ANNALES
POLONICI MATHEMATICI
80 (2003)

Coefficients in some classes defined by subordination to multivalent majorants

by Renata Jurasińska and Jan Stankiewicz (Rzeszów)

Dedicated to Professor Józef Siciak on the occasion of his 70th birthday

Abstract. We present some results connected with estimation of the coefficients for some special class of functions holomorphic in the unit disc and defined by subordination to some multivalent functions.

1. Introduction. Let $U = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc and $H = H(U)$ the family of functions holomorphic in U. Let Ω denote the class of Schwarz functions $\omega \in H$ such that $|\omega(z)| \le |z|$ for $z \in U$.

We say that f is *subordinate* to g in U and write $f \prec g$ or $f(z) \prec g(z)$ if there exists $\omega \in \Omega$ such that

$$f(z) = g(\omega(z)) \quad \text{for } z \in U. \tag{1.1}$$

In the geometric theory of univalent and multivalent functions an important role is played by the class P of Carathéodory functions with positive real part,

$$P = \{p \in H : p(0) = 1, \operatorname{Re} p(z) > 0 \text{ for } z \in U\}. \tag{1.2}$$

This class may also be defined by subordination:

$$P = \left\{p \in H : \ p(z) \prec \frac{1+z}{1-z}\right\}, \tag{1.3}$$

where the majorant function

$$F_1(z) = \frac{1+z}{1-z} \tag{1.4}$$

is univalent and maps U onto the right half-plane $\{w \in \mathbb{C} : \operatorname{Re} w > 0\}$. For

2000 *Mathematics Subject Classification*: 30C45, 30C50.
Key words and phrases: subordination, multivalent majorants.

univalent majorants subordination (1.1) is equivalent to

$$f(0) = g(0) \quad \text{whenever} \quad f(U) \subset g(U). \tag{1.5}$$

Thus by (1.5) and the univalence of $F_1(z)$ it is obvious that (1.2) and (1.3) define the same class.

R. Jurasińska and J. Stankiewicz [3] and R. Jurasińska and A. Szpila [4] introduced and investigated the classes $P(n)$, $n \in \mathbb{N}$, defined by formula (1.3) with the function F_1 replaced by the n-valent function

$$F_n(z) = \frac{1+z^n}{1-z^n}, \qquad n = 1, 2, \ldots, \tag{1.6}$$

that is,

$$P(n) = \left\{ p \in H : p(z) \prec \frac{1+z^n}{1-z^n} \right\}. \tag{1.7}$$

For $n \geq 2$ the class $P(n)$ is a proper subclass of P.

The classes $P(n)$ do not coincide with the subclasses

$$P_n = \{p(z) = 1 + p_n z^n + p_{2n} z^{2n} + \ldots \in P\}$$

of n-symmetric functions with positive real part. We can easily check (see e.g. [3], [4]) that

$$P_n \not\subset P(n) \quad \text{and} \quad P(n) \not\subset P_n.$$

In [3], [4] some estimates of $|p(z)|$, $\operatorname{Re} p(z)$, $\arg p(z)$ and of some coefficients p_n, p_{n+1} were established. Among other results the following theorem was proved:

Theorem A ([4]). *Let $n \in \mathbb{N}$, $n \geq 2$, be fixed. If $p(z) = 1 + p_1 z + p_2 z^2 + \ldots \in P(n)$ then*

$$p_k = 0 \quad \text{for } k = 1, \ldots, n-1, \tag{1.8}$$

$$|p_k| \leq 2 \quad \text{for } k = n, n+2, \ldots, \tag{1.9}$$

$$|p_{n+1}| \leq \frac{4n}{n+1} \left(\frac{n-1}{n+1} \right)^{(n-1)/2} < 2. \tag{1.10}$$

The functions

$$p_1(z) = \frac{1 + \left(z \dfrac{a+z}{1+\overline{a}z} \right)^n}{1 - \left(z \dfrac{a+z}{1+\overline{a}z} \right)^n}, \qquad |a| = \sqrt{\frac{n-1}{n+1}},$$

are extremal for (1.10), *and the functions*

$$p_2(z) = \frac{1+\eta z^{\nu n}}{1-\eta z^{\nu n}}, \qquad |\eta| = 1, \ \nu = 1, 2, \ldots,$$

are extremal for (1.8) *and* (1.9), $k = ln$, $l = 1, 2, \dots$ *For other* k *sharp estimates are not known.*

In this paper we give sharp estimates for the next coefficient p_{n+2}, $n \geq 3$.

