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Nagata submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$

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Dedicated to Professor Józef Siciak on his seventieth birthday

Abstract. The aim of this paper is to show that on $\mathbb{P}^1 \times \mathbb{P}^1$ with a polarization of type $(2, 1)$ there are no R-R expected submaximal curves through any $10 \le r \le 15$ points.

1. Introduction. The Nagata Conjecture has attracted a lot of attention recently [1], [3], [4]. It was originally formulated for $\mathbb{P}^2$. Recent work of Biran [1], [2] suggests that it should hold for a much broader class of algebraic varieties. In fact the Nagata Conjecture can be viewed as a statement on Seshadri constants at generic points. We adopt this point of view in this paper.

Even if the conjecture itself seems to demand new methods, it is reasonable to ask if there are some obvious counterexamples for a small number of points on a given surface. In the classical case of $\mathbb{P}^2$ the existence of counterexamples for $r \le 9$ points follows from the Riemann–Roch theorem. In this paper we show that, somewhat unexpectedly, this need not be the case even on a variety as simple as $\mathbb{P}^1 \times \mathbb{P}^1$.

Notation. For simplicity we denote the Néron–Severi group of a variety X tensored by $\mathbb{Q}$ by $\mathrm{NS}(X)$. By a *polarization of type* (a, b) or by a *curve of type* (a, b) in the product $\mathbb{P}^1 \times \mathbb{P}^1$ we mean a curve of bidegree a, b. We work throughout over the field $\mathbb{C}$ of complex numbers.

2. Seshadri constants and the Nagata–Biran Conjecture. Recall that a *polarized variety* is a pair (X, L) consisting of a smooth variety X and an ample line bundle L on X.

We assume that X is a smooth projective variety, L is a nef line bundle on X (that is, for all curves $C \subset X$ we have $L.C \ge 0$) and $x_1, \dots, x_r \in X$ are fixed points.

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Definition 1. The *Seshadri constant* of L at $x_1, \dots, x_r$ is the real number

$$\varepsilon(L; x_1, \dots, x_r) = \inf_{C \cap \{x_1, \dots, x_r\} \neq \emptyset} \frac{L.C}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} C},$$

where the infimum is taken over all (irreducible) curves C passing through at least one of the points $x_1, \dots, x_r$.

It follows from Kleiman's nefness criterion that $\varepsilon(L; x_1, \dots, x_r) \leq \sqrt[n]{L^n/r}$, where $n = \dim X$. If $\varepsilon(L; x_1, \dots, x_r) = \sqrt[n]{L^n/r}$ then we say that the Seshadri constant is *maximal*, otherwise, i.e. if $\varepsilon(L; x_1, \dots, x_r) < \sqrt[n]{L^n/r}$, it is *submaximal*.

Definition 2. We say that a curve C *computes the Seshadri constant* if

$$\varepsilon(L; x_1, \dots, x_r) = \frac{L.C}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} C}.$$

Note that if C computes $\varepsilon(L; x_1, \dots, x_r)$ then necessarily

$$\frac{L.C}{\sum \operatorname{mult}_{x_i} C} \leq \sqrt[n]{\frac{L^n}{r}}$$

by the above upper bound. This justifies the following

Definition 3. We say that a curve $C \subset X$ is *L-submaximal* at the points $x_1, \dots, x_r$ (or simply *submaximal*) if

$$\frac{L.C}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} C} < \sqrt[n]{\frac{L^n}{r}}.$$

Let $C \subset X$ be a curve passing through the points $x_1, \dots, x_r$ with multiplicities $m_1, \dots, m_r$, respectively. To the curve C we assign its *multiplicity vector* $M_C = (m_1, \dots, m_r) \in \mathbb{Z}^r$.

Definition 4. Let $C \subset X$ be a curve with multiplicity vector $M_C = (m_1, \dots, m_r)$. We say that C is *Riemann–Roch expected* (for short, *R-R expected*) if

$$h^0(\mathcal{O}_X(C)) - \sum_{i=1}^{r} \binom{m_i + 1}{2} > 0.$$

This simply means that a curve C is R-R expected if its existence follows from the naive dimension count (note that it takes at most $\binom{m+1}{2}$ independent linear conditions on a linear system to guarantee the existence of a member of this system passing through a given point with multiplicity at least m).

Now we are in a position to formulate

NAGATA–BIRAN CONJECTURE. *Let (X, L) be a polarized surface. Let k_0 be the smallest integer such that in the linear system $|k_0 L|$ there exists a smooth non-rational curve and let $N_0 = k_0^2 L^2$. With the above assumptions*

$$\varepsilon(L; x_1, \dots, x_r) = \sqrt{\frac{L^2}{r}}$$

for general $x_1, \dots, x_r \in X$ and $r \geq N_0$.

