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Regularity of domains of parameterized families of closed linear operators

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To Professor Józef Siciak on the occasion of his 70th birthday

Abstract. The purpose of this paper is to provide a method of reduction of some problems concerning families $A_t = (A(t))_{t\in\mathcal{T}}$ of linear operators with domains $(\mathcal{D}_t)_{t\in\mathcal{T}}$ to a problem in which all the operators have the same domain $\mathcal{D}$. To do it we propose to construct a family $(\Psi_t)_{t\in\mathcal{T}}$ of automorphisms of a given Banach space X having two properties: (i) the mapping $t \mapsto \Psi_t$ is sufficiently regular and (ii) $\Psi_t(\mathcal{D}) = \mathcal{D}_t$ for $t \in \mathcal{T}$. Three effective constructions are presented: for elliptic operators of second order with the Robin boundary condition with a parameter; for operators in a Hilbert space for which eigenspaces form a complete orthogonal system of closed linear subspaces; and for a class of closed operators having bounded inverses.

1. Introduction. Most of the results concerning differential operators with a parameter t in the coefficients have been obtained under the assumption that the operators $(A_t = A(t))_{t\in\mathcal{T}}$ of a given family have domains independent of t (see e.g. [2, 6, 7, 8]).

One of possible ways of handling some problems concerning operators $(A_t)_{t\in\mathcal{T}}$ with domains $\mathcal{D}_t \subset X$ depending on t is to find a sufficiently regular (with respect to $t \in \mathcal{T}$) family Ψ_t of automorphisms of the Banach space X such that $\Psi_t(\mathcal{D}_t) = \mathcal{D}$, where $\mathcal{D}$ is a fixed linear subspace of X.

In general, the domain of a differential operator is determined by some boundary conditions. Thus it would be useful to find an effective construction of a family Ψ_t using the boundary conditions only. Such a construction for a family of elliptic operators of order two with the Robin boundary condition with a parameter (i.e. $\partial u/\partial n + a(x,t)u = 0$ on $\partial\Omega$) is presented in 2.1. The problem of existence and construction of a family $(\Psi_t)_{t\in\mathcal{T}}$ for general types of boundary conditions is more delicate and still open.

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In Section 2.2 there is a construction of a continuous family Ψ_t for some families $(\mathcal{D}_t)_{t\in\mathcal{T}}$ of domains of operators in a Hilbert space H for which the corresponding eigenspaces form a complete orthogonal system of closed linear subspaces of H.

If $\mathcal{D}_t$ is the domain of a closed invertible operator $A_t : X \to X$ and $R_t = A_t^{-1}$, for $t \in \mathcal{T}$, then the natural candidate for $\Phi_t = \Psi_t^{-1}$ is $\overline{R_{t_0}A_t}$ whenever $R_{t_0}A_t$ is closable. If it is closable then we may use some results presented in [3] concerning the topology of generalized convergence to prove that the expected family is good (for more details see Section 3). Unfortunately, if $\mathcal{D}_t$ depends on t, it may happen that $R_{t_0}A_t$ is not closable.

2. Regularity of families of linear subspaces. Let X be a Banach space, $\mathcal{T}$ an interval in $\mathbb{R}$, and $(\mathcal{D}_t)_{t\in\mathcal{T}}$ a family of linear subspaces of X.

DEFINITION 1. We say that the family $(\mathcal{D}_t)_{t\in\mathcal{T}}$ is of *class* $\mathcal{C}_a{}^k$ (resp. *strongly of class* $\mathcal{C}_a{}^k$) if there exist a linear subspace $\mathcal{D}$ of X and a family $(\Psi_t)_{t\in\mathcal{T}}$ of automorphisms of X such that

- the mapping $\mathcal{T} \ni t \mapsto \Psi_t \in \mathrm{Aut}(X)$ is of class $\mathcal{C}^k$ (resp. strongly of class $\mathcal{C}^k$ (1)) and
- $\Psi_t(\mathcal{D}) = \mathcal{D}_t$ for $t \in \mathcal{T}$.

