Coefficients in some classes defined by subordination to multivalent majorants

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Dedicated to Professor Józef Siciak on the occasion of his 70th birthday

Abstract. We present some results connected with estimation of the coefficients for some special class of functions holomorphic in the unit disc and defined by subordination to some multivalent functions.

1. Introduction. Let $U = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc and $H = H(U)$ the family of functions holomorphic in $U$. Let $\Omega$ denote the class of Schwarz functions $\omega \in H$ such that $|\omega(z)| \leq |z|$ for $z \in U$.

We say that $f$ is subordinate to $g$ in $U$ and write $f \prec g$ or $f(z) \prec g(z)$ if there exists $\omega \in \Omega$ such that

\begin{equation}
(1.1) \quad f(z) = g(\omega(z)) \quad \text{for } z \in U.
\end{equation}

In the geometric theory of univalent and multivalent functions an important role is played by the class $P$ of Carathéodory functions with positive real part,

\begin{equation}
(1.2) \quad P = \{p \in H : p(0) = 1, \text{Re } p(z) > 0 \text{ for } z \in U\}.
\end{equation}

This class may also be defined by subordination:

\begin{equation}
(1.3) \quad P = \left\{ p \in H : p(z) \prec \frac{1+z}{1-z} \right\},
\end{equation}

where the majorant function

\begin{equation}
(1.4) \quad F_1(z) = \frac{1+z}{1-z}
\end{equation}

is univalent and maps $U$ onto the right half-plane $\{w \in \mathbb{C} : \text{Re } w > 0\}$. For
univalent majorants subordination (1.1) is equivalent to
\[(1.5) \quad f(0) = g(0) \quad \text{whenever} \quad f(U) \subset g(U).\]
Thus by (1.5) and the univalence of \(F_1(z)\) it is obvious that (1.2) and (1.3) define the same class.

R. Jurasińska and J. Stankiewicz [3] and R. Jurasińska and A. Szpila [4] introduced and investigated the classes \(P(n), n \in \mathbb{N},\) defined by formula (1.3) with the function \(F_1\) replaced by the \(n\)-valent function
\[(1.6) \quad F_n(z) = \frac{1 + z^n}{1 - z^n}, \quad n = 1, 2, \ldots,\]
that is,
\[(1.7) \quad P(n) = \left\{ p \in H : p(z) \prec \frac{1 + z^n}{1 - z^n} \right\}.
\]
For \(n \geq 2\) the class \(P(n)\) is a proper subclass of \(P.\)
The classes \(P(n)\) do not coincide with the subclasses
\[P_n = \{ p(z) = 1 + p_n z^n + p_{2n} z^{2n} + \ldots \in P \}\]
of \(n\)-symmetric functions with positive real part. We can easily check (see e.g. [3], [4]) that
\[P_n \not\subset P(n) \quad \text{and} \quad P(n) \not\subset P_n.\]
In [3], [4] some estimates of \(|p(z)|, \text{Re} \, p(z), \text{arg} \, p(z)\) and of some coefficients \(p_n, p_{n+1}\) were established. Among other results the following theorem was proved:

**Theorem A** ([4]). Let \(n \in \mathbb{N}, n \geq 2,\) be fixed. If \(p(z) = 1 + p_1 z + p_2 z^2 + \ldots \in P(n)\) then
\[(1.8) \quad p_k = 0 \quad \text{for} \quad k = 1, \ldots, n - 1,\]
\[(1.9) \quad |p_k| \leq 2 \quad \text{for} \quad k = n, n + 2, \ldots,\]
\[(1.10) \quad |p_{n+1}| \leq \frac{4n}{n+1} \left( \frac{n-1}{n+1} \right)^{(n-1)/2} < 2.\]
The functions
\[p_1(z) = \frac{1 + \left( \frac{z + a + z}{1 + \bar{a}z} \right)^n}{1 - \left( \frac{z + a + z}{1 + \bar{a}z} \right)^n}, \quad |a| = \sqrt{\frac{n-1}{n+1}},\]
are extremal for (1.10), and the functions
\[p_2(z) = \frac{1 + \eta z^n}{1 - \eta z^n}, \quad |\eta| = 1, \, \nu = 1, 2, \ldots,\]
are extremal for (1.8) and (1.9), \( k = ln, \ l = 1, 2, \ldots \). For other \( k \) sharp estimates are not known.