REMARK 1. (1) On $(\mathbb{P}^2, \mathcal{O}(1))$ we have $N_0 = 9$ and the curves computing the Seshadri constant for $r \leq N_0$ points are R-R expected.

(2) On $\mathbb{P}^1 \times \mathbb{P}^1$ with a polarization of type $(1,1)$ we have $N_0 = 8$ and again all curves computing the Seshadri constant for at most 8 points are R-R expected.

3. Submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$. In this paper we show that on $\mathbb{P}^1 \times \mathbb{P}^1$ with a polarization of type $(2,1)$ the curves computing the Seshadri constant for $r \leq 9$ points are R-R expected and submaximal unless $r = 4$ or 9. We do not know if for $r = 10, \dots, 16$ submaximal curves exist, but if they do, we show that they are not R-R expected.

Let (X, L) be a polarized surface with Picard number ϱ. Let $L_1, \dots, L_\varrho$ be a fixed basis of the Néron–Severi group $\mathrm{NS}(X)$ and let $x_1, \dots, x_r$ be fixed points on X. To a curve $C \subset X$ we assign a vector

$$v_C = (l_1, \dots, l_\varrho, m_1, \dots, m_r) \in \mathrm{NS}(X) \times \mathbb{Q}^r$$

such that $C \equiv l_1 L_1 + \ldots + l_\varrho L_\varrho$ and $M_C = (m_1, \dots, m_r)$.

The following lemma extends Propositions 1.8 and 4.5 of [4].

PROPOSITION 1. *Let (X, L) be a polarized surface with Picard number ϱ. Let $x_1, \dots, x_r \in X$ be such that the Seshadri constant $\varepsilon(L; x_1, \dots, x_r)$ is submaximal. Then there exist at most $\varrho + r$ irreducible and reduced submaximal curves passing through $x_1, \dots, x_r$.*

Proof. Let $C_1, \dots, C_s$ be irreducible and reduced submaximal curves. Each of them has a vector $v_{C_i} = v_i = (l_1^{(i)}, \dots, l_\varrho^{(i)}, m_1^{(i)}, \dots, m_r^{(i)}) \in \mathrm{NS}(X) \times \mathbb{Q}^r$ for $i = 1, \dots, s$.

If $s > \varrho + r$ then the equation

$$\sum_{i=1}^{s} \lambda_i v_i = 0 \quad \text{where } \lambda_i \in \mathbb{Q} \tag{1}$$

has a non-trivial solution. We may in fact assume that $\lambda_i \in \mathbb{Z}$ (because we can multiply both sides of this equation by the common denominator).

Now we define curves C_+ and C_- in the following way:

$$C_+ := \sum_{i=1}^{s} \beta_i C_i, \quad \text{where} \quad \beta_i = \begin{cases} \lambda_i & \text{if } \lambda_i \geq 0, \\ 0 & \text{if } \lambda_i < 0, \end{cases}$$

$$C_- := \sum_{i=1}^{s} \gamma_i C_i, \quad \text{where} \quad \gamma_i = \begin{cases} 0 & \text{if } \lambda_i \geq 0, \\ -\lambda_i & \text{if } \lambda_i < 0. \end{cases}$$

Then of course

$$C_+ \equiv C_- \tag{2}$$

and in particular the multiplicity vectors

$$M_+ = (m_1^+, \ldots, m_r^+), \qquad M_- = (m_1^-, \ldots, m_r^-)$$

are equal.

Let $M = (m_1, \ldots, m_r)$ be the multiplicity vector at $x_1, \ldots, x_r$ of both curves. The curves C_+ and C_- are submaximal (as combinations of submaximal curves with non-negative integer coefficients). Hence

$$\frac{L.C_+}{\sum_{i=1}^{r} m_i} < \sqrt{\frac{L^2}{r}} \tag{3}$$

and

$$\frac{L.C_-}{\sum_{i=1}^{r} m_i} < \sqrt{\frac{L^2}{r}} \tag{4}$$

By their definition, C_+ and C_- have no common components, thus

$$\begin{aligned} C_-^2 = C_+.C_- \geq \sum_{i=1}^{r} m_i^2 \geq \frac{1}{r}\Big(\sum_{i=1}^{r} m_i\Big)^2 &= \frac{1}{\sqrt{r}} \sum_{i=1}^{r} m_i \cdot \frac{1}{\sqrt{r}} \sum_{i=1}^{r} m_i \\ &> \frac{L.C_+}{\sqrt{L^2}} \cdot \frac{L.C_-}{\sqrt{L^2}} = \frac{(L.C_-)^2}{L^2} \geq C_-^2, \end{aligned}$$

where the last inequality follows from the Hodge index theorem. This is a contradiction, so s can be at most $\varrho + r$. ■

