), the coefficients $v_{jk}$ of the current $dd^c\nu$ for $\nu \in \operatorname{psh}(\Omega)$ are (locally finite, complex-valued) measures in $\Omega$, $v_{jj} \geq 0$ and $|v_{jk}| \leq \sum v_{\ell} = \frac{1}{2} \Delta u$, $j, k = 1, \ldots, n$.

Step 1. The theorem is local, so we can assume that $\Gamma$ is the zero-set of a function $\varrho \in C^1(\Omega)$ with $d\varrho \neq 0$ on $\Gamma$. By the Whitney extension theorem, we can also assume that $\varphi \in C^\infty(\Omega \setminus \Gamma)$, $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon = 0$ in a neighbourhood of $0$ and $\chi_\varepsilon(t) = 1$ for $|t| \geq \varepsilon > 0$. Then $dd^c\mu = \mu_\varepsilon + \sigma_\varepsilon$ where $\mu_\varepsilon = \lambda_\varepsilon dd^c u$ and

$$
\begin{align*}
\sigma_\varepsilon &= (1 - \lambda_\varepsilon)dd^c u = d((1 - \lambda_\varepsilon)d^c u) + (\chi'_\varepsilon \circ \varrho)d\varrho \wedge d^c u \\
(2) \quad &= -(1 - \lambda_\varepsilon)d^c du = -d^c((1 - \lambda_\varepsilon)du) - (\chi'_\varepsilon \circ \varrho)d^c \varrho \wedge du.
\end{align*}
$$

Step 2. As $u \in \operatorname{psh}(\Omega \setminus \Gamma)$ and $\lambda_\varepsilon = 0$ in a neighbourhood of $\Gamma$, the currents

$$
\mu_\varepsilon := \lambda_\varepsilon dd^c u = \lambda_\varepsilon i \sum u_{jk} dz_j \wedge d\zbar_k
$$

are well defined and positive in $\Omega$. As $u$ is subharmonic in $\Omega$, the measure $\Delta u$ is nonnegative and locally bounded in $\Omega$. As $|u_{jk}| \leq \Delta u$, $j, k \leq n$, it follows that

$$
\lim_{\varepsilon \to 0} \mu_\varepsilon =: \mu = i \sum \mu_{jk} dz_j \wedge d\zbar_k \text{ exists and is a positive current in } \Omega,
$$

and its coefficients $\mu_{jk}$ are locally finite measures in $\Omega$. By the construction, $\mu$ is carried by $\Omega \setminus \Gamma$, i.e., $\mu_{jk}(E) = 0$, $j, k \leq n$, for any set $E \subset \Gamma$. Moreover, $dd^c \mu = \mu + \sigma$, where

$$
\sigma = \lim_{\varepsilon \to 0} (1 - \lambda_\varepsilon)dd^c u
$$

is a current in $\Omega$ supported on $\Gamma$.

Step 3. As $u \in \operatorname{sh}(\Omega)$, the 1-currents $du, d^c u$ have coefficients in $L^1_{\text{loc}}(\Omega)$ (this follows obviously from the Riesz decomposition). Thus for every $(2n - 3)$-form $\Psi$ of class $C^1_0(\Omega)$ (the coefficients belong to $C^1(\Omega)$ and have
Removal of subharmonic singularities

compact supports) we see, according to (1), that the value

$$(dd^c u, d\omega \wedge \Psi) := \lim_{\delta \to 0} (dd^c u, d\omega_{\delta} \wedge \Psi)$$

$$= \lim_{\delta \to 0} ((\mu_\varepsilon, d\omega_{\delta} \wedge \Psi) + ((\chi_\varepsilon' \circ \varrho)d\varrho \wedge d^c u, d\omega_{\delta} \wedge \Psi)$$

$$+ ((1 - \lambda_\varepsilon)d^c u, d\omega_{\delta} \wedge d\Psi))$$

$$= (\mu_\varepsilon, d\varrho \wedge \Psi) + o_\varepsilon(1) = (\mu, d\varrho \wedge \Psi)$$

is well defined. (Here the index \(\delta\) means \(\delta\)-regularization, that is, convolution with a nonnegative function in \(C_0^\infty(\mathbb{C}^n)\) supported in the ball \(|z| < \delta\) and having Lebesgue integral 1.)