In this paper we give sharp estimates for the next coefficient \( p_{n+2}, n \geq 3 \).

2. Main result

THEOREM 1. Let \( n \in \mathbb{N}, n \geq 3 \), be fixed. If \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \in P(n) \), then

\[
|p_{n+2}| \leq \frac{4n}{n+1} \left( \frac{n-1}{n+2} \right)^{(n-1)/2} < 2.
\]

The estimate (2.1) is sharp. The extremal functions have the following form:

\[
p_3(z) = \frac{1 + \left( z \frac{a + z^2}{1 + az^2} \right)^n}{1 - \left( z \frac{a + z^2}{1 + az^2} \right)^n}, \quad |a| = \sqrt{\frac{n-1}{n+1}}.
\]

Proof. By definition \( p \in P(n) \) if and only if there exists \( \omega(z) = \alpha_1 z + \alpha_2 z^2 + \ldots \in \Omega \) such that \( p(z) = (1 + \omega(z))/(1 - \omega(z)) \). Thus we have

\[
1 + p_1 z + p_2 z^2 + \ldots = \left( 1 + (\alpha_1 z + \alpha_2 z^2 + \ldots)^n \right) / \left( 1 - (\alpha_1 z + \alpha_2 z^2 + \ldots)^n \right).
\]

Multiplying and equating the coefficients in (2.3) we obtain

\[
p_k = 0 \quad \text{for} \quad k = 1, \ldots, n-1,
\]
\[
p_n = 2\alpha_1^n,
\]
\[
p_{n+1} = 2n\alpha_1^{n-1}\alpha_2,
\]
\[
p_{n+2} = 2n\alpha_1^{n-2} \left[ \alpha_1 \alpha_3 + \frac{n-1}{2} \alpha_2^2 \right].
\]

For \( \omega \in \Omega \) the following estimates are known (see e.g. [1], [2], [5]–[7]):

\[
|\alpha_1| \leq 1,
\]
\[
|\alpha_2| \leq 1 - |\alpha_1|^2,
\]
\[
|\alpha_3(1 - |\alpha_1|^2) + \alpha_1 \alpha_2^2| \leq (1 - |\alpha_1|^2) - |\alpha_2|^2.
\]

Using the identity

\[
|p_{n+2}| = 2n|\alpha_1|^{n-2} \left| \alpha_1 \alpha_3 + \frac{n-1}{2} \alpha_2^2 \right|
\]
\[
= 2n|\alpha_1|^{n-2} \times \left| \frac{\alpha_1}{1 - |\alpha_1|^2} \left[ \alpha_3(1 - |\alpha_1|^2) + \alpha_1 \alpha_2^2 \right] + \frac{\alpha_2^2[(n-1)(1 - |\alpha_1|^2) - 2|\alpha_1|^2]}{2(1 - |\alpha_1|^2)} \right|
\]

\[
\]
and the above estimates we obtain
\[ |p_{n+2}| \leq 2n|\alpha_1|^{n-2} \left[ |\alpha_1|(1-|\alpha_1|^2) + \frac{|\alpha_2|^2((n-1) - (n+1)|\alpha_1|^2 - 2|\alpha_1|)}{2(1-|\alpha_1|^2)} \right]. \]

Hence we need to find the maximum of the bracketed expression with respect to $|\alpha_2|$ and next with respect to $|\alpha_1|$ in the interval $[0, 1]$.

To find the first maximum we have to investigate the following cases:

(2.4) \quad n - 1 - (n + 1)|\alpha_1|^2 - 2|\alpha_1| \geq 0,

(2.5) \quad n - 1 - (n + 1)|\alpha_1|^2 \geq 0, \quad n - 1 - (n + 1)|\alpha_1|^2 - 2|\alpha_1| < 0,

(2.6) \quad n - 1 - (n + 1)|\alpha_1|^2 < 0.