Considering a family $(A_t)_{t\in\mathcal{T}}$ of closed linear operators with the family of domains $(\mathcal{D}_t = \mathcal{D}(A_t))_{t\in\mathcal{T}}$ of class $\mathcal{C}_a{}^k$ we may reduce some problems to a family with a constant domain. For example, suppose that u is a classical solution of the evolution equation

$$\frac{du}{dt} = A(t)u + f(t) \tag{1}$$

in which the family of domains $(\mathcal{D}_t = \mathcal{D}(A_t) = \mathcal{D}(A(t)))_{t\in\mathcal{T}}$ is of class $\mathcal{C}_a{}^1$. Let $(\Psi_t)_{t\in\mathcal{T}}$ be a family of automorphisms of X as above and $(\Phi_t = \Psi_t^{-1})_{t\in\mathcal{T}}$ the family of inverses.

Since $u(t) \in \mathcal{D}_t$, there exists $v(t) \in \mathcal{D}$ such that $\Psi_t(v(t)) = u(t)$. We have

$$\frac{du}{dt} = \frac{d\Psi_t}{dt}v(t) + \Psi_t\frac{dv}{dt}$$

and after a standard calculation we obtain

$$\frac{dv}{dt} = \underbrace{\left(\Phi_t A(t)\psi_t - \frac{d\psi_t}{dt}\right)}_{B(t)} v(t) + \underbrace{\Phi_t f(t)}_{F(t)}.$$

Thus v is a classical solution of the evolution equation

$$\frac{dv}{dt} = B(t)v + F(t) \tag{2}$$

with the family $(B_t = B(t))_{t\in\mathcal{T}}$ of operators having domains independent of t.

(1) This means that for any $x \in X$ the mapping $t \mapsto \Psi_t x$ is of class $\mathcal{C}^k$.

2.1. *Construction using boundary conditions.* Now we produce an example of a family $(\mathcal{D}_t)_{t\in\mathcal{T}}$ of class ${\mathcal{C}_a}^k$, $k \geq 1$, of the domains for elliptic operators of order two which is a nonconstant family of linear subspaces of $X = \mathcal{L}^2(\Omega)$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $S = \partial\Omega$ of class $\mathcal{C}^{k+1}$, $\mathcal{T} = [0, T]$, and let $a : \overline{\Omega} \times \mathcal{T} \to \mathbb{R}$ be a function of class $\mathcal{C}^k$ nonvanishing on S. The sets

$$\mathcal{D}_t = \left\{u \in \mathcal{L}^2(\Omega) : u \in H^2(\Omega) \text{ and } \frac{\partial u}{\partial n} + a(x,t)u = 0 \text{ on } \partial\Omega\right\}, \tag{3}$$

$$\mathcal{D} = \left\{u \in \mathcal{L}^2(\Omega) : u \in H^2(\Omega) \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\} \tag{4}$$

are dense linear subspaces of $\mathcal{L}^2(\Omega)$, where n is the interior unit normal vector field on S.

Let $\eta : \overline{\Omega} \times \mathcal{T} \to \mathbb{R}$ be a function of class $\mathcal{C}^k$ such that

$$1/2 \leq \eta_t(x) = \eta(x,t) \quad \text{for } x \in \overline{\Omega},\ t \in \mathcal{T}, \tag{5}$$

$$\eta_t(x) = 1 \quad \text{and} \quad \frac{\partial \eta_t(x)}{\partial n} = a(x,t) \quad \text{for } x \in \partial\Omega,\ t \in \mathcal{T}. \tag{6}$$

The function η can be constructed in the following way. We consider S as the retract of class $\mathcal{C}^k$ (for $\varepsilon > 0$ small enough) of the open ε-tube

$$\mathrm{TUB}^{\varepsilon}(S) = \{x + \tau n(x) : x \in S,\ |\tau| < \varepsilon\}.$$

Then we take a function h_ε of class $\mathcal{C}^\infty$ in $\mathbb{R}^n$ satisfying the following conditions:

$$\begin{aligned} h_\varepsilon(x) &= 0 && \text{for } x \in \mathbb{R}^n \setminus \mathrm{TUB}^{\varepsilon}(S),\\ h_\varepsilon(x) &= 1 && \text{for } x \in \mathrm{TUB}^{\varepsilon/2}(S),\\ h_\varepsilon(x) &\in [0,1] && \text{for } x \in \mathbb{R}^n. \end{aligned}$$

The function

$$f_\varepsilon : \mathrm{TUB}^{\varepsilon}(S) \ni x + \tau n(x) \mapsto a(t,x)\tau \in \mathbb{R} \tag{7}$$

is of class $\mathcal{C}^k$, and for ε small enough, the function $\eta = h_\varepsilon f_\varepsilon + 1$ is one we have been looking for.