2. Main result

THEOREM 1. *Let* $n \in \mathbb{N}$, $n \geq 3$, *be fixed. If* $p(z) = 1 + p_1 z + p_2 z^2 + \ldots \in P(n)$, *then*

$$|p_{n+2}| \leq \frac{4n}{n+1}\left(\frac{n-1}{n+2}\right)^{(n-1)/2} < 2. \tag{2.1}$$

The estimate (2.1) *is sharp. The extremal functions have the following form*:

$$p_3(z) = \frac{1 + \left(z\,\dfrac{a+z^2}{1+\overline{a}z^2}\right)^n}{1 - \left(z\,\dfrac{a+z^2}{1+\overline{a}z^2}\right)^n}, \qquad |a| = \sqrt{\frac{n-1}{n+1}}. \tag{2.2}$$

Proof. By definition $p \in P(n)$ if and only if there exists $\omega(z) = \alpha_1 z + \alpha_2 z^2 + \ldots \in \Omega$ such that $p(z) = (1 + \omega^n(z))/(1 - \omega^n(z))$. Thus we have

$$1 + p_1 z + p_2 z^2 + \ldots = \frac{1 + (\alpha_1 z + \alpha_2 z^2 + \ldots)^n}{1 - (\alpha_1 z + \alpha_2 z^2 + \ldots)^n}. \tag{2.3}$$

Multiplying and equating the coefficients in (2.3) we obtain

$$\begin{aligned}
p_k &= 0 \quad \text{for } k = 1, \ldots, n-1,\\
p_n &= 2\alpha_1^n,\\
p_{n+1} &= 2n\alpha_1^{n-1}\alpha_2,\\
p_{n+2} &= 2n\alpha_1^{n-2}\left[\alpha_1\alpha_3 + \frac{n-1}{2}\,\alpha_2^2\right].
\end{aligned}$$

For $\omega \in \Omega$ the following estimates are known (see e.g. [1], [2], [5]–[7]):

$$\begin{aligned}
|\alpha_1| &\leq 1,\\
|\alpha_2| &\leq 1 - |\alpha_1|^2,\\
|\alpha_3(1 - |\alpha_1|^2) + \overline{\alpha}_1\alpha_2^2| &\leq (1 - |\alpha_1|^2) - |\alpha_2|^2.
\end{aligned}$$

Using the identity

$$\begin{aligned}
|p_{n+2}| &= 2n|\alpha_1|^{n-2}\left|\alpha_1\alpha_3 + \frac{n-1}{2}\alpha_2^2\right|\\
&= 2n|\alpha_1|^{n-2}\\
&\quad\times\left|\frac{\alpha_1}{1-|\alpha_1|^2}\,[\alpha_3(1-|\alpha_1|^2) + \overline{\alpha}_1\alpha_2^2] + \frac{\alpha_2^2[(n-1)(1-|\alpha_1|^2) - 2|\alpha_1|^2]}{2(1-|\alpha_1|^2)}\right|
\end{aligned}$$

and the above estimates we obtain

$$|p_{n+2}| \le 2n|\alpha_1|^{n-2}\left[|\alpha_1|(1-|\alpha_1|^2)+\frac{|\alpha_2|^2(|(n-1)-(n+1)|\alpha_1|^2|-2|\alpha_1|)}{2(1-|\alpha_1|^2)}\right].$$

Hence we need to find the maximum of the bracketed expression with respect to $|\alpha_2|$ and next with respect to $|\alpha_1|$ in the interval $[0,1]$.

To find the first maximum we have to investigate the following cases:

$$n-1-(n+1)|\alpha_1|^2-2|\alpha_1| \ge 0, \tag{2.4}$$

$$n-1-(n+1)|\alpha_1|^2 \ge 0, \qquad n-1-(n+1)|\alpha_1|^2-2|\alpha_1| < 0, \tag{2.5}$$

$$n-1-(n+1)|\alpha_1|^2 < 0. \tag{2.6}$$

The cases (2.5) and (2.6) are equivalent to

$$\frac{n-1}{n+1} \le |\alpha_1| \le \sqrt{\frac{n-1}{n+1}}, \qquad \sqrt{\frac{n-1}{n+1}} < |\alpha_1| \le 1,$$

respectively. In both cases we have

$$|p_{n+2}| \le 2n|\alpha_1|^{n-1}(1-|\alpha_1|^2) \le \frac{4n}{n+1}\left(\frac{n-1}{n+1}\right)^{(n-1)/2} = g(n)$$

for all $|\alpha_1| \in [0,1]$.

