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Cyclic coverings of Fano threefolds

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Abstract. We describe a series of Calabi–Yau manifolds which are cyclic coverings of a Fano 3-fold branched along a smooth divisor. For all the examples we compute the Euler characteristic and the Hodge numbers. All examples have small Picard number $\varrho = h^{1,1}$.

1. Introduction. In [4, 2] we constructed a class of Calabi–Yau manifolds that can be realized as a double covering of $\mathbb{P}^3$ branched along an octic surface. If the octic was singular we constructed its embedded resolution getting a smooth 3-fold which contains a smooth (reduced) divisor, equivalent to the -2 multiple of the canonical divisor.

We can generalize this construction studying cycle coverings of Fano 3-folds. A smooth manifold Y is called *Fano* if the divisor $-K_Y$ is ample; the *index* of Y is the greatest integer k such that $\frac{1}{k}K_Y$ is a Cartier divisor.

The main goal of the paper is to prove the following

THEOREM 1. *Let Y be a Fano 3-fold of index k and let d be an integer such that $d-1$ divides k. Assume that D is a smooth* (*reduced*) *divisor in the linear system*

$$\left|-\frac{d}{d-1}\,K_Y\right|.$$

Then there exists a cyclic covering $\pi : X \to Y$ of order d branched along D such that:

(1) *X is a smooth Calabi–Yau manifold*,

(2) $e(X) = de(Y) + \dfrac{d^2}{(d-1)^2}\,K_Y^3 - 24d$,

(3) $h^{1,1}(X) = h^{1,1}(Y)$.

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Using this theorem we can construct a series of examples of Calabi–Yau manifolds. The construction is based on the classification of Fano 3-folds given by Iskovskikh ([10, 11]) and Mori and Mukai ([12]). The main difficulty in constructing an explicit example is to find a smooth divisor in the appropriate linear system on the Fano 3-fold from the list. On the other hand the existence of such a divisor follows from [13] so it is easy to make a list of numerical data of 89 Calabi–Yau manifolds. Since the examples with Picard number 1 are well known, we only include examples with Picard number ≥ 2, i.e. resulting from the list of Mori and Mukai.

For the construction it is not important that the anticanonical divisor is ample; it is enough to assume that the linear system $\left|-\frac{d}{d-1}K_Y\right|$ contains a smooth divisor. Moreover we point out that the divisor D need not a priori be irreducible.

In the case of a double octic we produce new examples by allowing certain types of singularities; we can do the same also in the case of cyclic coverings of Fano 3-folds. Again we can define the notion of an admissible blow-up, i.e. a blow-up with smooth center which does not affect the canonical divisor and the first Betti number of the double covering. There are exactly seven types of admissible blow-ups, four in the case of double covering (blow-up of a fourfold and fivefold point and of a double and triple curve). The only new types are blow-ups of a triple, fourfold and fivefold point in a triple covering. A blow-up of a fourfold or fivefold point in a triple covering leads to a singularity that cannot be resolved by admissible blow-ups. For every type of admissible blow-ups we compute the effect on the Euler characteristic of the smooth model of the double covering.

2. Cyclic coverings. In this section we collect some information about cyclic coverings of smooth projective manifolds; the details and proofs can be found in [7].

Let Y be a non-singular complex algebraic variety and let $D = \sum_i D_i$ be a reduced divisor on Y which is divisible by d as an element of the Picard group, i.e. there exists a line bundle $\mathcal{L} \in \operatorname{Pic} Y$ such that $\mathcal{O}_Y(D) \cong \mathcal{L}^{\otimes d}$. Fixing a section $s \in \Gamma(\mathcal{L}^{\otimes d})$ whose zero-divisor is D we can define on $\mathcal{O}_Y \oplus \mathcal{L}^{-1} \oplus \ldots \oplus \mathcal{L}^{-(d-1)}$ a structure of $\mathcal{O}_Y$-module. Let $X \xrightarrow{\pi} Y$ be the spectrum of $\mathcal{O}_Y \oplus \mathcal{L}^{-1} \oplus \ldots \oplus \mathcal{L}^{-(d-1)}$. Then $X \xrightarrow{\pi} Y$ is a cyclic covering of Y of degree d branched along D.

PROPOSITION 2.1 ([7, Lemma 3.16]).

(1) $\pi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{L}^{-1} \oplus \ldots \oplus \mathcal{L}^{-(d-1)}$,
(2) $\pi_*\Omega^1_X \cong \Omega^1_Y \oplus \Omega^1_Y(\log D) \otimes \mathcal{L}^{-1} \oplus \ldots \oplus \Omega^1_Y(\log D) \otimes \mathcal{L}^{-(d-1)}$,
(3) $K_X \cong \pi^*(K_Y \otimes \mathcal{L}^{d-1})$.