Let $\Phi_t : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ be given by

$$\Phi_t(u) = \eta_t \cdot u \quad \text{for } u \in \mathcal{L}^2(\Omega),\ t \in [0, T] \tag{8}$$

and let $\Psi_t = \Phi_t^{-1}$. One can verify that

- $\Phi_t \in \mathrm{Aut}(\mathcal{L}^2(\Omega))$,
- $\Phi_t(\mathcal{D}_t) = \mathcal{D}$ and $\Psi_t(\mathcal{D}) = \mathcal{D}_t$,
- the mapping $\mathcal{T} \ni t \mapsto \Phi_t \in \mathcal{B}(\mathcal{L}^2(\Omega))$ is of class $\mathcal{C}^k$. Thus the mapping $\mathcal{T} \ni t \to \Psi_t \in \mathcal{B}(\mathcal{L}^2(\Omega))$ is also of class $\mathcal{C}^k$.

Considering parametrized boundary conditions of the form

$$\frac{\partial u}{\partial \mu_t} + a(x,t)u = 0 \quad \text{on } \partial\Omega, \tag{9}$$

where μ_t is a vector field on S parametrized by $t \in \mathcal{T}$, one can look for Φ_t of the form

$$\Phi_t u = \eta_t \cdot (u \circ \varphi_t) \quad \text{for } u \in \mathcal{L}^2(\Omega), \tag{10}$$

where φ_t is a diffeomorphism of $\overline{\Omega}$ such that $\varphi_t'(x).n(x) = \mu_t(x)$, $\varphi_t(x) = x$ for $x \in S$, $t \in \mathcal{T}$, and η is as in (8). Indeed, if u satisfies (9) then

$$\frac{\partial(u\circ\varphi)}{\partial n}(x) + a(x,t)(u\circ\varphi)(x) = \frac{\partial u}{\partial \mu_t} + a(x,t)u = 0 \quad \text{on } \partial\Omega.$$

Thus, $v = u \circ \varphi_t \in \mathcal{D}_t$, $\eta_t v \in \mathcal{D}$ and vice versa.

Let us remark that the boundary conditions (9) parametrized by t are natural, for example, when we consider the family $(A_t = tA + (1-t)\Delta)_{t\in[0,1]}$ in which A is a strongly elliptic operator of the second order. For an application see the second part of the proof of Theorem 3.4 in [1].

2.2. *Construction using eigenspaces.* Let H be a Hilbert space, and H_j, $j = 1,2,\ldots$, a complete orthogonal sequence of closed linear subspaces of H. We will use the following well known facts from the theory of Fourier series.

LEMMA 1. *If $a_j \in H_j$ for $j = 1,2,\ldots$ then the series $\sum_{j=1}^{\infty} a_j$ converges to a point $a \in H$ if and only if the series $\sum_{j=1}^{\infty} \|a_j\|^2$ is convergent. Moreover, if*

$$a = \sum_{j=1}^{\infty} a_j$$

then

$$\|a\|^2 = \sum_{j=1}^{\infty} \|a_j\|^2 \quad \text{and} \quad a_j = p_j(a) \text{ for } j = 1,2,\ldots,$$

where $p_j : H \to H_j$ is the orthogonal projection of H onto H_j for $j = 1,2,\ldots$

To any sequence $\lambda = \{\lambda_j\}_{j=1}^{\infty}$ of real (complex if H is a complex space) numbers corresponds a closed linear operator $A = A_\lambda(t) : H \to H$ given by

$$Ax = A_\lambda x = \sum_{j=1}^{\infty} \lambda_j p_j(x).$$

The operator A with domain

$$\mathcal{D} := \mathcal{D}(A) = \Big\{x \in H : \sum_{j=1}^{\infty} \lambda_j p_j(x) \text{ is convergent}\Big\}$$

is a closed densely defined linear operator and λ_j is an eigenvalue of A corresponding to the eigenspace H_j.

From now on we assume that H_j and λ_j depend on the parameter $t \in \mathcal{T}$. This implies that the projections $p_j,\ j = 1, 2, \ldots$, also depend on t. Thus, $H_j(t), \lambda_j(t), p_j(t),\ j = 1, 2, \ldots$, are sequences of closed subspaces, numbers and projections, respectively, parametrized by $t \in \mathcal{T}$.

