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A note on Rosay's paper

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To Professor Józef Siciak on his 70th birthday

Abstract. We give a simplified proof of J. P. Rosay's result on plurisubharmonicity of the envelope of the Poisson functional [10].

1. Introduction. Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$. In [10], J. P. Rosay proved the following result.

Theorem 1.1. *Let u be an upper semicontinuous function on a complex manifold X. Then*

$$E_u(x) = \inf\left\{ \frac{1}{2\pi}\int_0^{2\pi} u(f(e^{i\theta}))\,d\theta : f \in \mathcal{O}(\overline{\mathbb{D}}, X),\ f(0) = x \right\}$$

is plurisubharmonic on X.

Here, $\mathcal{O}(\overline{\mathbb{D}}, X)$ denotes the set of all holomorphic mappings $\mathbb{D} \to X$ which extend holomorphically to a neighborhood of $\overline{\mathbb{D}}$.

Special cases of Theorem 1.1 have been treated by E. Poletsky (see [8]), Lárusson–Sigurdsson (see [6]), and by the author (see [1]).

As a corollary we have the following characterization of Liouville manifolds.

Corollary 1.2. *Let X be a complex manifold. Then any plurisubharmonic function on X bounded from above is constant (i.e., X is a Liouville manifold) if and only if for any $x \in X$, any open set $U \subset X$, and any $\varepsilon > 0$ there exists a holomorphic mapping $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(0) = x$ and the measure of the set $\{\theta \in [0, 2\pi) : f(e^{i\theta}) \in U\}$ is at least $2\pi - \varepsilon$.*

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For a complex manifold for which E_u is plurisubharmonic for any upper semicontinuous function u on X (by Theorem 1.1 on *any* complex manifold) Corollary 1.2 is given in [1]. The "only if" part is mentioned in Rosay's paper [10] (see also [6]). Some other applications of Theorem 1.1 can be found in [1].

The main purpose of the paper is to give a simplified proof of Theorem 1.1.

2. Preliminary results. Let us start with the following modification of a well known result (see e.g. [1], [6], [10]). For completeness we give the proof.

THEOREM 2.1. *Let X be a complex manifold. Let $f : \mathbb{D}_R \to X$, $R > 0$, be a holomorphic mapping, where $\mathbb{D}_R = \{\xi \in \mathbb{C} : |\xi| < R\}$. Then for any $r \in (0, R)$ there exists a holomorphic mapping $F : \mathbb{D}_r \times \mathbb{D}_r^n \to X$ ($n = \dim X$) such that*:

(i) $F(\xi, 0) = f(\xi)$, $\xi \in \mathbb{D}_r$,

(ii) $F_\xi = F(\xi, \cdot)$ *is an injective holomorphic mapping for any* $\xi \in \mathbb{D}_r$ (*note that* $F_\xi : \mathbb{D}_r^n \to F_\xi(\mathbb{D}_r^n) \subset X$ *is a biholomorphic mapping*).

For the proof of Theorem 2.1 we need the following simple result, which follows immediately from the implicit function theorem.

LEMMA 2.2. *Let Ω be a domain in $\mathbb{C}^n$ and let $T : \mathbb{D}_\varrho \times \Omega \to \mathbb{C}$ be a holomorphic function, where $\varrho > 0$. Assume that the following conditions are fulfilled*:

- $T'_\xi(\xi, w) \neq 0$ *for any* $(\xi, w) \in \mathbb{D}_\varrho \times \Omega$;
- *for any* $w \in \Omega$ *there exists exactly one* $\xi = \xi(w) \in \mathbb{D}_\varrho$ *such that* $T(\xi(w), w) = 0$.

