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A regularity theorem for the complex Monge–Ampère equation in $\mathbb{CP}^n$

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Dedicated to Professor Józef Siciak

Abstract. $C^{1,1}$ regularity of the solutions of the complex Monge–Ampère equation in $\mathbb{CP}^n$ with the n-root of the right hand side in $C^{1,1}$ is proved.

0. Introduction. The purpose of this paper is to prove $C^{1,1}$ regularity of solutions of the complex Monge–Ampère equation in $\mathbb{CP}^n$ when the n-root of the function on the right hand side belongs to $C^{1,1}(\mathbb{CP}^n)$ with an extra assumption on the zero-set of this function. If we denote by $[z_0, z_1, \ldots, z_n]$ the homogeneous coordinates in $\mathbb{CP}^n$ then the closed positive $(1,1)$-form

$$\omega = \frac{i}{2}\,\partial\overline{\partial}\log\|z\|^2 = \frac{1}{4}\,dd^c(\log\|z\|^2) \qquad (d^c := i(\overline{\partial} - \partial))$$

induces the Fubini–Study metric. This is a Kähler metric invariant under holomorphic rotations of $\mathbb{CP}^n$. One can change the metric to obtain a given volume form $g\omega^n$, with some positive function g satisfying

$$\int_{\mathbb{CP}^n} g\omega^n = \int_{\mathbb{CP}^n} \omega^n = \pi^n. \tag{0.1}$$

For this we need to solve the Monge–Ampère equation

$$(\omega + dd^c u)^n = g\omega^n, \tag{0.2}$$

with unknown function u such that $\omega + dd^c u$ is a positive form. By the Calabi–Yau theorem [Y], if $g > 0$, $g \in C^k(\mathbb{CP}^n), k \geq 3$, then there exists a solution $u \in C^{k+1,\alpha}(\mathbb{CP}^n)$, where α is any number from the interval $(0,1)$. The existence part of the Calabi–Yau theorem was generalized by the author in [K1], [K2]. In particular, if $g \geq 0$ and $g \in L^p(\mathbb{CP}^n)$, $p > 1$, satisfying

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(0.1) then one can find a continuous solution (in the weak sense) of (0.2). As observed by Z. Błocki (cf. [B2]), the following result follows from the second order estimates in [Y], the stability theorem in [K2] and Trudinger's estimates [T] adapted to the complex Monge–Ampère equation in [B1].

THEOREM. *Let M be a compact Kähler manifold. If $g^{1/n} \in C^{1,1}(M)$, $g > 0$, satisfies* (0.1) *then the solution of* (0.2) *belongs to $C^{3,\alpha}(M)$ for some $\alpha \in (0,1)$.*

The regularity of solutions for the degenerate case $g \geq 0$ is harder. A particular case, when the set $\{g = 0\}$ is analytic, has been dealt with in [Y]. Here we show a partial result for $M = \mathbb{CP}^n$.

THEOREM 1. *If $g^{1/n} \in C^{1,1}(\mathbb{CP}^n)$, $g \geq 0$, satisfies* (0.1) *and the set $\{g = 0\}$ has volume* 0 *then the solution of* (0.2) *belongs to $C^{1,1}(\mathbb{CP}^n)$.*

In fact from the proof one can extract that it is enough to assume that the volume of $\{g = 0\}$ is bounded from above by some fixed positive constant (smaller than $\pi^n/2$). A related result has been obtained in [K3], where the regularity of the solution is shown in the standard coordinate chart: $\mathbb{C}^n$ embedded in $\mathbb{CP}^n$.

From [B2] it follows that for g as in Theorem 1 except for the assumption on the set $\{g = 0\}$, the solution has bounded Laplacian and so $g \in C^{1,\alpha}$, $\alpha < 1$.

I am indebted to Z. Błocki for helpful discussions on the subject.