In the case (2.4) we have $0 \le |\alpha_1| \le \frac{n-1}{n+1}$ and so

$$|p_{n+2}| \le 2n|\alpha_1|^{n-2} \times \left\{|\alpha_1|(1-|\alpha_1|^2)+\frac{(1-|\alpha_1|^2)^2[n-1-(n+1)|\alpha_1|^2-2|\alpha_1|]}{2(1-|\alpha_1|^2)}\right\}$$

or equivalently

$$|p_{n+2}| \le n|\alpha_1|^{n-2}[(n+1)|\alpha_1|^4-2n|\alpha_1|^2+n-1].$$

In order to find the global maximum of the right hand side of the last expression in the interval $0 \le |\alpha_1| \le \frac{n-1}{n+1}$ we consider the function

$$\begin{aligned}\varphi(t) &= nt^{n-2}[(n+1)t^4-2nt^2+n-1] \\ &= n[(n+1)t^{n+2}-2nt^n+(n-1)t^{n-2}]\end{aligned} \tag{2.7}$$

for $t \in \left[0, \frac{n-1}{n+1}\right]$. Its derivative is

$$\varphi'(t) = nt^{n-3}[(n+1)(n+2)t^4-2n^2t^2+(n-2)(n-1)].$$

Thus $\varphi'(t)=0$ if and only if

$$t=0 \quad \text{or} \quad (n+1)(n+2)t^4-2n^2t^2+(n-2)(n-1)=0.$$

Hence the roots of $\varphi'(t)$ are $t_0=0$, t_1, $-t_1$, t_2, $-t_2$ where

$$t_1 = \sqrt{\frac{n^2-\sqrt{5n^2-4}}{(n+1)(n+2)}}, \qquad t_2 = \sqrt{\frac{n^2+\sqrt{5n^2-4}}{(n+1)(n+2)}}.$$

We can check that $t_1 \in [0, \frac{n-1}{n+1}]$ and $t_2 \notin [0, \frac{n-1}{n+1}]$. The global maximum of $\varphi(t)$, $t \in \left[0, \frac{n-1}{n+1}\right]$, is attained at $t = t_1$:

$$\begin{aligned}\varphi(t) \le \varphi(t_1) &= h(n)\\ &= n\left(\frac{n^2-\sqrt{5n^2-4}}{(n+1)(n+2)}\right)^{(n-2)/2}\left(1-\frac{n^2-\sqrt{5n^2-4}}{(n+1)(n+2)}\right)\\ &\quad\times\left(n-1-\frac{n^2-\sqrt{5n^2-4}}{n+2}\right)\end{aligned}$$

for $t \in \left[0, \frac{n-1}{n+1}\right]$. Finally we have

$$|p_{n+2}| \le \max\{g(n), h(n)\}. \tag{2.8}$$

Some computer calculations and graphs suggest that $g(n) \ge h(n)$ for $n \ge 3$ or equivalently that

$$\max\{g(n), h(n)\} = g(n). \tag{2.9}$$

To prove (2.9) we first find an estimate for $h(n)$. Observe that the function $\varphi(t)$ given by (2.7) may be written as a product $\varphi(t) = \varphi_1(t) \cdot \varphi_2(t)$, where

$$\begin{aligned}\varphi_1(t) &= nt^{n-2},\\ \varphi_2(t) &= (n+1)t^4 - 2nt^2 + n - 1 = (n+1)(1-t^2)\left(\frac{n-1}{n+1} - t^2\right).\end{aligned}$$

The functions φ_1, φ_2 are nonnegative, and φ_1 is increasing and φ_2 decreasing in the interval $0 \le t \le \frac{n-1}{n+1}$.