Then $\Omega \ni w \mapsto \xi(w) \in \mathbb{D}_\varrho$ *is a holomorphic function.*

Proof of Theorem 2.1. Consider the graph

$$\Gamma = \{(\xi, f(\xi)) : \xi \in \mathbb{D}_R\} \subset \mathbb{D}_R \times X.$$

Then Γ is a Stein submanifold of $\mathbb{D}_R \times X$. By Siu's theorem (see [11, Corollary 1]) there exist a Stein neighborhood $\widetilde{W} \subset \mathbb{D}_R \times X$ of Γ and a biholomorphic mapping $\widetilde{\Psi}$ of $\widetilde{W}$ onto a neighborhood of the zero section of the normal bundle of Γ, which identifies Γ with the zero section. It is well known that the normal bundle of Γ is holomorphically trivial (see e.g. [2, Theorem 30.4]) and therefore it is biholomorphic to $\Gamma \times \mathbb{C}^n$. From this we conclude that there exists a biholomorphic mapping $\Psi^{-1} : \widetilde{W} \to W$ such that $\Psi^{-1}(\xi, f(\xi)) = (\xi, 0)$ for all $\xi \in \mathbb{D}_R$, where W is a neighborhood of $\mathbb{D}_R \times \{0\}$.

Fix $r \in (0, R)$ and $\varrho \in (r, R)$. Note that $\{(\xi, 0)) : \xi \in \mathbb{D}_\varrho\}$ is relatively compact in W. Therefore, there exists $C > 0$ such that

$$U := \{(\xi, z_1, \ldots, z_n) : \xi \in \mathbb{D}_\varrho,\ |z_j| < C,\ j = 1, \ldots, n\} \subset W.$$

Put $\widetilde{U} = \Psi(U)$. We define $\Phi : \mathbb{D}_\varrho \times \mathbb{D}_\varrho^n \ni (\xi, z) \mapsto \Psi(\xi, (C/\varrho)z) \in \widetilde{U} \subset \mathbb{C} \times X$. Note that Φ is a biholomorphic mapping such that $\Phi(\xi, 0) = (\xi, f(\xi))$ for any $\xi \in \mathbb{D}_\varrho$.

Let $s \in (r, \varrho)$. There exists $\varepsilon > 0$ such that

$$(2.1) \qquad |\Phi_1(\xi, z) - \Phi_1(\xi, 0)| < s - r \quad \text{for } \xi \in \overline{\mathbb{D}}_s,\ \|z\| \le \varepsilon,$$

where $\Phi = (\Phi_1, \Phi_2) : \mathbb{D}_r^{n+1} \to \mathbb{C} \times X$ and $\|z\| = \max\{|z_1|, \ldots, |z_n|\}$.

Consider the holomorphic function

$$T(\xi, \zeta, z) = \Phi_1(\xi, z) - \zeta, \quad (\zeta, \xi, z) \in \mathbb{C} \times \mathbb{D}_\varrho \times \mathbb{D}_\varrho^n.$$

Note that $T(\xi, \zeta, 0) = \xi - \zeta$ has a single zero for any fixed $\zeta \in \mathbb{D}_\varrho$. So, by Rouché's theorem (use (2.1)), the function $T(\xi, \zeta, z)$ has a single zero for any fixed $\zeta \in \overline{\mathbb{D}}_r$ and $z \in \overline{\mathbb{D}}_\varepsilon^n$. Hence, for any $(\zeta, z) \in \overline{\mathbb{D}}_r \times \overline{\mathbb{D}}_\varepsilon^n$ there exists exactly one $\xi \in \mathbb{D}_s$ (which we denote $S(\zeta, z)$) such that $\Phi_1(\xi, z) = \zeta$.

Note that $T'_\xi(\xi, \zeta, 0) = 1$ for any $(\xi, \zeta) \in \mathbb{D}_\varrho \times \mathbb{C}$. Hence, there exist $\varepsilon' \in (0, \varepsilon)$ and $\varrho' \in (r, \varrho)$ such that $T'_\xi(\xi, \zeta, z) \neq 0$ for any $(\xi, \zeta, z) \in \mathbb{D}_{\varrho'} \times \mathbb{D}_{\varrho'} \times \mathbb{D}_{\varepsilon'}^n$. By Lemma 2.2 we see that $S : \mathbb{D}_r \times \mathbb{D}_{\varepsilon'}^n \to \mathbb{C}$ is a holomorphic function. Set $F(\xi, z) = \Phi_2(S(\xi, \delta z), \delta z)$, where $\delta > 0$ is sufficiently small. ∎

Proposition 2.3. *Let J be a closed subset of the unit circle $\mathbb{T}$ in $\mathbb{C}$ such that $J \neq \mathbb{T}$. Then $H = (J \times \overline{\mathbb{D}}) \cup (\overline{\mathbb{D}} \times \{0\})$ is a polynomially convex compact set in $\mathbb{C}^2$. Therefore, there exists a smooth plurisubharmonic function $\varrho : \mathbb{C}^2 \to [0, \infty)$ such that $\{z \in \mathbb{C}^2 : \varrho(z) = 0\} = H$.*

Proof. Fix $(\xi_0, \zeta_0) \in \mathbb{C}^2 \setminus H$. We have to show that there exists a polynomial p on $\mathbb{C}^2$ such that $|p(\xi_0, \zeta_0)| > \|p\|_H$.