1. Proof of Theorem 1. We denote the homogeneous coordinates in $\mathbb{CP}^n$ by $[z_0, z_1, \ldots, z_n]$. Fix a coordinate chart $w(z) = (z_1/z_0, \ldots, z_n/z_0)$ in $\{z : z_0 \neq 0\}$ and in this chart consider the two balls $B = \{w : |w| < 1/2\}$, $B_1 = \{w : |w| < 1\}$. The Lebesgue measure in $\mathbb{C}^n$ will be denoted by dV.

Orthogonal transformations in $\mathbb{C}^{n+1}$ of the form

$$\widetilde{F}_t(z) = (\cos t\, z_0 + \sin t\, z_1, -\sin t\, z_0 + \cos t\, z_1, z'), \qquad z' = (z_2, z_3, \ldots, z_n),$$
$$\widetilde{G}_t(z) = (\cos t\, z_0 + i \sin t\, z_1, -i \sin t\, z_0 + \cos t\, z_1, z'),$$

induce automorphisms on $\mathbb{CP}^n$ which we denote by F_t and G_t respectively.

By means of F_t and G_t we shall define "difference quotients" for functions defined on $\mathbb{CP}^n$. Let $w_t(z)$ (resp. $w_t'(z)$) be the midpoint of the interval $[w(F_t(z)), w(F_{-t}(z))]$ (resp. $[w(G_t(z)), w(G_{-t}(z))]$). Thus

$$w_t(z) = \frac{w(F_t(z)) + w(F_{-t}(z))}{2} = \frac{z_0}{\cos^2 t\, z_0^2 - \sin^2 t\, z_1^2}\,(z_1, \cos t\, z'),$$
$$w_t'(z) = \frac{w(G_t(z)) + w(G_{-t}(z))}{2} = \frac{z_0}{\cos^2 t\, z_0^2 + \sin^2 t\, z_1^2}\,(z_1, \cos t\, z').$$

Observe that

$$w_t(z) - w(z_0, z_1, \cos t\, z') = \frac{-\sin^2 t\,(z_0^2 + z_1^2)}{\cos^2 t\, z_0^2 + \sin^2 t\, z_1^2}\left(\frac{z_1}{z_0}, \frac{\cos t\, z'}{z_0}\right)$$
$$= \frac{-\sin^2 t\,(1 + w_1^2)}{\cos^2 t\, - \sin^2 t\, w_1^2}\,(w_1, \cos t\; w'),$$
$$w'_t(z) - w(z_0, z_1, \cos t\; z') = \frac{\sin^2 t\,(z_0^2 - z_1^2)}{\cos^2 t\, z_0^2 + \sin^2 t\, z_1^2}\left(\frac{z_1}{z_0}, \frac{\cos t\, z'}{z_0}\right)$$
$$= \frac{\sin^2 t\,(1 - w_1^2)}{\cos^2 t\, + \sin^2 t\, w_1^2}\,(w_1, \cos t\, w'),$$

with $w = w(z)$ and $w' = (w_2, w_3, \ldots, w_n)$. Therefore there exists $C_0 > 0$ such that for any $z \in w^{-1}(B_1)$ we have

(1.1) $$|w_t(z) - w(z)| \le C_0 t^2, \quad |w'_t(z) - w(z)| \le C_0 t^2, \quad 0 < t < 1.$$

By similar computation,

$$w(F_t(z)) - w(F_{-t}(z)) = \frac{-2\sin t}{\cos^2 t\, z_0^2 - \sin^2 t\, z_1^2}\,(\cos t\,(z_0^2 + z_1^2), z_1 z')$$
$$= \frac{-2\sin t}{\cos^2 t\, - \sin^2 t\, w_1^2}\,(\cos t\,(1 + w_1^2), w_1 w'),$$
$$w(G_t(z)) - w(G_{-t}(z)) = \frac{-2i\sin t}{\cos^2 t\, z_0^2 + \sin^2 t\, z_1^2}\,(\cos t\,(z_0^2 + z_1^2), z_1 z')$$
$$= \frac{-2i\sin t}{\cos^2 t\, + \sin^2 t\, w_1^2}\,(\cos t\,(1 + w_1^2), w_1 w').$$