3. Proof of Theorem 1. Since the map π is finite, for $i > 0$,

$$H^i\mathcal{O}_X \cong H^i(\pi_*\mathcal{O}_X) \cong H^i\mathcal{O}_Y \oplus H^i\mathcal{L}^{-1} \oplus \ldots \oplus H^i\mathcal{L}^{-(d-1)} = 0$$

by the Kodaira vanishing. In particular, we have $H^1\mathcal{O}_X = 0$. Moreover, by Proposition 2.1(3) and the choice of the branch divisor, $K_X \cong \mathcal{O}_X$. Consequently, X is a Calabi–Yau manifold.

To prove the formula for the Euler characteristic of X, observe that

$$\#\pi^{-1}(y) = \begin{cases} d & \text{for } x \notin D, \\ 1 & \text{for } x \in D. \end{cases}$$

Hence,

$$e(X) = de(Y) - (d-1)e(D).$$

Since $-K_Y$ is ample, $H^i(\mathcal{O}_Y)=0$ for $i > 0$, and consequently, $\chi(\mathcal{O}_Y)=1$. By the Riemann–Roch theorem,

$$\chi(\mathcal{O}_Y) = \frac{1}{24}\, c_1(Y)c_2(Y),$$

and so

$$c_1(Y)c_2(Y) = 24.$$

On the other hand, by adjunction,

$$c_1(D) = -\frac{1}{d-1}\, c_1(Y),$$
$$c_2(D) = \frac{d}{d-1}\, c_1(Y)c_2(Y) - \frac{d^2}{(d-1)^3}\, K_Y^3$$

and finally

$$e(X) = de(Y) - (d-1)\left(\frac{d}{d-1}\, c_1(Y)c_2(Y) - \frac{d^2}{(d-1)^3}\, K_Y^3\right)$$
$$= de(Y) - \frac{d^2}{(d-1)^2}\, K_Y^3 - 24d.$$

Assertion (3) follows from Proposition 2.1(2) and the Kawamata–Viehweg vanishing theorem ([7, Thm. 6.2]). ■

4. Cyclic coverings of Fano manifolds with $h^{1,1} \geq 2$. In [12] the numerical data of all Fano 3-folds with $h^{1,1} \geq 2$ were collected; with three exceptions, all these examples have index 1. The three exceptions have index 2, each leading to two examples of Calabi–Yau manifolds, a double and triple covering. Using those data we have been able to compute the Euler and Hodge numbers of the resulting Calabi–Yau manifolds and compile Table 1 (in the table we denote a triple covering by n^1, where n denotes the double covering of the same Fano 3-fold).

Table 1. Coverings of Fano 3-folds with $h^{1,1,}(Y) \geq 2$

No.	K_Y^3	$e(Y)$	d	$h^{1,1}(X)$	$e(X)$	$h^{1,2}(X)$
1	−4	−38	2	2	−140	72
2	−6	−34	2	2	−140	72
3	−8	−16	2	2	−112	58
4	−10	−14	2	2	−116	60
5	−12	−6	2	2	−108	56
6	−12	−12	2	2	−120	62
7	−14	−4	2	2	−112	58
8	−14	−12	2	2	−128	66
9	−16	−4	2	2	−120	62
10	−16	0	2	2	−112	58
11	−18	−4	2	2	−128	66
12	−20	0	2	2	−128	66
13	−20	2	2	2	−124	64
14	−20	4	2	2	−120	62
15	−22	−2	2	2	−140	72
16	−22	2	2	2	−132	68
17	−24	4	2	2	−136	70
18	−24	2	2	2	−140	72
19	−26	2	2	2	−148	76
20	−26	6	2	2	−140	72
21	−28	4	2	2	−152	78
22	−28	6	2	2	−148	76
23	−30	4	2	2	−160	82
24	−30	6	2	2	−156	80
25	−32	4	2	2	−168	86
26	−34	6	2	2	−172	88
27	−38	6	2	2	−188	96
28	−40	4	2	2	−200	102
29	−40	6	2	2	−196	100
30	−46	6	2	2	−220	112
31	−46	6	2	2	−220	112
32	−48	6	2	2	−228	116
32^1	−48	6	3	2	−162	83
33	−54	6	2	2	−252	128
34	−54	6	2	2	−252	128
35	−56	6	2	2	−260	132
35^1	−56	6	3	2	−180	92
36	−62	6	2	2	−284	144
37	−12	−8	2	3	−112	59
38	−14	2	2	3	−100	53
39	−18	2	2	3	−116	61
40	−18	4	2	3	−112	59
41	−20	8	2	3	−112	59

Table 1 (cont.)

