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Intersection of analytic curves

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Dedicated to Professor Józef Siciak
on the occasion of his 70th birthday

Abstract. We give a relation between two theories of improper intersections, of Tworzewski and of Stückrad–Vogel, for the case of algebraic curves. Given two arbitrary quasiprojective curves V_1, V_2, the intersection cycle $V_1 \bullet V_2$ in the sense of Tworzewski turns out to be the rational part of the Vogel cycle $v(V_1, V_2)$. We also give short proofs of two known effective formulae for the intersection cycle $V_1 \bullet V_2$ in terms of local parametrizations of the curves.

1. Introduction. For two arbitrary purely dimensional analytic subsets V_1, V_2 (or, in general, for two analytic cycles V_1, V_2) of a complex manifold M, Tworzewski [T] defined an intersection product $V_1 \bullet V_2$ which is an analytic cycle. This theory was initiated in the case of improper isolated intersections by Achilles, Winiarski and Tworzewski [ATW], who made use of Draper's ideas (cf. [D]) concerning proper intersections in complex analytic geometry. The intersection cycle $V_1 \bullet V_2$ coincides with the classical one for every proper intersection of V_1 and V_2. On the other hand, for two arbitrary purely dimensional quasiprojective varieties V_1, V_2 over a field K we have the Vogel intersection cycle $v(V_1, V_2)$ (see [CFV, G1, G2]), which is an algebraic cycle defined over a pure transcendental extension of K. So, we may distinguish in $v(V_1, V_2)$ its rational part $v_{\mathrm{rat}}(V_1, V_2)$ (i.e. the part defined over K) and transcendental part $v_{\mathrm{tr}}(V_1, V_2)$:

$$v(V_1, V_2) = v_{\mathrm{rat}}(V_1, V_2) + v_{\mathrm{tr}}(V_1, V_2).$$

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The main result of the paper (Theorem 1) asserts that if V_1, V_2 are arbitrary quasiprojective curves over $\mathbb{C}$, then

$$V_1 \bullet V_2 = v_{\mathrm{rat}}(V_1, V_2).$$

For plane algebraic curves, this was obtained by the first author in [K2] by other methods. The proof of the main theorem is based on the method of deformation to the normal cone for analytic improper intersections, investigated by the second author in [N1, N2, N3]. Namely, we analyze the normal cone

$$C := C_{(V_1 \times V_2)\cap \Delta}(V_1 \times V_2)$$

to $(V_1 \times V_2) \cap \Delta$ in $V_1 \times V_2$, where Δ is the diagonal.

The other results are new short proofs of two known effective formulae for the intersection cycle $V_1 \bullet V_2$ of analytic curves V_1 and V_2 in terms of local parametrizations of V_1 and V_2 (Theorem 2). Whereas the first formula—due to Chądzyński, Krasiński and Tworzewski [CKT]—refers to an isolated intersection, the second one—due to the first author [K2]—is concerned with self-intersection. What makes it possible to simplify the reasonings is a theorem by the second author [N4] to the effect that the generalized intersection index $\widetilde{g}$ is realized by every collection of hyperplanes admissible with respect to the tangent cone B at P to the support of the normal cone C.

2. Normal cones. For two arbitrary purely dimensional analytic subsets V_1, V_2 of a complex manifold M, Tworzewski [T] defined, by means of the analytic intersection algorithm, a generalized intersection index $\tilde{g}$ of V_1 and V_2 at a point P (which is a sequence of non-negative integers), an intersection index g (the sum of the components of $\tilde{g}$) and an intersection product $V_1 \bullet V_2$ (an analytic cycle such that $\mathrm{mult}_P(V_1 \bullet V_2) = g(P)$ for $P \in M$). The intersection product $V_1 \bullet V_2$ coincides with the classical one for every proper analytic intersection. The intersection index g at a point P for an analytic set V and a submanifold S of the ambient manifold M coincides with the multiplicity at P of the normal cone $C := C_{V\cap S}V$ (cf. [AR, N1, N2, N3]), where the intersection $V \cap S$ is understood in the ideal-theoretic sense, i.e. as a possibly non-reduced analytic subspace of M. One may regard the normal cone C both as an analytic space and an analytic cycle. Throughout the paper we adopt the latter meaning.