If $|\xi_0| > 1$ (resp. $|\zeta_0| > 1$), put $p(\xi, \zeta) = \xi$ (resp. $p(\xi, \zeta) = \zeta$).

Let $\xi_0 \in \overline{\mathbb{D}} \setminus J$. Note that J is a polynomially convex set in $\mathbb{C}$ (see e.g. [9]). Consider a polynomial $p_n(\xi, \zeta) = \zeta q^n(\xi)$, $n \in \mathbb{N}$, where q is a polynomial such that $|q(\xi_0)| > 1$ and $\|q\|_J = 1$. If $\zeta_0 \neq 0$, then for sufficiently large n we have $|p_n(\xi_0, \zeta_0)| = |\zeta_0| \cdot |q(\xi_0)|^n > 1 = \|p_n\|_H$.

The existence of a smooth plurisubharmonic function ϱ is well known (see e.g. [3]). ∎

Proposition 2.4. *Let Ω be a domain in $\mathbb{C}^n$ and let $\overline{\mathbb{D}} \times \{0\}_{\mathbb{C}^{n-1}} \subset \Omega$. Assume that $F : \Omega \to \mathbb{C}^n$ is an injective holomorphic mapping such that $F(\xi, 0) = (\xi, 0)$, $\xi \in \mathbb{D}$. Then there exist $C \ge 1$ and $r > 0$ such that F and F^{-1} are well defined on $\mathbb{D}_{1+r} \times \mathbb{D}_r^{n-1}$ and*

$$\frac{1}{C\sqrt{n}}\,\|z\| \le \frac{1}{C}\,\|z\| \le \|F_2(\xi, z)\| \le C\|z\| \le C\|z\|, \quad (\xi, z) \in \mathbb{D}_{1+r} \times \mathbb{D}_r^{n-1},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{C}^n$.

Proof. There exists a $\delta > 0$ such that $F(\mathbb{D}_{1+\delta} \times \mathbb{D}_\delta^{n-1}) \subset \mathbb{D}_2 \times \mathbb{D}^{n-1}$. For any $(\xi, z) \in \mathbb{D}_{1+\delta} \times \mathbb{D}_\delta^{n-1}$ we have

$$\begin{aligned}\frac{1}{\delta}\|z\| &= c^*_{\mathbb{D}_\delta^{n-1}}(z,0) = c^*_{\mathbb{D}_{1+\delta}\times\mathbb{D}_\delta^{n-1}}((\xi,z),(\xi,0))\\ &\geq c^*_{\mathbb{D}_2\times\mathbb{D}^{n-1}}(F(\xi,z),F(\xi,0)) \geq c^*_{\mathbb{D}^{n-1}}(F_2(\xi,z),F_2(\xi,0)) = \|F_2(\xi,z)\|,\end{aligned}$$

where c^*_D is the Carathéodory pseudodistance of a domain D (see e.g. [4]). Hence, $\|F_2(\xi,z)\| \leq (1/\delta)\|z\|$.

Put $\widetilde{\Omega} = F(\Omega)$. Note that $F : \Omega \to \widetilde{\Omega}$ is biholomorphic and $F^{-1}(\xi,0) = (\xi,0)$ for any $\xi \in \mathbb{D}$. From the first part of the proof there exists a $\delta' \in (0,\delta)$ such that $F^{-1}(\mathbb{D}_{1+\delta'} \times \mathbb{D}_{\delta'}^{n-1}) \subset \mathbb{D}_{1+\delta} \times \mathbb{D}_\delta^{n-1}$ and $\|(F^{-1})_2(\xi,z)\| \leq (\delta/\delta')\|z\|$. So, $\|F_2(\xi,z)\| \geq (\delta'/\delta)\|z\|$. Now it suffices to put $r = \delta'$. ∎