Hence

$$\gamma(z) := \lim_{t\to 0}\frac{w(F_t(z)) - w(F_{-t}(z))}{2} = -(1 + w_1^2, w_1 w'),$$
$$\gamma'(z) := \lim_{t\to 0}\frac{w(G_t(z)) - w(G_{-t}(z))}{2} = -i(1 + w_1^2, w_1 w').$$

Note that since

$$|\gamma(z)| = |\gamma'(z)| = (|1 + w_1^2|^2 + |w_1|^2|w'|^2)^{1/2}$$

we have

(1.2) $$3/4 \le |\gamma(z)| \le 3/2 \quad \text{for } z \in w^{-1}(B).$$

Let u be a smooth function on $\mathbb{CP}^n$ with $dd^c u + \omega \ge 0$. Then $U = u \circ w^{-1}$ is defined and smooth on $\mathbb{C}^n$. Observe that there exists a constant C_1 independent of u such that

(1.3) $$\int_{B_1} \Delta U\, dV \le C_1,$$

where Δ denotes the Laplacian with respect to the Euclidean metric. Indeed, from $dd^c u \wedge \omega^{n-1} \geq -\omega^n$ and $\int_{\mathbb{CP}^n} dd^c u \wedge \omega^{n-1} = 0$ we have

$$\int_{w^{-1}(B_1)} dd^c u \wedge \omega^{n-1} = - \int_{\mathbb{CP}^n \setminus w^{-1}(B_1)} dd^c u \wedge \omega^{n-1} \leq \int_{\mathbb{CP}^n \setminus w^{-1}(B_1)} \omega^n \leq \pi^n.$$

Since the Euclidean metric and the pull-back of the Fubini–Study metric are equivalent we thus get

$$\int_{B_1} \Delta\Big(U + \frac{1}{4}\log(1+|w|^2)\Big)\, dV \leq C_2 \int_{w^{-1}(B_1)} (dd^c u + \omega) \wedge \omega^{n-1} \leq 2C_2\pi^n,$$

from which (1.3) follows. By (1.2) and the fact that $\gamma(z) = i\gamma'(z)$ we can estimate

$$\begin{aligned} \Phi_u(z) &:= D_{\gamma(z)\gamma(z)}U(w(z)) + D_{\gamma'(z)\gamma'(z)}U(w(z)) \\ &\leq \frac{9}{4}\Delta U(w(z)), \qquad w(z) \in B, \end{aligned} \tag{1.4}$$

where $D_{\gamma\gamma}$ denotes the second derivative in the direction of vector γ. Consider the sets

$$\begin{aligned} \Omega(M) &= \{w \in B : D_{\gamma(z)\gamma(z)}U(w) \geq M\}, \\ \Omega'(M) &= \{w \in B : D_{\gamma'(z)\gamma'(z)}U(w) \geq M\}. \end{aligned}$$

By (1.3) and (1.4),

$$2MV(\Omega(M) \cap \Omega'(M)) \leq \int_{\Omega(M)\cap\Omega'(M)} \Phi_u \circ w^{-1} dV \leq \frac{9}{4}\int_B \Delta U\, dV \leq \frac{9}{4}C_1.$$

Take M_0 so large that $V(\Omega(M_0) \cap \Omega'(M_0)) < \frac{1}{4}V(B)$. Then either

$$V(\Omega(M_0)) < \frac{3}{4}V(B) \tag{1.5}$$

or

$$V(\Omega'(M_0)) < \frac{3}{4}V(B).$$

From now on we assume that (1.5) holds. The proof for the other case is analogous.