A diagonal procedure reduces the study of intersections of two analytic subsets V_1 and V_2 of a complex manifold M to that of intersections of the product $V_1 \times V_2$ with the diagonal Δ which is a submanifold of the product $M \times M$. Therefore, the index g of intersection at P of two analytic sets V_1 and V_2 is the multiplicity at (P, P) of the normal cone

$$C := C_{(V_1 \times V_2)\cap \Delta}(V_1 \times V_2).$$

We now recall a geometric construction of C (see e.g. [N3, Ch. II, Sects. 3 and 4]). Define the following family of analytic sets parametrized by $\lambda \in \mathbb{C}\setminus\{0\}$:

$$\{(x,y,\lambda(x-y);1:\lambda) : x\in V_1,\ y\in V_2,\ \lambda\in\mathbb{C},\ \lambda\neq 0\}\subset \mathbb{C}^m_x\times\mathbb{C}^m_y\times\mathbb{C}^m_v\times\mathbb{P}_1.$$

The closure $\mathcal{V}$ of this family in $\mathbb{C}^m_x\times\mathbb{C}^m_y\times\mathbb{C}^m_v\times\mathbb{P}_1$ is an analytic set, and the fibre $\mathcal{V}_\infty$ of $\mathcal{V}$ over $\lambda=\infty$ (which may be regarded both as an analytic subspace and an analytic cycle) is the normal cone C. The normal cone $C=\mathcal{V}_\infty$ is the limit of the cycles $\mathcal{V}_\lambda$ (the fibres of $\mathcal{V}$ over λ) in the topology of locally uniform convergence of positive cycles. From the set-theoretical viewpoint, the support of the cone C is the analytic set

$$|C| = \{(x,x,v)\in V_1\times V_2\times\mathbb{C}^m_v : x_n,y_n\to x \text{ for some } x_n\in V_1,\ y_n\in V_2, \text{ and } \lambda_n(x_n-y_n)\to v \text{ for some } \lambda_n\to\infty\}. \tag{1}$$

From now on we shall consider two analytic curves V_1 and V_2. Then the normal cone C is an analytic cone of pure dimension 2. Since the intersection problem in question is local, we may view V_1, V_2 as germs at a fixed point $P=0\in\mathbb{C}^m$.

The constructions of normal cones and intersection indices are additive (see e.g. [N3, Ch. II, Sect. 3]). Consequently, if V_{1i} $(i=1,\dots,k)$ and V_{2j} $(j=1,\dots,l)$ are the irreducible components of the germs V_1 and V_2 respectively, the (generalized) intersection index for the pair V_1, V_2 is the sum of those for the kl pairs V_{1i}, V_{2j}, and

$$C=\sum_{i,j} C_{(V_{1i}\times V_{2j})\cap\Delta}(V_{1i}\times V_{2j}).$$

One may thus confine oneself to the case where the analytic curve germs V_1 and V_2 are irreducible. There are two different cases:

- $V_1\neq V_2$, and then $C=C_0$ is an algebraic cone with vertex at the origin;
- $V_1=V_2=V$, and then C is the sum of an analytic cone C' over $V^\Delta := (V\times V)\cap\Delta$ (counted with coefficient 1) and an algebraic cone C_0 with vertex at the origin.

For the case of self-intersection we show that the multiplicity at the origin of the analytic cone C' is equal to the multiplicity at $P=0$ of the curve V (Proposition 1). From this we will deduce the main theorem of the paper (Theorem 1).

We should mention that—in view of the geometric description (1) of the normal cone C (see also [N3, Ch. II, Sects. 3 and 4])—for arbitrary analytic sets V_1 and V_2, the support of the fibre over (P,P) of the cone C is the relative tangent cone to V_1 and V_2 at P (defined in [ATW]). The relative tangent cones to analytic curves were studied by Ciesielska [C] and the first author [K1]; in particular, the paper [K1] describes the relative tangent cone in terms of parametrizations of the curves.