3. Proof of Theorem 1.1. First recall the following well known result (see [1], [6], [8], [10]).

PROPOSITION 3.1. *Let X be a complex manifold and let u be an upper semicontinuous function on X. Then E_u is also upper semicontinuous on X.*

According to Proposition 3.1 it suffices to show that for any $h \in \mathcal{O}(\overline{\mathbb{D}}, X)$ we have

$$E_u(h(0)) \leq \frac{1}{2\pi}\int_0^{2\pi} E_u(h(e^{i\theta}))\,d\theta.$$

From [1], [6] we know that for this it suffices to construct a special Stein neighborhood (see below). The following important result is a main tool in this construction (see [7, Theorem II]).

THEOREM 3.2. *A complex manifold X is a Stein manifold if and only if there exists a continuous strongly plurisubharmonic function q defined on X with*

$$X_\alpha = \{x \in X : q(x) < \alpha\} \Subset X \quad \text{for each } \alpha \geq 0.$$

Recall that a plurisubharmonic function v defined in a neighborhood of $z_0 \in \mathbb{C}^n$ is called *strongly plurisubharmonic* at z_0 if there exist $r > 0$ and $\alpha > 0$ such that $v(z) - \alpha\|z - z_0\|^2$ is a plurisubharmonic function on $\{z \in \mathbb{C}^n : \|z - z_0\| < r\}$. We say that v is strongly plurisubharmonic in an open set Ω if it is strongly plurisubharmonic at any point of Ω. Note that strong plurisubharmonicity is a local property. So, we may define it on a complex manifold via local coordinates. Note that the maximum of two strongly plurisubharmonic functions is strongly plurisubharmonic.

A C^2 plurisubharmonic function v is strongly plurisubharmonic at $z_0 \in \mathbb{C}^n$ iff

$$\mathcal{L}_v(z_0, X) = \sum_{j,k=1}^n \frac{\partial^2 v(z_0)}{\partial z_j \partial \overline{z}_k} X_j \overline{X}_k > 0 \quad \text{for any } X \in \mathbb{C}^n \setminus \{0\}.$$

The following simple result will be useful in the proof of Theorem 1.1.

LEMMA 3.3. *Let $\beta : \mathbb{C} \to \mathbb{R}$ be a smooth subharmonic function and let u be a strongly plurisubharmonic function on a domain $\Omega \subset \mathbb{C}^n$. Then $v(\xi, z) = |\xi|^2 + e^{\beta(\xi)} u(z)$ is a strongly plurisubharmonic function on $\mathbb{C} \times \Omega$.*

Proof. Fix $(\xi_0, z_0) \in \mathbb{C} \times \Omega$. Since u is strongly plurisubharmonic at z_0, there exist $r > 0$ and $\alpha > 0$ such that $u(z) - \alpha\|z - z_0\|^2$ is plurisubharmonic on $\{z \in \mathbb{C}^n : \|z - z_0\| < r\}$. So, it suffices to note that $\widetilde{v}(\xi, z) = |\xi|^2 + \alpha e^{\beta(\xi)}\|z - z_0\|^2$ is strongly plurisubharmonic at (ξ_0, z_0). ■

Proof of Theorem 1.1. Step 1. Fix an $x_0 \in X$. Let h be a holomorphic mapping from a neighborhood of the closed unit disk $\overline{\mathbb{D}}$ into X with $h(0) = x_0$. We have to show that

$$E_u(x_0) \le \frac{1}{2\pi} \int_0^{2\pi} E_u(h(e^{i\theta}))\, d\theta.$$

Let $\varepsilon > 0$. Since E_u is upper semicontinuous, there exists a continuous function $\Gamma : \mathbb{T} \to \mathbb{R}$ such that $\Gamma(e^{i\theta}) > E_u(h(e^{i\theta}))$ and

$$\frac{1}{2\pi} \int_0^{2\pi} \Gamma(e^{i\theta})\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} E_u(h(e^{i\theta}))\, d\theta + \varepsilon.$$