Define

$$\gamma_t(z) = \frac{1}{t}\frac{-\sin t}{\cos^2 t - \sin^2 t\, w_1^2}(\cos t\,(1+w_1^2), w_1 w'), \qquad w = w(z).$$

So

$$\lim_{t\to 0} \gamma_t(z) = \gamma(z). \tag{1.6}$$

For smooth u we have, by Taylor expansion,

$$D_{\zeta\zeta}U(w) = \lim_{t\to 0} \frac{U(w+t\zeta) + U(w-t\zeta) - 2U(w)}{t^2}$$

and the convergence is uniform on the set $\{(w,\zeta)\in\mathbb{C}^2 : |w|\le 1,\ |\zeta|\le 3\}$. So for any $\varepsilon>0$ there exists $t_0>0$ such that

$$|\delta_u(t,z) - D_{\gamma_t(z)\gamma_t(z)}U(w_t(z))| < \varepsilon, \quad t<t_0, \tag{1.7}$$

where

$$\delta_u(t,z) = \frac{u(F_t(z)) + u(F_{-t}(z)) - 2U(w_t(z))}{t^2} = \frac{U(w_t(z)+t\gamma_t(z)) + U(w_t(z)-t\gamma_t(z)) - 2U(w_t(z))}{t^2}.$$

Using (1.1) and (1.6) we can decrease t_0 (recall that we work with smooth u) so that

$$|D_{\gamma_t(z)\gamma_t(z)}U(w_t(z)) - D_{\gamma(z)\gamma(z)}U(w(z))| < \varepsilon. \tag{1.8}$$

Combining (1.7) and (1.8) we conclude that given $\varepsilon>0$ we have

$$|\delta_u(t,z) - D_{\gamma(z)\gamma(z)}U(w(z))| < 2\varepsilon \quad \text{for } t<t_0,\ |w(z)|<1. \tag{1.9}$$

Define

$$u_t(z) = \frac{u(F_t(z)) + u(F_{-t}(z))}{2}$$

and $U_t = u_t\circ w^{-1}$. Note that $u_t(z) - u(z) = (t^2/2)\delta_u(t,z)$. Set

$$\Omega_t(M) = \{z\in\mathbb{CP}^n : u(z) < u_t(z) - Mt^2\} = \{\delta_u(t,z) > 2M\}.$$

By (1.9) for small t we have $w(\Omega_t(M_0))\cap B_1 \subset \Omega(M_0)$. Therefore, using (1.5) we obtain $V(w(\Omega_t(M_0))\cap B) \le \frac{3}{4}V(B)$. Hence, given g as in the assumptions, there exist $c>1$ and $c_0>0$ such that

$$\int_{\Omega_t(M_0)} (cg+c_0)\omega^n \le \int_{\mathbb{CP}^n} g\omega^n, \quad t<t_1,$$

for some $t_1>0$.

Let h_t be the solution of

$$(\omega+dd^c h_t)^n = \begin{cases} (cg+c_0)\omega^n & \text{on } \Omega_t(M_0),\\ c_1\omega^n & \text{on } \mathbb{CP}^n\setminus\Omega_t(M_0),\end{cases}$$

satisfying $\max h_t = 0$, where $t<t_1$ and $c_1\ge 0$ is chosen so that the integral of the right hand side over $\mathbb{CP}^n$ is equal to $\int_{\mathbb{CP}^n}\omega^n$. The solution exists by [K1] and moreover there exists c_2 independent of t such that

$$-c_2 < h_t \le 0.$$

One can increase c_2 and add a constant to u to have also

$$-c_2 < u_t \le 0.$$

Set

$$\Omega'(t,A) = \{u < (1-At^2)u_t + At^2h_t - (M_0+c_2A)t^2\}$$

for $A > 0$ and $t < t_1$ so small that $2nAt_1^2 < 1$. Note that $\Omega'(t,A) \subset \Omega_t(M_0)$, since $h_t - u_t - c_2 < 0$.