The cases (2.5) and (2.6) are equivalent to

\[ \frac{n - 1}{n + 1} \leq |\alpha_1| \leq \sqrt{\frac{n - 1}{n + 1}}, \quad \sqrt{\frac{n - 1}{n + 1}} < |\alpha_1| \leq 1, \]

respectively. In both cases we have

\[ |p_{n+2}| \leq 2n|\alpha_1|^{n-1}(1 - |\alpha_1|^2) \leq \frac{4n}{n + 1} \left( \frac{n - 1}{n + 1} \right)^{(n-1)/2} = g(n) \]

for all $|\alpha_1| \in [0, 1]$.

In the case (2.4) we have $0 \leq |\alpha_1| \leq \frac{n - 1}{n + 1}$ and so

\[ |p_{n+2}| \leq 2n|\alpha_1|^{n-2} \times \left\{ |\alpha_1|(1 - |\alpha_1|^2) + \frac{(1 - |\alpha_1|^2)^2[n - 1 - (n + 1)|\alpha_1|^2 - 2|\alpha_1|]}{2(1 - |\alpha_1|^2)} \right\} \]

or equivalently

\[ |p_{n+2}| \leq n|\alpha_1|^{n-2}[(n + 1)|\alpha_1|^4 - 2n|\alpha_1|^2 + n - 1]. \]

In order to find the global maximum of the right hand side of the last expression in the interval $0 \leq |\alpha_1| \leq \frac{n - 1}{n + 1}$ we consider the function

(2.7) \quad \varphi(t) = nt^{n-2}[(n + 1)t^4 - 2nt^2 + n - 1] = n[(n + 1)t^{n+2} - 2nt^n + (n - 1)t^{n-2}]

for $t \in \left[0, \frac{n - 1}{n + 1}\right]$. Its derivative is

\[ \varphi'(t) = nt^{n-3}[(n + 1)(n + 2)t^4 - 2n^2t^2 + (n - 2)(n - 1)]. \]

Thus $\varphi'(t) = 0$ if and only if

\[ t = 0 \quad \text{or} \quad (n + 1)(n + 2)t^4 - 2n^2t^2 + (n - 2)(n - 1) = 0. \]

Hence the roots of $\varphi'(t)$ are $t_0 = 0$, $t_1$, $-t_1$, $t_2$, $-t_2$ where

\[ t_1 = \sqrt{\frac{n^2 - \sqrt{5n^2 - 4}}{(n + 1)(n + 2)}}, \quad t_2 = \sqrt{\frac{n^2 + \sqrt{5n^2 - 4}}{(n + 1)(n + 2)}}. \]
We can check that \( t_1 \in [0, \frac{n-1}{n+1}] \) and \( t_2 \not\in [0, \frac{n-1}{n+1}] \). The global maximum of \( \varphi(t), t \in [0, \frac{n-1}{n+1}] \), is attained at \( t = t_1 \):

\[
\varphi(t) \leq \varphi(t_1) = h(n) = n\left(\frac{n^2 - \sqrt{5n^2 - 4}}{(n+1)(n+2)}\right)^{(n-2)/2} \left(1 - \frac{n^2 - \sqrt{5n^2 - 4}}{(n+1)(n+2)}\right) \times \left(n - 1 - \frac{n^2 - \sqrt{5n^2 - 4}}{n+2}\right)
\]

for \( t \in [0, \frac{n-1}{n+1}] \). Finally we have

\[
(2.8) \quad |p_{n+2}| \leq \max\{g(n), h(n)\}.
\]

Some computer calculations and graphs suggest that \( g(n) \geq h(n) \) for \( n \geq 3 \) or equivalently that

\[
(2.9) \quad \max\{g(n), h(n)\} = g(n).
\]

To prove (2.9) we first find an estimate for \( h(n) \). Observe that the function \( \varphi(t) \) given by (2.7) may be written as a product \( \varphi(t) = \varphi_1(t) \cdot \varphi_2(t) \), where

\[
\varphi_1(t) = nt^{n-2},
\]

\[
\varphi_2(t) = (n+1)t^4 - 2nt^2 + n - 1 = (n+1)(1-t^2)\left(\frac{n-1}{n+1} - t^2\right).
\]

The functions \( \varphi_1, \varphi_2 \) are nonnegative, and \( \varphi_1 \) is increasing and \( \varphi_2 \) decreasing in the interval \( 0 \leq t \leq \frac{n-1}{n+1} \).