Proposition 1. *Suppose V is an irreducible curve germ at $P = 0 \in \mathbb{C}^m$ and $C := C_{V^\Delta}(V \times V)$. Let C' be the part of the normal cone C that lies over V^Δ. Then the multiplicity of C' at the origin coincides with the multiplicity of V at $P = 0$.*

Proof. Indeed, V can be parametrized, in suitable coordinates near the origin, as follows:

$$x_1 = t^p, \quad x_2 = \varphi_2(t), \ \ldots, \ x_m = \varphi_m(t)$$

with $\operatorname{ord}_0 \varphi_k > p$. Obviously, the multiplicity of V at $P = 0$ is p and the tangent line to V at P is $\{(t, 0, \ldots, 0) \in \mathbb{C}^m : t \in \mathbb{C}\}$. Put $\varphi(t) = (\varphi_2(t), \ldots, \varphi_m(t))$; then it is easy to check that

$$(t^p, \varphi(t), t^p, \varphi(t), pst^{p-1}, s\varphi'(t))$$

is a parametrization of the cone C'. Since

$$\operatorname{ord}_{(0,0)}(t^p, \varphi(t), t^p, \varphi(t), pst^{p-1}, s\varphi'(t)) = p,$$

it follows that $\operatorname{mult}_P C' = p$, as desired. ∎

3. The main result. We apply Proposition 1 to show that the intersection cycle $V_1 \bullet V_2$ is the rational part of the Vogel cycle $v(V_1, V_2)$.

Theorem 1. *For any quasiprojective curves V_1 and V_2, the intersection cycle $V_1 \bullet V_2$ is the rational part $v_{\mathrm{rat}}(V_1, V_2)$ of the Vogel cycle $v(V_1, V_2)$.*

Proof. Since the contructions of the intersection cycle and Vogel cycle are local, we can confine ourselves to the affine case. By additivity, we may assume that the curves V_1, V_2 are irreducible. There are two cases: $V_1 \neq V_2$ or $V_1 = V_2 = V$. We keep the foregoing notation: the normal cone C is the sum of the cones C_0 and C'. While in the first case $C' = 0$ and C_0 is a finite sum of algebraic cones with vertices at (P, P), $P \in V_1 \cap V_2$, in the second C' is a cone over V^Δ (counted with coefficient 1) and C_0 is a finite sum of algebraic cones with vertices at (P, P), $P \in$ singular locus of V.

In the first case both $V_1 \bullet V_2$ and $v(V_1, V_2) = v_{\mathrm{rat}}(V_1, V_2)$ are 0-cycles on $V_1 \cap V_2$:

$$V_1 \bullet V_2 = \sum i(V_1 \bullet V_2; P) \cdot [P], \qquad v_{\mathrm{rat}}(V_1, V_2) = \sum \varepsilon_P \cdot [P],$$

where P ranges over $V_1 \cap V_2$. But both the coefficients ε_P and $i(V_1 \bullet V_2; P)$ are equal to the multiplicity at the vertex (P, P) of the normal cone C; these equalities follow from the fact that the Vogel cycle is invariant under deformation to the normal cone (cf. [G2]) and from the description of the intersection index by normal cones (cf. [AR, N1, N2, N3]). Hence $V_1 \bullet V_2 = v_{\mathrm{rat}}(V_1, V_2)$, as desired.

In the case of self-intersection,

$$V \bullet V = [V] + \sum i(V \bullet V; P) \cdot [P],$$
$$v_{\mathrm{rat}}(V, V) = [V] + \sum \varepsilon_P \cdot [P],$$

where P ranges over the singular locus of V. Again, as before, ε_P is equal to the multiplicity at (P, P) of the algebraic cone C_0 because the Vogel cycle is invariant under deformation to the normal cone (cf. [G2] and also [G1, Chap. 3, Sect. 2A]). On the other hand, we have

$$\mathrm{mult}_P(V \bullet V) = \mathrm{mult}_P V + i(V \bullet V; P),$$

and $\mathrm{mult}_P(V \bullet V)$ is, by definition, equal to the intersection index $g(P)$, which again coincides with the multiplicity of the normal cone C. Therefore

$$\mathrm{mult}_P V + i(V \bullet V; P) = \mathrm{mult}_{(P,P)} C = \mathrm{mult}_{(P,P)} C' + \mathrm{mult}_{(P,P)} C_0.$$