Before proceeding, we need some more notation. For a vector $M = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ we define

$$\begin{aligned} |M| &:= \sum_{i=1}^{r} m_i, \\ \alpha(M) &:= \max\{|m_i - m_j| : i, j = 1, \ldots, r\}, \\ l(M) &:= \sum_{i=1}^{r} \binom{m_i + 1}{2}. \end{aligned}$$

LEMMA 1. *If* $M_1, M_2 \in \mathbb{Z}^r$ *are of the form* $M_1 = (m, \ldots, m, m, m+2)$ *and* $M_2 = (m, \ldots, m, m+1, m+1)$ *then* $l(M_2) < l(M_1)$.

Proof. This is a simple computation:

$$l(M_2) - l(M_1) = 2\binom{m+2}{2} - \left[\binom{m+3}{2} + \binom{m+1}{2}\right]$$
$$= (m+2)(m+1) - \frac{1}{2}[(m+3)(m+2) + (m+1)m] = -1. \blacksquare$$

An obvious consequence of this lemma is

COROLLARY 1. *Let* $\mathcal{M}_p = \{M \in \mathbb{Z}^r : |M| = p\}$. *If not all multiplicities are equal then vectors which impose the least theoretical number of conditions on a curve* C (*see remark after Definition* 4) *are the ones with* $\alpha(M) = 1$.

Hence if M is a vector in $\mathcal{M}_p$, then up to a permutation

$$M = (\underbrace{m, \ldots, m}_{i}, \underbrace{m+\delta, \ldots, m+\delta}_{r-i}),$$

where $\delta \in \{-1, 1\}$. Obviously $l(M)$ is independent of the way the multiplicities are ordered, i.e. $l((m_1, \ldots, m_r)) = l((m_{\sigma(1)}, \ldots, m_{\sigma(r)}))$ for any permutation $\sigma \in S_r$.

COROLLARY 2. *If* $p = rm + (r-i)\delta$ *then*

$$\#\{M \in \mathbb{Z}^r : |M| = p \text{ and } \alpha(M) = 1\} = \binom{r}{i}.$$

We proved in Proposition 1 that the number of curves computing the submaximal Seshadri constant is at most $\varrho + r$. Then analyzing the inequality $\binom{r}{i} \le \varrho + r$ for $\varrho = 2$ we conclude that $i = 0, 1, r-1$ or r, hence we have the following:

COROLLARY 3. *Let* (X, L) *be a polarized surface with Picard number* $\varrho = 2$ *and let* $x_1, \ldots, x_r \in X$ *be fixed generic points. If* $M = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ *is the multiplicity vector of a submaximal reduced and irreducible curve* C *at* $x_1, \ldots, x_r$, *then* M *is almost homogeneous*, *i.e. up to a permutation* M *is of the form* $(m, \ldots, m, m+\delta)$ *with* $\delta \in \{-1, 0, 1\}$.

Now we can formulate the main result of this paper:

THEOREM. *Let* $X = \mathbb{P}^1 \times \mathbb{P}^1$. *If* L *is a polarization of type* $(2,1)$ *then there are no R-R expected submaximal curves on* X *through* $10 \le r \le 15$ *points.*

Proof. Fix r and suppose to the contrary that $C \subset X$ of type (a, b) is R-R expected and submaximal. We can assume that the multiplicity vector of C is $M = (m, \ldots, m, m+\delta)$, where $\delta \in \{-1, 0, 1\}$, $m \in \mathbb{Z}$ (by Corollary 3). Hence the number of conditions imposed by M is

$$l(M) = (r-1)\binom{m+1}{2} + \binom{m+\delta+1}{2}$$
$$= \frac{1}{2}\,[rm^2 + rm + 2m\delta + \delta^2 + \delta].$$