By the definition of E_u, for any $\theta_0 \in [0, 2\pi)$ there exists a holomorphic disk $\phi_{\theta_0} : \mathbb{D}_{\varrho_0} \to X$, $\varrho_0 > 1$, such that $\phi_{\theta_0}(0) = h(e^{i\theta_0})$, and

$$\frac{1}{2\pi} \int_0^{2\pi} u(\phi_{\theta_0}(e^{i\nu}))\, d\nu < \Gamma(e^{i\theta_0}).$$

Fix an $r^0 \in (1, \varrho_0)$. By Theorem 2.1, there exists a holomorphic mapping $F^o : \mathbb{D}_{r^0} \times \mathbb{D}^n_{r^o} \to X$ such that

(i) $F^0(\xi, 0) = \phi_{\theta_0}(\xi)$, $\xi \in \mathbb{D}_{r^0}$,
(ii) $F^0(\xi, \cdot)$ is an injective holomorphic mapping for any $\xi \in \mathbb{D}_{r^0}$.

Put $G^0 = F^0(0, \cdot)$. Let $\widetilde{r}^0 \in (1, r^0)$. We know that $G^0(0) = h(e^{i\theta_0})$. Hence, there exists a neighborhood $\omega^0 \subset \mathbb{D}_{\widetilde{r}^0}$ of $e^{i\theta_0}$ such that

$$\|(G^0)^{-1}(h(\xi))\| < r^0 - \widetilde{r}^0 \quad \text{for any } \xi \in \omega^0.$$

We put

$$T^0 : \omega^0 \times \mathbb{D}_{\widetilde{r}^0} \times \mathbb{D}^n_{\widetilde{r}^0} \ni (\xi, \zeta, z) \mapsto (\xi, \zeta, F^0(\zeta, z + (G^0)^{-1}(h(\xi)))) \in \mathbb{C}^2 \times X.$$

Note that $T^0(\xi, 0, 0) = (\xi, 0, h(\xi))$, $\xi \in \omega^0$, and

$$T^0(e^{i\theta_0}, \zeta, 0) = (e^{i\theta_0}, \zeta, \phi_{\theta_0}(\zeta)).$$

Let $\Pi : \mathbb{C}^2 \times X \to X$ be the natural projection. Put $\widetilde{u} = u \circ \Pi$. We have

$$\frac{1}{2\pi} \int_0^{2\pi} \widetilde{u} \circ T^0(e^{i\theta_0}, e^{i\nu}, 0)\, d\nu < \Gamma(e^{i\theta_0}).$$

By the upper semicontinuity of $\widetilde{u}$ for $\theta \approx \theta_0$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} \widetilde{u}\circ T^0(e^{i\theta}, e^{i\nu}, 0)\, d\nu < \Gamma(e^{i\theta}).$$

By a compactness argument there exist disjoint closed arcs $J_1, \ldots, J_N$ on $\mathbb{T}$ and open disks $\omega_1, \ldots, \omega_N$ in $\mathbb{C}$ such that $J_j \subset \omega_j$, $\overline{\omega}_k \cap \overline{\omega}_j = \emptyset$ if $j \neq k$, and

$$\frac{1}{2\pi}\int_{\mathbb{T}\setminus\bigcup_j J_j} \Gamma(e^{i\theta})\, d\theta < \varepsilon.$$

Put $\Omega_j = \omega_j \times \mathbb{D}_{r_j} \times \mathbb{D}^n_{r_j}$, $r_j > 1$, and

$$T_j : \Omega_j \ni (\xi, \zeta, z) \mapsto (\xi, \zeta, F_j(\zeta, z + G_j^{-1}(h(\xi)))) \in \mathbb{C}^2 \times X.$$

We have

$$\frac{1}{2\pi}\int_0^{2\pi} \widetilde{u}\circ T_j(e^{i\theta}, e^{i\nu}, 0)\, d\nu < \Gamma(e^{i\theta}).$$

By Theorem 2.1, there exists a holomorphic mapping $F_0 : \mathbb{D}_{r_0} \times \mathbb{D}^n_{r_0} \to X$ such that $F_0(\xi, 0) = h(\xi)$ and $F_0(\xi, \cdot)$ is an injective holomorphic mapping for any $\xi \in \mathbb{D}_{r_0}$. We may assume that $1 < r_0 < \min\{r_j : j = 1, \ldots, N\}$. Put $\Omega_0 = \mathbb{D}_{r_0} \times \mathbb{D}_{r_0} \times \mathbb{D}^n_{r_0}$ and

$$T_0 : \Omega_0 \ni (\xi, \zeta, x) \mapsto (\xi, \zeta, F_0(\xi, x)) \in \mathbb{C}^2 \times X.$$

Note that $T_0(\xi, 0, 0) = (\xi, 0, h(\xi))$.