The product \( \varphi_1(t) \cdot \varphi_2(t) = \varphi(t) \) attains its maximum at the point

\[
t_1 = \sqrt{\frac{n^2 - \sqrt{5n^2 - 4}}{(n+1)(n+2)}}.
\]

If we consider two values \( t_1^+ \) and \( t_1^- \) close to \( t_1 \) and such that

\[
0 < t_1^- < t_1 < t_1^+ < \frac{n-1}{n+1},
\]

then

\[
\varphi(t_1) = \varphi_1(t_1) \cdot \varphi_2(t_1) \leq \varphi_1(t_1^+) \cdot \varphi_2(t_1^-).
\]

Now we put

\[
t_1^- = \sqrt{\frac{n^2 - \sqrt{5n}}{(n+1)(n+2)}}, \quad t_1^+ = \sqrt{\frac{n^2 - 2n}{(n+1)(n+2)}}
\]
to obtain

\[ \varphi(t_1) \leq \varphi_1(t_1^+) \cdot \varphi_2(t_1^-) \]

\[ = n \left[ \frac{n(n-2)}{(n+1)(n+2)} \right]^{(n-2)/2} (n+1) \times \left[ \frac{n^2 - \sqrt{5}n}{(n+1)(n+2)} - 1 \right] \left[ \frac{n^2 - 5n}{(n+1)(n+2)} - \frac{n-1}{n+2} \right] \]

\[ = \frac{4n}{n+1} \left[ \frac{n(n-2)}{(n+1)(n+2)} \right]^{(n-2)/2} \left( 2 + \sqrt{5} \right) \frac{n^2 - n - 1}{(n+1)(n+2)} = l(n). \]

Since \( h(n) = \varphi(t_1) \) we have

\[ h(n) \leq l(n) \]

and the inequality \( l(n) \leq g(n) \) is equivalent to

\[ \frac{l(n)}{g(n)} = \left[ \frac{n(n-2)}{(n+2)(n-1)} \right]^{(n-2)/2} \left( \frac{n+1}{n-1} \right)^{1/2} \left( 2 + \sqrt{5} \right) \frac{n^2 - n - 1}{(n+2)^2} \leq 1. \]

Thus to prove that \( h(n) \leq g(n) \) it is enough to show that

\[ \left[ \frac{n(n-2)}{(n+2)(n-1)} \right]^{(n-2)/2} \left( \frac{n+1}{n-1} \right)^{1/2} \left( 2 + \sqrt{5} \right) \frac{n^2 - n - 1}{(n+2)^2} = L(n) \leq 1. \]

For \( n \geq 3 \) we have

\[ \frac{L'(n)}{L(n)} = \frac{d}{dn} \left\{ \frac{n-2}{2} \left[ \ln n + \ln(n-2) - \ln(n+2) - \ln(n-1) \right] \right. \]

\[ + \frac{1}{2} \left( \ln(n+1) - \ln(n-1) \right) + 2 \ln n - 2 \ln(n+2) + \ln(2 + \sqrt{5}) \right\} \]

\[ = \frac{1}{2} \left[ \ln n + \ln(n-2) - \ln(n+2) - \ln(n-1) \right] \]

\[ + \frac{n-2}{2} \left[ \frac{1}{n} + \frac{1}{n-2} - \frac{1}{n+2} - \frac{1}{n-1} \right] \]

\[ + \frac{1}{2(n+1)} - \frac{1}{2(n-1)} + \frac{2}{n} - \frac{2}{n+2} \]

\[ = \frac{1}{2} \left[ \ln n + \ln(n-2) - \ln(n+2) - \ln(n-1) + \frac{2}{n} + \frac{1}{n+1} \right]. \]

Hence

\[ \frac{d}{dn} \left[ \frac{L'(n)}{L(n)} \right] = \frac{1}{2} \left[ \frac{1}{n} + \frac{1}{n-2} - \frac{1}{n+2} - \frac{1}{n-1} - \frac{2}{n^2} - \frac{1}{(n+1)^2} \right] \]

\[ = \frac{n^4 + 13n^3 + 10n^2 - 4n - 8}{2n^2(n+1)^2(n-1)(n-2)(n+2)} > 0 \text{ for } n \geq 3. \]
Therefore the logarithmic derivative $L'(n)/L(n)$ is an increasing function in $[3, \infty)$. Since
\[
\lim_{n \to \infty} \frac{L'(n)}{L(n)} = 0,
\]
we conclude that $L'(n) < 0$ for $n > 3$ and hence $L(n)$ is a decreasing function in the interval $(3, \infty)$.