No.	K_Y^3	$e(Y)$	d	$h^{1,1}(X)$	$e(X)$	$h^{1,2}(X)$
42	−22	6	2	3	−124	65
43	−24	6	2	3	−132	69
44	−24	8	2	3	−128	67
45	−26	2	2	3	−148	77
46	−26	8	2	3	−136	71
47	−28	6	2	3	−148	77
48	−28	8	2	3	−144	75
49	−30	8	2	3	−152	79
50	−32	6	2	3	−164	85
51	−32	8	2	3	−160	83
52	−34	8	2	3	−168	87
53	−36	8	2	3	−176	91
54	−36	8	2	3	−176	91
55	−38	8	2	3	−184	95
56	−38	8	2	3	−184	95
57	−38	8	2	3	−184	95
58	−40	8	2	3	−192	99
59	−42	8	2	3	−200	103
60	−42	8	2	3	−200	103
61	−44	8	2	3	−208	107
62	−46	8	2	3	−216	111
63	−48	8	2	3	−224	115
63^1	−48	8	3	3	−156	81
64	−50	8	2	3	−232	119
65	−50	8	2	3	−232	119
66	−52	8	2	3	−240	123
67	−24	8	2	4	−128	68
68	−28	8	2	4	−144	76
69	−30	10	2	4	−148	78
70	−32	10	2	4	−156	82
71	−32	10	2	4	−156	82
72	−34	10	2	4	−164	86
73	−36	10	2	4	−172	90
74	−38	10	2	4	−180	94
75	−40	10	2	4	−188	98
76	−42	10	2	4	−196	102
77	−44	10	2	4	−204	106
78	−46	10	2	4	−212	110
79	−28	12	2	5	−136	73
80	−36	12	2	5	−168	89
81	−36	12	2	5	−168	89
82	−36	14	2	6	−164	88
83	−24	16	2	7	−112	63
84	−18	18	2	8	−84	50
85	−12	20	2	9	−56	37
86	−6	22	2	10	−28	24

5. Singularities. In [4, 2] we constructed a family of Calabi–Yau manifolds by considering double coverings of $\mathbb{P}^3$ branched along a singular octic. The class of singularities allowed was described by means of admissible blow-ups, i.e. blow-ups that do not affect the first Betti number and the canonical class of the double covering. For a d-sheeted cyclic covering of Y branched along D we can resolve the singularities by taking a blow-up $\sigma : \widetilde{Y} \to Y$ with a smooth center C (or a sequence of such blow-ups) and considering the d-sheeted covering of $\widetilde{Y}$ branched along the divisor $D^* := \sigma^* D - d[m/d] \cdot E$, where E is the exceptional divisor of σ and m is the multiplicity of D along C.

Since $\sigma^* D = \widetilde{D} + mE$ (where $\widetilde{D}$ is the strict transform of D), we have $D^* = \widetilde{D} + qE$ with $q \in \{0, \dots, d-1\}$. For $q > 1$ we consider a cyclic covering branched along a non-reduced divisor, and the construction in this case is more complicated (but we shall not describe it any further as we shall in fact never use it in the present paper—for details see [7]).

Since $K_{\widetilde{Y}} = \sigma^* K_Y + (r-1)E$, where r is the codimension of C in Y, we can formulate the following

Definition 5.1. A blow-up σ is called *admissible* if

$$(d-1)\left[\frac{m}{d}\right] = r - 1.$$

From the above definition we easily get the following

Lemma 5.2. *We have the following admissible blow-ups*:

(1) **double coverings**:
- (a) *blow-up of a* **fourfold point**,
- (b) *blow-up of a* **fivefold point**,
- (c) *blow-up of a* **double curve**,
- (d) *blow-up of a* **triple curve**;

(2) **triple coverings**:
- (a) *blow-up of a* **triple point**,
- (b) *blow-up of a* **fourfold point**,
- (c) *blow-up of a* **fivefold point**.

Remark 5.3. The blow-ups of type (1a), ..., (1d) were described in [4, 2]. On the other hand, blow-ups of type (2b) and (2c) lead to a multiple curve in a triple covering which cannot be resolved by admissible blow-ups.