LEMMA. *For $g \ge 0$ with $g^{1/n} \in C^2(\mathbb{CP}^n)$ and u which is the solution of $(dd^c u + \omega)^n = g\omega^n$, define u_t as above and let g_t be the functions satisfying $(dd^c u_t + \omega)^n = g_t\omega^n$. Then there exists c_3 independent of t such that*

$$g_t^{1/n} \ge g^{1/n} - c_3 t^2, \qquad t < t_1,$$

with c_3 depending only on $\|D^2 g^{1/n}\|$.

Proof of Lemma. Since F_t are isometric with respect to the Fubini–Study metric we have

$$dd^c u_t + \omega = \frac{1}{2}\,[F_t^*(dd^c u + \omega) + F_{-t}^*(dd^c u + \omega)].$$

From the concavity of the mapping $A \mapsto \det^{1/n} A$ defined on the set of positive definite Hermitian matrices we have

$$g_t^{1/n} \ge \frac{1}{2}\,[g^{1/n} \circ F_t + g^{1/n} \circ F_{-t}].$$

By Taylor expansion,

$$\left|\frac{g^{1/n} \circ F_t(w) + g^{1/n} \circ F_{-t}(w)}{2} - g^{1/n}(w_t)\right| \le c_3' t^2,$$

where c_3' depends only on $\|D^2 g^{1/n}\|$. Combining this inequality with (1.1) we get the statement. ■

In what follows we can assume that $c_2 = c_3$ by just taking the larger of the two numbers. Choose $A > 0$ so that

$$A > 2nc_2 c_0^{-1/n} \quad \text{and} \quad A > \sup_{[0,\sup g]} f(x),$$

where $f(x) = c_2 x^{-1/n}[(c + c_0/x)^{1/n} - 1]^{-1}$. Note that $\sup_{[0,\sup g]} f(x)$ is finite since $\lim_{x\to 0} f(x) = c_2 c_0^{-1/n}$.

Reasoning by contradiction suppose that $\Omega' = \Omega'(t,A) \ne \emptyset$ for fixed small $t < t_1$. Set

$$\Omega'' = \Omega''(t,A) = \Omega' \cap \{g^{1/n} > 2nc_2 t^2\}.$$

For brevity, in the estimates below we write $a_t = 1 - At^2$, $b_t = g^{1/n} - c_2 t^2$. Applying, in turn, the comparison principle from [K2], Lemma 1.2 from [K2] and the above Lemma we obtain

$$\begin{aligned}\int_{\Omega'} g\omega^n &\geq \int_{\Omega'} a_t^n (dd^c u_t + \omega)^n + nAt^2 a_t^{n-1} (dd^c u_t + \omega)^{n-1} \wedge (dd^c h_t + \omega) \\ &\quad + A^n t^{2n} (dd^c h_t + \omega)^n \\ &\geq \int_{\Omega'} [a_t^n g_t + nAt^2 a_t^{n-1} g_t^{(n-1)/n} (cg + c_0)^{1/n} + A^n t^{2n} c_0]\omega^n \\ &= \int_{\Omega' \setminus \Omega''} \ldots + \int_{\Omega''} \ldots \\ &\geq \int_{\Omega' \setminus \Omega''} A^n t^{2n} c_0 \omega^n + \int_{\Omega''} [a_t^{n-1} b_t^{n-1} (a_t b_t + nAt^2 (cg + c_0)^{1/n}) + c_0]\omega^n.\end{aligned}$$

A contradiction is reached when the following two inequalities hold:

$$A^n t^{2n} (cg + c_0) > g \quad \text{on } \Omega' \setminus \Omega''$$

and

$$a_t^{n-1} b_t^{n-1} (a_t b_t + nAt^2 (cg + c_0)^{1/n}) \geq g \quad \text{on } \Omega''. \tag{1.10}$$