We can calculate that $L(8) = 0.983 \ldots < 1$, hence $L(n) \leq 1$ for all $n \geq 8$.

For $n = 3, 4, 5, 6, 7$ we see immediately that
\[
\max\{g(n), h(n)\} = g(n),
\]
namely:
\[
\begin{align*}
\max\{g(3), h(3)\} &= \max\{1.500, 1.392 \ldots\} = 1.500, \\
\max\{g(4), h(4)\} &= \max\{1.487 \ldots, 1.313 \ldots\} = 1.48723 \ldots, \\
\max\{g(5), h(5)\} &= \max\{1.481 \ldots, 1.283 \ldots\} = 1.48148 \ldots, \\
\max\{g(6), h(6)\} &= \max\{1.478 \ldots, 1.267 \ldots\} = 1.4784 \ldots, \\
\max\{g(7), h(7)\} &= \max\{1.476 \ldots, 1.258 \ldots\} = 1.476 \ldots
\end{align*}
\]

In this way we have proved that
\[
\frac{h(n)}{g(n)} \leq \frac{l(n)}{g(n)} \leq L(n) \leq 1
\]
for all $n \in \mathbb{N}$, $n \geq 3$. This proves inequality (2.1).

The function $p_3(z)$ given by (2.2) has the power series expansion
\[
p_3(z) = 1 + 2a^n z^n + 2a_n a^{n-1}(1 - a^2) z^{n+2} + \ldots
\]
and for $|a| = \sqrt{\frac{n-1}{n+1}}$ its $(n+2)$th coefficient $p_{n+2}^*$ satisfies
\[
|p_{n+2}^*| = \frac{4n}{n+1} \left(\frac{n-1}{n+1}\right)^{(n-1)/2}.
\]
Thus the result is sharp and the proof is complete.

3. Remarks. If we put
\[
p_k^*(z) = 1 + \left(z \frac{a + z^k}{1 + az^k}\right)^n = 1 + p_1^* z + \ldots, \quad z \in U, \quad k = 1, \ldots, n-1,
\]
then for $|a| = \sqrt{\frac{n-1}{n+1}}$ we have
\[
|p_{n+k}^*| = \frac{4n}{n+1} \left(\frac{n-1}{n+1}\right)^{(n-1)/2}, \quad k = 1, \ldots, n-1.
\]
This suggests that the following conjecture may be true:
Conjecture. Let $n \geq 4$. If $p(z) = 1 + p_1 z + \ldots \in P(n)$ then for all $k = 1, \ldots, n - 1$,

$$|p_{n+k}| \leq \frac{4n}{n+1} \left(\frac{n-1}{n+1}\right)^{(n-1)/2}. $$

This conjecture is proved for $k = 1$ and 2. For $k \geq 3$ it is open. The estimate of $|p_{n+k}|$ for $k = 1, 2$ has the following interesting property:

$$\lim_{n \to \infty} \frac{4n}{n+1} \left(\frac{n-1}{n+1}\right)^{(n-1)/2} = \frac{4}{e} \approx 1.47 \ldots$$

and

$$\frac{4}{e} < \frac{4n}{n+1} \left(\frac{n-1}{n+1}\right)^{(n-1)/2} \leq 1.5, \quad n \geq 3.$$

In the theory of univalent and multivalent functions many classes are defined via the class $P = P(1)$ of Carathéodory functions (e.g. starlike, convex, spirallike functions). If in these definitions we replace $P$ by $P(n)$ we obtain new classes which have many interesting properties (see e.g. [3], [4]).

References