Proposition 5.4. *If $\sigma : Y_0 \to Y$ is a sequence of admissible blow-ups $(\sigma = \sigma_1 \circ \dots \circ \sigma_t)$ and D_0 is a smooth (reduced) divisor divisible by d such that $\sigma(D_0) = D$, then the d-sheeted cyclic covering X_0 of Y_0 branched along*

D_0 is a Calabi–Yau manifold. Moreover

$$e(X_0) = de(Y) + \frac{d^2}{(d-1)^2} K_Y^3 - 24d + \sum_{i=1}^{t} \operatorname{eff}(\sigma_i),$$

where $\operatorname{eff}(\sigma_i)$ *is the effect of* σ_i*, defined by*

$$\operatorname{eff}(\sigma_i) = \begin{cases} 36 & \text{if } \sigma_i \text{ is of type (1a) or (1b)}, \\ 14\deg(\mathcal{L}|C) - 6\deg(\bigwedge^2 \mathcal{N}) & \text{if } \sigma_i \text{ is of type (1c) or (1d)}, \\ 24 & \text{if } \sigma_i \text{ is of type (2)}. \end{cases}$$

REMARK 5.5. If σ is a sequence of blow-ups that are *not* admissible it does not follow in general that X has no smooth model which is a Calabi–Yau manifold. If σ_i is a blow-up of a point and $(d-1)[m/d] < 2 < r-1$, then there can exist a small resolution (e.g., A_1 point in the cases $d = 2$ or 4, A_3 point in the case $d = 2$).

On the other hand, if σ_i is a blow-up with $(d-1)[m/d] > r-1$ then the exceptional divisor cannot be blown down because it is nef along the fibers. Consequently, also in this situation no smooth model of X is a Calabi–Yau manifold.

EXAMPLE 1. If $D \subset \mathbb{P}^3$ is a sextic with ordinary triple points, then the triple covering of $\mathbb{P}^3$ branched along D has a smooth model $\widetilde{X}$ which is a Calabi–Yau manifold with Euler number

$$e(\widetilde{X}) = -204 + 24\mu,$$

where μ is the number of triple points. By [6] a sextic surface can have up to 10 triple points; all the possible sextics with ordinary triple points where classified in that paper.

EXAMPLE 2. Let $D = D_1 + D_2$ be a sum of two smooth anticanonical divisors in a Fano 3-fold Y intersecting transversally along a smooth curve C. Denote by $\widetilde{Y}$ the blow-up of Y along C. Let $\widetilde{D}_i$ be a strict transform of D_i. Then $\widetilde{D}_i$ is isomorphic to D_i. The double covering $\widetilde{X}$ of $\widetilde{Y}$ branched along $\widetilde{D} = \widetilde{D}_1 + \widetilde{D}_2$ is a Calabi–Yau manifold. The Euler characteristic $e(\widetilde{Y})$ equals $2e(\widetilde{Y}) - e(\widetilde{D})$. Now, $e(\widetilde{Y}) = e(Y) + e(C)$ and $e(\widetilde{D}) = 2e(D_1)$. Computations analogous to the proof of Theorem 1 give $e(C) = K_Y^3$, $e(D_1) = 24$ and finally

$$e(\widetilde{X}) = 2e(Y) + 2K_Y^3 - 48.$$

Moreover, by [3],

$$h^{1,1}(\widetilde{X}) = h^{1,1}(Y) + 1.$$

References

[1] C. H. Clemens, *Double solids*, Adv. Math. 47 (1983), 107–230.

[2] S. Cynk, *Double coverings of octic arrangements with isolated singularities*, Adv. Theor. Math. Phys. 3 (1999), 217–225.
[3] —, *Cohomologies of a double covering of a non-singular algebraic 3-fold*, Math. Z., to appear.
[4] S. Cynk and T. Szemberg, *Double covers and Calabi–Yau varieties*, in: Banach Center Publ. 44, Inst. Math., Polish Acad. Sci., 1998, 93–101.
[5] A. Dimca, *Betti numbers of hyperplanes and defects of linear systems*, Duke Math. J. 60 (1990), 285–294.
[6] S. Endraß, U. Persson and J. Stevens, *Surfaces with triple points*, preprint math.AG/0010163.
[7] H. Esnault and E. Viehweg, *Lectures on Vanishing Theorems*, DMV Seminar 20, Birkhäuser, 1992.
[8] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.
[9] R. Hartshorne, *Algebraic Geometry*, Springer, 1977.
[10] V. A. Iskovskikh, *Fano 3-folds*, *I*, Math. USSR-Izv. 11 (1977), 485–527.
[11] —, *Fano 3-folds*, *II*, ibid. 12 (1978), 469–506.
[12] S. Mori and S. Mukai, *Classification of Fano 3-folds with* $B_2 \geq 2$, Manuscripta Math. 36 (1981), 147–162.
[13] V. V. Shokurov, *Smoothness of a general anticanonical divisor on a Fano variety*, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 430–441 (in Russian).

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