The first one follows from the choice of A and the fact that $g \leq (2nc_2)^n t^{2n}$ away from Ω''. To get the second one, divide both sides by g and use the inequalities of the type $a_t^{n-1} \geq 1 - (n-1)At^2$ to conclude that (1.10) follows from

$$\begin{aligned}(1 - (n-1)At^2)\left(1 - \frac{(n-1)c_2 t^2}{g^{1/n}}\right) \\ \times \left[\left(1 - \frac{c_2 t^2}{g^{1/n}}\right)(1 - At^2) + nAt^2 \left(c + \frac{c_0}{g}\right)^{1/n}\right] \geq 1.\end{aligned} \tag{1.11}$$

The left hand side of (1.11) is not smaller than

$$\left(1 - (n-1)At^2 - \frac{(n-1)c_2 t^2}{g^{1/n}}\right)\left[1 - \frac{c_2 t^2}{g^{1/n}} - At^2 + nAt^2 \left(c + \frac{c_0}{g}\right)^{1/n}\right]$$

and the last expression is not less than 1 if

$$nAt^2 \left(c + \frac{c_0}{g}\right)^{1/n} \left[1 - (n-1)At^2 - \frac{(n-1)c_2 t^2}{g^{1/n}}\right] \geq nAt^2 + \frac{nc_2 t^2}{g^{1/n}}.$$

Since the expression in the square brackets tends to 1 as $t \to 0$ we reach a contradiction as soon as

$$A > \frac{c_2}{g^{1/n}[(c + c_0/g)^{1/n} - 1]}.$$

The last inequality follows from the choice of A. The contradiction proves that Ω' is empty for t sufficiently small. So

$$u > (1 - At^2)u_t + At^2 h_t - (M_0 + c_2 A)t^2.$$

Therefore there exists A_0 such that for t small enough $u > u_t - A_0 t^2$, or equivalently,

$$\delta_u(t,z) \le 2A_0.$$

In view of (1.9) the last inequality implies

$$D_{\gamma(z)\gamma(z)}U(w(z)) \le \text{const} \tag{1.10}$$

for $|w(z)| \le 1/2$. The last estimate has been obtained for smooth u. It remains valid for all directions at a given point if we apply automorphisms of $\mathbb{CP}^n$.

To get the general case let us approximate a given g (in $C^{1,1}$ norm) by a sequence of smooth g_j normalized by $\int g_j\omega^n = \int \omega^n$ and such that the same constant c in the proof works for all j. By the above the solutions of

$$(dd^c u_j + \omega)^n = g_j\omega^n$$

have pure second order derivatives uniformly upper bounded. Thus (1.10) holds also for the original g. By the argument from [BT] a bound for pure second order derivatives also gives an upper bound for mixed second order derivatives of a plurisubharmonic function. But $U(w) + \log(1 + w^2)$ is plurisubharmonic and the second term is smooth. Thus U is $C^{1,1}$.

References

[BT] E. Bedford and B. A. Taylor, *The Dirichlet problem for the complex Monge–Ampère operator*, Invent. Math. 37 (1976), 1–44.

[B1] Z. Błocki, *On regularity of the complex Monge–Ampère operator*, in: Contemp. Math. 222, Amer. Math. Soc., 1999, 181–189.

[B2] —, *Regularity of the degenerate Monge–Ampère equation on compact Kähler manifolds*, Math. Z., to appear.

[K1] S. Kołodziej, *The complex Monge–Ampère equation*, Acta Math. 180 (1998), 69–117.

[K2] —, *Stability of solutions to the complex Monge–Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. J., to appear.

[K3] —, *Regularity of entire solutions to the complex Monge–Ampère equation*, Comm. Anal. Geom., to appear.

[T] N. S. Trudinger, *Regularity of solutions of fully nonlinear elliptic equations*, Boll. Un. Mat. Ital. 3 (1984), 421–430.

[Y] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation*, Comm. Pure Appl. Math. 31 (1978), 339–411.

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