Hence and by Proposition 1, we obtain

$$i(V \bullet V; P) = \mathrm{mult}_{(P,P)} C_0 = \varepsilon_P,$$

which completes the proof. ∎

Example 1. Consider the self-intersection of the curve

$$V := \{(x, y) : y^2 - x^3 = 0\}$$

in $\mathbb{C}^2$ (for simplicity we do not consider this intersection in $\mathbb{P}^2$). Using the intersection algorithms of both theories we obtain

$$V \bullet V = 1 \cdot [V] + 3 \cdot [(0,0)] \quad \text{(a cycle over the field } \mathbb{C}),$$
$$v(V, V) = 1 \cdot [V] + 3 \cdot [(0,0)] + 1 \cdot \left[\left(\frac{4a^2}{9b^2}, -\frac{8a^3}{27b^3}\right)\right]$$
$$\text{(a cycle over the field } \mathbb{C}(a, b)).$$

The construction of the Vogel cycle imposes a lower bound on the dimensions of its components. Since no such bound exists for intersection cycles, the cycles $V_1 \bullet V_2$ and $v_{\mathrm{rat}}(V_1, V_2)$ need not coincide for higher dimensional sets V_1, V_2. We illustrate this by Example 2 (Ex. 5 from [N3, Ch. III, Sect. 3]) concerning the self-intersection of a projective surface. We should also emphasize that while the Vogel cycle $v(V_1, V_2)$ is effective, $V_1 \bullet V_2$ may not be an effective cycle, as demonstrated in Example 3.

Example 2. Let V be the cubic surface in $\mathbb{P}^3$ given by the equation

$$z_0 z_1 z_2 + z_1 z_2 z_3 + z_2 z_3 z_0 + z_3 z_0 z_1 = 0.$$

Its singular locus consists of only four nodes (i.e. non-degenerate isolated singular points of degree 2)

$$P_0 = (1 : 0 : 0 : 0), \quad P_1 = (0 : 1 : 0 : 0),$$
$$P_2 = (0 : 0 : 1 : 0), \quad P_3 = (0 : 0 : 0 : 1).$$

We have (cf. [N3, Ch. III, Sect. 3, Ex. 5])

$$V \bullet V = 1 \cdot [V] + 2 \cdot [P_0] + 2 \cdot [P_1] + 2 \cdot [P_2] + 2 \cdot [P_3].$$

On the other hand, the Vogel cycle $v(V_1, V_2)$ is the sum of $[V]$ and a transcendental curve of degree 6 through P_0, P_1, P_2, P_3. Hence

$$v_{\mathrm{rat}}(V_1, V_2) = 1 \cdot [V] \neq V \bullet V.$$

Example 3. Consider two affine sets in $\mathbb{C}^6 = \mathbb{C}^4_{\mathbf{x}} \times \mathbb{C}^2_{y,z}$, $\mathbf{x} = (x_1, x_2, x_3, x_4)$, defined by

$$V_1 := \{(\mathbf{x}, y, z) : z x_1 x_2 = y x_3 x_4\}, \qquad V_2 := \{(\mathbf{x}, y, z) : z = y = 0\}.$$

By straightforward calculations we obtain

$$\begin{aligned} V_1 \bullet V_2 = 1 \cdot [V_2] &+ 1 \cdot [\mathbb{C}^2_{x_1,x_3}] + 1 \cdot [\mathbb{C}^2_{x_1,x_4}] + 1 \cdot [\mathbb{C}^2_{x_2,x_3}] + 1 \cdot [\mathbb{C}^2_{x_2,x_4}] \\ &-1 \cdot [\mathbb{C}_{x_1}] - 1 \cdot [\mathbb{C}_{x_2}] - 1 \cdot [\mathbb{C}_{x_3}] - 1 \cdot [\mathbb{C}_{x_4}] + 2 \cdot [0], \\ v(V_1, V_2) = 1 \cdot [V_2] &+ v^2, \end{aligned}$$

where v^2 is the cycle of dimension 3 on $V_2 \otimes_{\mathbb{C}} \mathbb{C}(a, b) = \mathbb{C}(a, b)^4$ determined by the equation

$$a x_1 x_2 + b x_3 x_4 = 0.$$

Hence

$$v_{\mathrm{rat}}(V_1, V_2) = 1 \cdot [V_2] \neq V_1 \bullet V_2.$$

4. Formulae for intersection cycles. For two analytic curve germs V_1, V_2 at $P = 0 \in \mathbb{C}^m$, put

$$C := C_{(V_1 \times V_2) \cap \Delta}(V_1 \times V_2),$$

and let C' and C_0 be the parts of C that lie over $(V_1 \times V_2) \cap \Delta$ and the origin, respectively. From the geometric description (1) of the normal cone C (see also [N3, Ch. II, Sects. 3 and 4]), it follows that the support of the fibre over the origin of the cone C coincides with the relative tangent cones to V_1 and V_2 at P. For the definition and properties of relative tangent cones we refer the reader to [ATW, C, K1]. From these papers we can thus deduce the following