Since $h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)) = ab + a + b + 1$ and C is R-R expected, and by Proposition 1 there is no continuous family of submaximal curves, we must have

$$ab + a + b = \frac{1}{2}\,[rm^2 + rm + 2m\delta + \delta^2 + \delta],$$

or equivalently,

$$2b = \frac{rm^2 + rm + 2m\delta + \delta^2 + \delta - 2a}{a+1}. \tag{5}$$

The submaximality of C means that

$$\frac{a+2b}{\sum_{i=1}^r m_i} < \frac{2}{\sqrt{r}}. \tag{6}$$

Conditions (5) and (6) give the inequality

$$\sqrt{r}\,a^2 - (\sqrt{r} + 2rm + 2\delta)a + \sqrt{r}(rm^2 + rm + 2m\delta + \delta^2 + \delta) - 2rm - 2\delta < 0.$$

We view it as an inequality in the variable a. We know that the set of solutions is non-empty, hence

$$r + 12r\sqrt{r}\,m + 12\sqrt{r}\,\delta + 4\delta^2 - 4r^2m - 4r\delta^2 - 4r\delta > 0.$$

Substituting $\sqrt{r} = t$ we obtain

$$-4t^4m + 12t^3m + (1 - 4\delta^2 - 4\delta)t^2 + 12\delta t + 4\delta^2 > 0. \tag{7}$$

This inequality has the simplest form for $\delta = 0$. In this case we have

$$-4t^4m + 12t^3m + t^2 > 0,$$

or equivalently,

$$4tm(3-t) + 1 > 0,$$

which of course implies that $t \le 3$. So we have shown that if $\delta = 0$ then $r \in [1, 9] \cap \mathbb{Z}$.

It remains to check (7) for $\delta \in \{-1, 1\}$. More precisely, we try to estimate m. Obviously

$$m(-4t^4 + 12t^3) > (4\delta^2 + 4\delta - 1)t^2 - 12\delta t - 4\delta^2. \tag{8}$$

Since $t > 3$ by assumption, the inequality (8) yields

$$m < \frac{(4\delta^2 + 4\delta - 1)t^2 - 12\delta t - 4\delta^2}{-4t^4 + 12t^3}.$$

For $\delta = 1$ we obtain

$$m < \frac{7t^2 - 12t - 4}{-4t^4 + 12t^3} < 0,$$

a contradiction; for $\delta = -1$,

$$m < \frac{t^2 - 12t + 4}{4t^4 - 12t^3} < \frac{1}{4t^2 - 12t} < 1,$$

and again there is no m such that (8) holds. ■

REMARK 2. Observe that if $0 < t \leq 3$, then from (8) we have

$$m > \frac{(4\delta^2 + 4\delta - 1)t^2 - 12\delta t - 4\delta^2}{-4t^4 + 12t^3},$$

which for $\delta = 1$ gives a lower bound for m:

$$m > \frac{7t^2 - 12t - 4}{-4t^4 + 12t^3}, \tag{9}$$

and for $\delta = -1$,

$$m > \frac{-t^2 + 12t - 4}{-4t^4 + 12t^3}. \tag{10}$$

Analyzing the sign of the numerator in (9) and (10) it is not difficult to check that we can take m and t such that for individual values of δ the inequality becomes true. Thus if there exists an R-R expected submaximal curve with a multiplicity vector $M = \widetilde{M}_\sigma$ (where $\widetilde{M} = (m, \dots, m, m + \delta)$, $\sigma \in S_r, m \in \mathbb{Z}_+, \delta \in \{-1, 0, 1\}$) then necessarily $r \in \{1, \dots, 9\}$.

More exactly we have the following curves and Seshadri constants:

r	Type of the curve	$\varepsilon(L; x_1, \dots, x_r)$	$\sqrt{L^2/2}$	Remarks
1	$(1,0)$	1	2	
2	$(2,0)$	1	$\sqrt{2}$	
3	$(3,0)$	1	$2\sqrt{3}/3$	
4	$(4,0)$	1	1	R-R expected but not submaximal
5	$(2,1)$	$4/5$	$2\sqrt{5}/5$	
6	$(2,1)$	$4/5$	$\sqrt{6}/3$	
7	$(3,1)$	$5/7$	$2\sqrt{7}/7$	
8	$(6,3)$	$12/17$	$\sqrt{2}/2$	
9	$(4,1)$	$2/3$	$2/3$	R-R expected but not submaximal

It might seem that on rational surfaces the lower bound for the number of points for which the Nagata–Biran Conjecture holds is $N = 9$. This is not the case, as the following example shows.

EXAMPLE 1. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ with a polarization L of type $(k, 1)$. Then for any $r = 2k + 4$ points there exists an R-R expected submaximal curve C

of type $(k^2+k,k+1)$ with multiplicity vector $M_C=(k,\ldots,k,k+1)$. This is an easy computation.

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