In the same way, using (2) we find that the values \((dd^c u, d^c \varrho \wedge \Psi)\) are well defined and \((dd^c u, d^c \varrho \wedge \Psi) = (\mu, d^c \varrho \wedge \Psi)\), which implies that

$$(\sigma, d\varrho \wedge \Psi) := \lim_{\delta \to 0} (\sigma, d\omega_{\delta} \wedge \Psi) = 0,$$

$$(\sigma, d^c \varrho \wedge \Psi) := \lim_{\delta \to 0} (\sigma, d^c \omega_{\delta} \wedge \Psi) = 0.$$

**Step 4.** Let \(\varphi \in C_0(\Omega), \varphi \geq 0\), and let \(\omega = \frac{1}{4}(dd^c |z|^2)\) be the fundamental form in \(\mathbb{C}^n\). Then

$$(dd^c u, \varphi \omega^{n-1}) := \lim_{\delta \to 0} (dd^c u, \varphi \omega^{n-1}) = 4(n-1)! (\Delta u, \varphi) \geq 0,$$

hence,

$$\lim_{\delta \to 0} (\mu + \sigma, (1 - \lambda_\varepsilon)\varphi \omega^{n-1}) = (\mu, (1 - \lambda_\varepsilon)\varphi \omega^{n-1}) + \lim_{\delta \to 0} (\sigma, \varphi \omega^{n-1})$$

is nonnegative. As \(\mu\) is carried by \(\Omega \setminus \Gamma\) and \(1 - \lambda_\varepsilon \to 0\) there, we can pass to the limit as \(\varepsilon \to 0\) and obtain

$$(\sigma, \varphi \omega^{n-1}) := \lim_{\delta \to 0} (\sigma, \varphi \omega^{n-1}) \geq 0. $$

**Step 5.** Let \(\Phi\) be an arbitrary \((n-1, n-1)\)-form of class \(C_0^\infty(\Omega)\) and \(\Phi_\tau\) be its projection onto the complex tangent planes \(T_z^c\) to the levels \(\{\varrho(z) = \text{const}\}\). (Representing \(\Phi = \sum_{j,k=1}^n \Phi_{jk} \prod_{\alpha \neq j} dz_\alpha \wedge \prod_{\beta \neq k} d\bar{z}_\beta\) and assuming that \(T_a^c = \{z_n = 0\}\), which can be done by a unitary transform, we see that \(\Phi_\tau|\alpha\) is the same sum from 1 to \(n-1\) at the point \(a\).) It is obvious that \(\Phi_\tau\) is a \((2n-2)\)-form of class \(C_0(\Omega)\). Moreover, \(\Phi = \Phi_\tau + \Phi_{\nu}\), where \(\Phi_{\nu}\) is in the same class and \(\Phi_{\nu}(\tau_1 \wedge \ldots \wedge \tau_{n-1}) = 0\) for any vector fields \(\tau_1, \ldots, \tau_{n-1}\) which are complex orthogonal to \(\nabla \varrho\).

By the Wirtinger theorem, the restrictions \(\omega^{n-1}/(n-1)!|T_z^c\) coincide with the usual volume form on the planes \(T_z^c\), hence

$$\Phi|T_z^c = \Phi_\tau|T_z^c = \varphi \omega^{n-1}|T_z^c$$

for some \(\varphi \in C_0(\Omega)\). Moreover, \(\varphi \geq 0\) if the form \(\Phi\) is positive (because \(T_z^c\) are complex planes).
Decomposing $\omega^{n-1} = (\omega^{n-1})_\tau + (\omega^{n-1})_\nu$, we obtain, for arbitrary $\Phi$, the decomposition $\Phi = \varphi \omega^{n-1} + \Phi_0$, where $\Phi_0$ is orthogonal to $T^c_z$ in the sense that $\Phi_0(\tau_1 \wedge \ldots \wedge \tau_{n-1}) = 0$ for any fields $\tau_j$ as above. As $d\varrho, d^c\varrho$ constitute a basis of 1-covectors annihilating all such $\tau_j$, the form $\Phi_0$ is represented as a sum $d\varrho \wedge \Psi_1 + d^c\varrho \wedge \Psi_2$ with continuous $\Psi_1, \Psi_2$ in $\Omega$. Finally, we have the decomposition

$$\Phi = \varphi \omega^{n-1} + \Phi_0 = \varphi \omega^{n-1} + d\varrho \wedge \Psi_1 + d^c\varrho \wedge \Psi_2,$$

where all the terms belong to the class $C^0(\Omega)$, and $\varphi \geq 0$ if $\Phi$ is a positive form of bidegree $(n - 1, n - 1)$.

**Step 6.** Let now

$$\Phi^\delta := \varphi_\delta \omega^{n-1} + \Phi_0^\delta := \varphi_\delta \omega^{n-1} + d\varrho \wedge (\Psi_1)^\delta + d^c\varrho \wedge (\Psi_2)^\delta$$

for a positive $\Phi$ of class $C^\infty_0(\Omega)$ and bidegree $(n - 1, n - 1)$. Then

$$(dd^c u, \Phi^\delta) = (\mu, \Phi^\delta) + (\sigma, \varphi_\delta \omega^{n-1}),$$

according to (3) and (3'). As $\varphi \geq 0$, we have $(\sigma, \varphi_\delta \omega^{n-1}) \geq 0$ by (4). Thus

$$(dd^c u, \Phi) = \lim_{\delta \to 0} (dd^c u, \Phi^\delta) \geq \lim_{\delta \to 0} (\mu, \Phi^\delta) = (\mu, \Phi) \geq 0,$$

as $\Phi$ has bidegree $(n - 1, n - 1)$ and $\mu, \Phi$ are positive.

The theorem is proved.

**References**