Proposition 2. *Let V_1, V_2 be two analytic curve germs at the point $P = 0 \in \mathbb{C}^m$. Then the cone C_0 is a finite sum of planes*; *it is the zero cycle iff $V_1 = V_2 = V$ and P is a regular point of V. Moreover*, *if one of the germs*, *say V_1*, *is irreducible*, *all those planes contain the tangent line to V_1 at P.*

We are now in a position to present short proofs of two effective formulae (from [CKT, K2]) for the intersection indices of two analytic curves V_1 and V_2 in terms of their parametrizations at a point P.

By additivity one may confine oneself to the case where V_1 and V_2 are two irreducible curve germs at $P = 0 \in \mathbb{C}^m$. The problem is evident whenever V_1 and V_2 do not have a common tangent line at P, as then the intersection index g of V_1 and V_2 at P is equal to $\mathrm{mult}_P V_1 \cdot \mathrm{mult}_P V_2$.

THEOREM 2. *Consider two irreducible curve germs V_1, V_2 at $0 \in \mathbb{C}^m$, having a common tangent line, parametrized as follows:*

$$(x_1 = t^p,\, x_k = \varphi_k(t)), \qquad (x_1 = t^q,\, x_k = \psi_k(t)), \qquad k = 2, \dots, m,$$

with $\mathrm{ord}_0\, \varphi_k > p$ and $\mathrm{ord}_0\, \psi_k > q$. Then the generalized intersection index $\widetilde{g} = (g_0, g_1, g_2)$ of V_1 and V_2 at 0 is equal to

$$\widetilde{g} = \begin{cases} \Big(0, 0, q^{-1} \sum_{i=1}^{q} \mathrm{ord}_0(\varphi(t^q) - \psi(\varepsilon^i t^p))\Big) & \text{if } V_1 \neq V_2, \\ \Big(0, p, \sum_{i=1}^{p-1} \mathrm{ord}_0(\varphi(t) - \varphi(\varepsilon^i t))\Big) & \text{if } V_1 = V_2 = V, \end{cases}$$

where ε is a primitive root of unity of degree q (in the second case $p = q$ and $\varphi(t) = \psi(t)$).

Proof. For the proof, observe that for instance, via the linear change of coordinates $u = x + y$, $v = x - y$, the normal cone

$$C := C_{(V_1 \times V_2) \cap \Delta}(V_1 \times V_2)$$

can be treated as a cone in the product $\mathbb{C}^m_x \times \mathbb{C}^m_y$ of the ambient spaces (see e.g. [N3]). In view of Proposition 2, the part C_0 of C that lies over the origin is a finite union of planes containing the line

$$L := \{(x_1, 0, \dots, 0, y_1, 0, \dots, 0) \in \mathbb{C}^m_x \times \mathbb{C}^m_y : x_1 + y_1 = 0\}.$$

CASE I: $V_1 \neq V_2$. Then $C = C_0$ and the tangent cone B at $0 \in \mathbb{C}^m$ to the support of C is a finite union of planes containing L.

The generalized intersection index $\widetilde{g}$ is realized by every collection of hyperplanes (H_1, H_2) admissible with respect to B (cf. [N4]). Since the hyperplane $\{x_1 - y_1 = 0\}$ meets the cone B properly, it can be taken as the first element H_1 of such an admissible collection of hyperplanes. Proceed now with the intersection algorithm:

Step 0: the result $\varrho_0 = 0$, the remainder $\alpha_0 = V_1 \times V_2$;
Step 1: the result $\varrho_1 = 0$, the remainder $\alpha_1 = \alpha_0 \cdot H_1$;
Step 2: for a generic hyperplane H_2 of the form

$$\Big\{ l(x, y) := \sum_{k=2}^{m} a_k (x_k - y_k) = 0 \Big\},$$

we have $\alpha_2 = 0$ and $\varrho_2 = \alpha_1 \cdot H_2 = (V_1 \times V_2) \cdot H_1 \cdot H_2$, and the index g_2 is the degree at the origin of this 0-cycle. But

$$(x_1 = t^{pq}, x_k = \varphi_k(t^q)), \quad (y_1 = \tau^{pq}, y_k = \psi_k(\tau^p)), \quad k = 2, \ldots, m,$$

is a parametrization with multiplicity pq of the product $V_1 \times V_2$. Hence and using parametric multiplicity (see e.g. [TW]), we get the required result

$$\begin{aligned} g_2 &= \frac{1}{pq} \operatorname{mult}_{(0,0)}(t^{pq} - \tau^{pq}, l(\varphi(t^q) - \psi(\tau^p))) \\ &= \frac{1}{pq} \sum_{i=1}^{pq} \operatorname{mult}_{(0,0)}(\eta^i t - \tau, l(\varphi(t^q) - \psi(\tau^p))) \\ &= \frac{1}{pq} \sum_{i=1}^{pq} \operatorname{ord}_0(\varphi(t^q) - \psi((\eta^i t)^p)) = \frac{1}{q} \sum_{i=1}^{q} \operatorname{ord}_0(\varphi(t^q) - \psi(\varepsilon^i t^p)); \end{aligned}$$

here η is a primitive root of unity of degree pq.

Case II: $V_1 = V_2 = V$. Then the normal cone C is the sum of C_0 and an analytic cone C' that lies over $V^{\Delta} = (V \times V) \cap \Delta$; C' can be parametrized by

$$(t^p + pst^{p-1}, \varphi(t) + s\varphi'(t), t^p - pst^{p-1}, \varphi(t) - s\varphi'(t)).$$

The tangent cone B to the support of C at the origin is thus the union of the plane

$$\{(x_1, 0, \ldots, 0, y_1, 0, \ldots, 0) \in \mathbb{C}^m_x \times \mathbb{C}^m_y : x_1, y_1 \in \mathbb{C}\}$$

and a finite number of planes containing the line L (which form the support of C_0). Therefore, as in Case I, the hyperplane $H_1 := \{x_1 - y_1 = 0\}$ can be picked up as the first element H_1 of a collection of hyperplanes which realizes the generalized intersection index. Proceed again with the intersection algorithm:

Step 0: the result $\varrho_0 = 0$, the remainder $\alpha_0 = V \times V$;

Step 1: $\alpha_0 \cdot H_1 = \sum_{i=1}^{p} W_i$, where $W_i = \{(t^p, \varphi(t), t^p, \varphi(\varepsilon^i t))\}$, $i = 1, \ldots, p$; the result $\varrho_1 = W_p = V^{\Delta}$, $g_1 = p$ and the remainder $\alpha_1 = \sum_{i=1}^{p-1} W_i$;

Step 2: for a generic hyperplane H_2, $\alpha_1 \cdot H_2 = \varrho_2$, $\alpha_2 = 0$ and the index g_2 of the result ϱ_2 at (P, P) is equal to

$$g_2 = \sum_{i=1}^{p-1} \operatorname{ord}_0(\varphi(t) - \varphi(\varepsilon^i t)),$$

and the proof of the theorem is complete. ■

We immediately obtain the following formulae for the intersection cycles and indices:

COROLLARY ([CKT, Th. 1], [K2, Th. 4]). 1) *If* $V_1 \neq V_2$, *then*

$$V_1 \bullet V_2 = i(V_1 \bullet V_2; 0) \cdot [0],$$

$$i(V_1 \bullet V_2; 0) = \frac{1}{q} \sum_{i=1}^{q} \mathrm{ord}_0(\varphi(t^q) - \psi(\varepsilon^i t^p)); \tag{2}$$

2) *if* $V_1 = V_2 = V$, *then*

$$V \bullet V = 1 \cdot [V] + i(V \bullet V; 0) \cdot [0],$$

$$i(V \bullet V; 0) = \sum_{i=1}^{p-1} \mathrm{ord}_0(\varphi(t) - \varphi(\varepsilon^i t)). \tag{3}$$

REMARK. By the additivity of the intersection index, formulae (2) and (3) can be applied to calculate $i(V_1 \bullet V_2; P)$ for arbitrary analytic curve germs V_1, V_2 at P.

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