Set $H := \bigcup_{j=1}^N T_j((J_j \times \overline{\mathbb{D}} \times \{0\}_{\mathbb{C}^n})) \cup T_0(\overline{\mathbb{D}} \times \{0\}_{\mathbb{C}^{n+1}})$.

Step 2. We claim that H has a Stein neighborhood in $Y := \bigcup_{j=0}^N T_j(\Omega_j)$.

Note that $T_j^{-1} \circ T_0(\xi, 0, 0) = (\xi, 0, 0)$, $\xi \in J_j$, $j = 1, \ldots, N$. Hence, by Proposition 2.4 there exist $C \geq 1$ and $\delta > 0$ such that $T_j^{-1} \circ T_0$ and $T_0^{-1} \circ T_j$ are well defined on $\omega'_j \times \mathbb{D}^{n+1}_\delta$ for any $j = 1, \ldots, N$, and

$$\text{(3.1)}\qquad \frac{1}{C}(|\zeta|^2 + \|z\|^2) \leq |\zeta|^2 + \|\pi \circ T_j^{-1} \circ T_0(\xi, \zeta, z)\|^2 \leq C(|\zeta|^2 + \|z\|^2),$$

where $J_j \subset \omega'_j \Subset \omega_j$ is an open disk and $\pi : \mathbb{C}^{n+2} \ni (\xi, \zeta, z) \mapsto z \in \mathbb{C}^n$ is the natural projection.

Taking even smaller $\delta > 0$ we may assume that $\delta < 1/\sqrt{C}$. Take open disks $J_j \subset \omega'''_j \Subset \omega''_j \Subset \omega'_j$.

Let β be a smooth subharmonic function on $\mathbb{C}$ such that $e^\beta \geq \delta^2/3$ on $\mathbb{C}$,

$$\text{(3.2)}\qquad e^\beta = \frac{\delta^2}{3} \quad \text{on } \bigcup_{j=1}^N \overline{\omega}'''_j, \quad \text{and} \quad e^\beta \geq 1 \quad \text{on } \bigcup_{j=1}^N \partial\omega''_j.$$

Put $M := \sup_{\xi \in \bigcup_{j=1}^N \overline{\omega}'_j} e^{\beta(\xi)}$.

Let γ be a smooth subharmonic function on $\mathbb{C}$ such that $e^{\gamma} \geq 1/C$ on $\mathbb{C}$,

$$(3.3)\qquad e^{\gamma(\xi)} = \frac{1}{C} \quad \text{on} \quad \bigcup_{j=1}^{N} \overline{\omega}''_j, \quad \text{and} \quad e^{\gamma(\xi)} \geq M + C \quad \text{on} \quad \bigcup_{j=1}^{N} \partial\omega'_j.$$

According to Proposition 2.3 there exists a smooth plurisubharmonic function $\widehat{\varrho}\colon \mathbb{C}^2 \to [0,\infty)$ such that

$$(3.4)\qquad \{\widehat{\varrho} = 0\} = (J \times \overline{\mathbb{D}}) \cup (\overline{\mathbb{D}} \times \{0\}),$$

where $J = \bigcup_{j=1}^{N} J_j$.

Fix an $r \in (1, r_0)$. By the smoothness of $\widehat{\varrho}$ and (3.4) there exists a positive number κ with the following property: if $(\xi,\zeta) \in \mathbb{D}^2_{r_0}$ is such that $\widehat{\varrho}(\xi,\zeta) < \kappa$ then either $\xi \in \bigcup_{j=1}^{N} \omega''_j$ and $|\zeta| < r$, or $|\xi| < r$ and $|\zeta| < \delta$.