The product $\varphi_1(t) \cdot \varphi_2(t) = \varphi(t)$ attains its maximum at the point

$$t_1 = \sqrt{\frac{n^2-\sqrt{5n^2-4}}{(n+1)(n+2)}}.$$

If we consider two values t_1^+ and t_1^- close to t_1 and such that

$$0 < t_1^- < t_1 < t_2^+ < \frac{n-1}{n+1},$$

then

$$\varphi(t_1) = \varphi_1(t_1) \cdot \varphi_2(t_1) \le \varphi_1(t_1^+) \cdot \varphi_2(t_1^-).$$

Now we put

$$t_1^- = \sqrt{\frac{n^2-\sqrt{5}n}{(n+1)(n+2)}}, \qquad t_1^+ = \sqrt{\frac{n^2-2n}{(n+1)(n+2)}}$$

to obtain

$$\begin{aligned}\varphi(t_1) &\le \varphi_1(t_1^+)\cdot\varphi_2(t_1^-)\\ &= n\left[\frac{n(n-2)}{(n+1)(n+2)}\right]^{(n-2)/2}(n+1)\\ &\quad\times\left[\frac{n^2-\sqrt{5}n}{(n+1)(n+2)}-1\right]\left[\frac{n^2-5n}{(n+1)(n+2)}-\frac{n-1}{n+2}\right]\\ &= \frac{4n}{n+1}\left[\frac{n(n-2)}{(n+1)(n+2)}\right]^{(n-2)/2}\frac{(2+\sqrt{5})n^2-n-1}{(n+2)^2}=l(n).\end{aligned}$$

Since $h(n)=\varphi(t_1)$ we have

$$h(n)\le l(n)$$

and the inequality $l(n)\le g(n)$ is equivalent to

$$\frac{l(n)}{g(n)}=\left[\frac{n(n-2)}{(n+2)(n-1)}\right]^{(n-2)/2}\left(\frac{n+1}{n-1}\right)^{1/2}\frac{(2+\sqrt{5})n^2-n-1}{(n+2)^2}\le 1.$$

Thus to prove that $h(n)\le g(n)$ it is enough to show that

$$\left[\frac{n(n-2)}{(n+2)(n-1)}\right]^{(n-2)/2}\left(\frac{n+1}{n-1}\right)^{1/2}\frac{(2+\sqrt{5})n^2}{(n+2)^2}=L(n)\le 1.$$

For $n\ge 3$ we have

$$\begin{aligned}\frac{L'(n)}{L(n)} &= \frac{d}{dn}\Big\{\frac{n-2}{2}\left[\ln n+\ln(n-2)-\ln(n+2)-\ln(n-1)\right]\\ &\quad+\frac{1}{2}\left(\ln(n+1)-\ln(n-1)\right)+2\ln n-2\ln(n+2)+\ln(2+\sqrt{5})\Big\}\\ &= \frac{1}{2}\left[\ln n+\ln(n-2)-\ln(n+2)-\ln(n-1)\right]\\ &\quad+\frac{n-2}{2}\left[\frac{1}{n}+\frac{1}{n-2}-\frac{1}{n+2}-\frac{1}{n-1}\right]\\ &\quad+\frac{1}{2(n+1)}-\frac{1}{2(n-1)}+\frac{2}{n}-\frac{2}{n+2}\\ &= \frac{1}{2}\left[\ln n+\ln(n-2)-\ln(n+2)-\ln(n-1)+\frac{2}{n}+\frac{1}{n+1}\right].\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dn}\left[\frac{L'(n)}{L(n)}\right] &= \frac{1}{2}\left[\frac{1}{n}+\frac{1}{n-2}-\frac{1}{n+2}-\frac{1}{n-1}-\frac{2}{n^2}-\frac{1}{(n+1)^2}\right]\\ &= \frac{n^4+13n^3+10n^2-4n-8}{2n^2(n+1)^2(n-1)(n-2)(n+2)}>0 \quad \text{for } n\ge 3.\end{aligned}$$

Therefore the logarithmic derivative $L'(n)/L(n)$ is an increasing function in $[3,\infty)$. Since

$$\lim_{n\to\infty}\frac{L'(n)}{L(n)}=0,$$

we conclude that $L'(n)<0$ for $n>3$ and hence $L(n)$ is a decreasing function in the interval $(3,\infty)$.

We can calculate that $L(8)=0.983\ldots<1$, hence $L(n)\le 1$ for all $n\ge 8$.