Now we define a function ϱ on Y as follows. For $j = 1,\dots,N$ we set

$$\begin{aligned}&\varrho \circ T_j(\xi,\zeta,z)\\ &\quad = \frac{1}{\kappa}\,\widehat{\varrho}(\xi,\zeta) + \frac{1}{3}\,|\xi|^2 + \frac{1}{\delta^2}\,(e^{\beta(\xi)}|\zeta|^2 + \|z\|^2) \qquad \text{for } \xi \in \omega''_j,\\ &\varrho \circ T_0(\xi,\zeta,z)\\ &\quad = \frac{1}{\kappa}\,\widehat{\varrho}(\xi,\zeta) + \frac{1}{3}\,|\xi|^2 + \frac{1}{\delta^2}\,e^{\gamma(\xi)}(|\zeta|^2 + \|z\|^2) \qquad \text{for } \xi \in \mathbb{D}_{r_0} \setminus \bigcup_{j=1}^{N} \omega'_j,\end{aligned}$$

and

$$\begin{aligned}\varrho \circ T_0(\xi,\zeta,z) &= \frac{1}{\kappa}\widehat{\varrho}(\xi,\zeta) + \frac{1}{3}|\xi|^2\\ &\quad + \frac{1}{\delta^2}\max\{e^{\beta(\xi)}|\zeta|^2 + \|\pi \circ T_j^{-1} \circ T_0(\xi,\zeta,z)\|^2, e^{\gamma(\xi)}(|\zeta|^2 + \|z\|^2)\}\\ &\qquad\qquad \text{for } \xi \in \omega'_j \setminus \omega''_j.\end{aligned}$$

For $\xi \in \bigcup_{j=1}^{N} \partial\omega''_j$, from (3.1)–(3.3) we have

$$\begin{aligned}e^{\beta(\xi)}|\zeta|^2 + \|\pi \circ T_j^{-1} \circ T_0(\xi,\zeta,z)\|^2 &\geq (e^{\beta(\xi)} - 1)|\zeta|^2 + \frac{1}{C}\,(|\zeta|^2 + \|z\|^2)\\ &\geq e^{\gamma(\xi)}(|\zeta|^2 + \|z\|^2).\end{aligned}$$

For $\xi \in \bigcup_{j=1}^{N} \partial\omega'_j$, again from (3.1)–(3.3) we have

$$\begin{aligned}e^{\beta(\xi)}|\zeta|^2 + \|\pi \circ T_j^{-1} \circ T_0(\xi,\zeta,z)\|^2 &\leq (M-1)|\zeta|^2 + C(|\zeta|^2 + \|z\|^2)\\ &\leq e^{\gamma(\xi)}(|\zeta|^2 + \|z\|^2).\end{aligned}$$

Therefore, ϱ is a continuous strongly plurisubharmonic function defined on Y (use Lemma 3.3). It is easy to see that $H \subset \{\varrho \leq 2/3\}$. Define $V = \bigcup_{j=1}^{N} T_j(\omega''_j \times \mathbb{D}^{n+1}_r) \cup T_0(\mathbb{D}^{n+2}_r)$. Note that $H \subset V \Subset Y$.

Assume for a while that $\varrho \geq 1$ on $Y \setminus V$. Then $1/(1-\varrho)$ is a continuous strongly plurisubharmonic exhaustion function of $\{\varrho < 1\} \subset V$. By Theorem 3.2 we see that $\{\varrho < 1\}$ is a Stein neighborhood of H in Y.

So, we have to show that $\varrho \geq 1$ on $Y \setminus V$. If $\xi \in \omega_j''$ and either $|\zeta| \geq r$ or $\|z\| \geq \delta$, then

$$\varrho \circ T_j(\xi, \zeta, z) \geq \frac{1}{\kappa}\,\widehat{\varrho}(\xi, \zeta) + \frac{1}{\delta^2}\,\|z\|^2 \geq 1.$$

If $\xi \notin \bigcup_{j=1}^{N} \omega_j''$ and either $|\zeta| \geq \delta$ or $\|z\| \geq r$, then

$$\varrho \circ T_0(\xi, \zeta, z) \geq \frac{1}{\kappa}\,\widehat{\varrho}(\xi, \zeta) + \frac{1}{C\delta^2}\,\|z\|^2 \geq 1$$

(recall that $\delta^2 C < 1 < r$).

Step 3. Having constructed a Stein neighborhood of H, one has to proceed as in Lárusson–Sigurdsson's paper (see [6], and also [1], [8]). ■

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