For $n=3,4,5,6,7$ we see immediately that

$$\max\{g(n),h(n)\}=g(n),$$

namely:

$$\begin{aligned}
\max\{g(3),h(3)\} &= \max\{1.500, 1.392\ldots\}=1.500,\\
\max\{g(4),h(4)\} &= \max\{1.487\ldots, 1.313\ldots\}=1.48723\ldots,\\
\max\{g(5),h(5)\} &= \max\{1.481\ldots, 1.283\ldots\}=1.48148\ldots,\\
\max\{g(6),h(6)\} &= \max\{1.478\ldots, 1.267\ldots\}=1.4784\ldots,\\
\max\{g(7),h(7)\} &= \max\{1.476\ldots, 1.258\ldots\}=1.476\ldots
\end{aligned}$$

In this way we have proved that

$$\frac{h(n)}{g(n)}\le\frac{l(n)}{g(n)}\le L(n)\le 1$$

for all $n\in\mathbb{N}$, $n\ge 3$. This proves inequality (2.1).

The function $p_3(z)$ given by (2.2) has the power series expansion

$$p_3(z)=1+2a^nz^n+2a_na^{n-1}(1-a^2)z^{n+2}+\ldots$$

and for $|a|=\sqrt{\frac{n-1}{n+1}}$ its $(n+2)$th coefficient p^*_{n+2} satisfies

$$|p^*_{n+2}|=\frac{4n}{n+1}\left(\frac{n-1}{n+1}\right)^{(n-1)/2}.$$

Thus the result is sharp and the proof is complete.

3. Remarks. If we put

$$p^*_k(z)=\frac{1+\left(z\,\dfrac{a+z^k}{1+\overline{a}z^k}\right)^n}{1-\left(z\,\dfrac{a+z^k}{1+\overline{a}z^k}\right)^n}=1+p^*_1z+\ldots,\qquad z\in U,\ k=1,\ldots,n-1,$$

then for $|a|=\sqrt{\frac{n-1}{n+1}}$ we have

$$|p^*_{n+k}|=\frac{4n}{n+1}\left(\frac{n-1}{n+1}\right)^{(n-1)/2},\qquad k=1,\ldots,n-1.$$

This suggests that the following conjecture may be true:

Conjecture. *Let $n \geq 4$. If $p(z) = 1 + p_1 z + \ldots \in P(n)$ then for all $k = 1, \ldots, n-1$,*

$$|p_{n+k}| \leq \frac{4n}{n+1}\left(\frac{n-1}{n+1}\right)^{(n-1)/2}.$$

This conjecture is proved for $k = 1$ and 2. For $k \geq 3$ it is open.

The estimate of $|p_{n+k}|$ for $k = 1, 2$ has the following interesting property:

$$\lim_{n\to\infty} \frac{4n}{n+1}\left(\frac{n-1}{n+1}\right)^{(n-1)/2} = \frac{4}{e} \approx 1.47\ldots$$

and

$$\frac{4}{e} < \frac{4n}{n+1}\left(\frac{n-1}{n+1}\right)^{(n-1)/2} \leq 1.5, \qquad n \geq 3.$$

In the theory of univalent and multivalent functions many classes are defined via the class $P = P(1)$ of Carathéodory functions (e.g. starlike, convex, spirallike functions). If in these definitions we replace P by $P(n)$ we obtain new classes which have many interesting properties (see e.g. [3], [4]).

References

[1] P. L. Duren, *Univalent Functions*, Springer, 1983.

[2] G. M. Golusin, *Geometrische Funktionentheorie*, Springer, 1957.

[3] R. Jurasińska and J. Stankiewicz, *Special subclasses of Carathéodory functions and their applications*, Folia Sci. Univ. Techn. Resov. Mat. 10 (1991), 37–45.

[4] R. Jurasińska and A. Szpila, *The classes of functions which can be defined by subordination*, ibid. 22 (1998), 41–49.

[5] C. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[6] J. Stankiewicz, *Some extremal methods in the theory of univalent functions*, TNR, Rzeszów, 1986.

[7] J. Szynal, *On some extremal problems for holomorphic functions and polynomials*, Univ. Mariae Curie-Skłodowska, Lublin, 1987 (in Polish).

Renata Jurasińska
Institute of Mathematics
University of Rzeszów
Rejtana 16A
35-959 Rzeszów, Poland
E-mail: rjuras@univ.rzeszow.pl

Jan Stankiewicz
Institute of Mathematics
University of Rzeszów
Rejtana 16A
35-959 Rzeszów, Poland

Department of Mathematics
Rzeszów University of Technology
W. Pola 2, P.O. Box 85
35-959 Rzeszów, Poland
E-mail: jan.stankiewicz@man.rzeszow.pl

Reçu par la Rédaction le 19.3.2002 (1394)