PROPOSITION 2. *Suppose that for given $t, t_0 \in \mathcal{T}$, $\Phi_j(t) : H \to H$, $j = 1, 2, \ldots$, are bounded linear mappings satisfying the following conditions*:

(i) $\Phi_j(t)(H_j(t)) = H_j(t_0)$ *and* $\Phi_j(t)|_{H_j(t)} : H_j(t) \to H_j(t_0)$ *is an isomorphism of Banach spaces for* $j = 1, 2, \ldots$,

(ii) *there exist positive constants* $M(t), m(t) > 0$ *such that*

$$m(t)\|x\| \le \|\Phi_j(t)x\| \le M(t)\|x\| \quad \text{for } x \in H_j(t),\ j = 1, 2, \ldots,$$

(iii) *there exist positive constants* $\delta(t), \Delta(t) > 0$ *such that*

$$\delta(t) \le \left|\frac{\lambda_j(t_0)}{\lambda_j(t)}\right| \le \Delta(t) \quad \text{for } j = 1, 2, \ldots$$

Then

$$\Phi_t := \Phi(t) = \sum_{j=1}^{\infty} \Phi_j(t) \circ p_j(t) \tag{11}$$

is an automorphism of H such that $\Phi_t(\mathcal{D}_t) = \mathcal{D}_{t_0}$.

Proof. We begin by proving that Φ_t is well defined. Since

$$\|\Phi_j(t)(p_j(t)x)\|^2 \le \|\Phi_j(t)\|^2\|p_j(t)x\|^2 \le M^2(t)\|p_j(t)x\|^2$$

and the series $\sum_{j=1}^{\infty}\|p_j(t)x\|^2$ is convergent (because $\sum_{j=1}^{\infty} p_j(t)x$ is convergent), the series defining $\Phi_t x$ is convergent for any $(t, x) \in I \times H$.

Since

$$\|\Phi_t x\|^2 = \sum_{j=1}^{\infty}\|\Phi_j(t)p_j(t)x\|^2 \le M^2(t)\|x\|^2,$$

the operator Φ_t is bounded.

Injectivity of Φ_t follows from Lemma 1. Indeed,

$$\ker\Phi_t = \{x \in H : \Phi_j(t)p_j(t)x = 0,\ j = 1, 2, \ldots\} = \{0\}.$$

Let $y \in H$ and

$$x = \sum_{j=1}^{\infty}(\Phi_j(t))^{-1}p_j(t_0)y. \tag{12}$$

To prove surjectivity we must prove that the series (12) defining x is convergent and that $\Phi_t x = y$.

Assuming the convergence for the moment, we have

$$\Phi_t x = \sum_{j=1}^{\infty} \Phi_j(t) p_j(t) x = \sum_{j=1}^{\infty} \Phi_j(t)(\Phi_j(t))^{-1} p_j(t_0) y = \sum_{j=1}^{\infty} p_j(t_0) y = y.$$

The convergence of $\sum_{j=1}^{\infty} (\Phi_j(t))^{-1} p_j(t_0) y$ follows from Lemma 1, because of the estimates

$$\|(\Phi_j(t))^{-1} p_j(t_0) y\|^2 \le \frac{1}{m(t)} \|p_j(t) y\|^2 \quad \text{for } j = 1, 2, \ldots,\ y \in X.$$

For $x \in \mathcal{D}_t$ the series $\sum_{j=1}^{\infty} \lambda_j(t) p_j(t) x$ is convergent and we have $\Phi_t x = y = \sum_{j=1}^{\infty} p_j(t_0) y$. Thus

$$\sum_{j=1}^{\infty} \Phi_j(t) p_j(t) x = \sum_{j=1}^{\infty} p_j(t_0) y,$$

which implies that

$$p_j(t_0) y = \Phi_j(t) p_j(t) x. \tag{13}$$

Since

$$\begin{aligned}\left\| \frac{\lambda_j(t_0)}{\lambda_j(t)} \Phi_j(t)(\lambda_j(t) p_j(t)) x \right\|^2 &\le \left| \frac{\lambda_j(t_0)}{\lambda_j(t)} \right|^2 M^2(t) \|\lambda_j(t) p_j(t)\|^2 \\ &\le \Delta^2(t) M^2(t) \|\lambda_j(t) p_j(t)\|^2,\end{aligned}$$

the series

$$\sum_{j=1}^{\infty} \frac{\lambda_j(t_0)}{\lambda_j(t)} \Phi_j(t)(\lambda_j(t) p_j(t)) x$$

is convergent and hence, because of (13), so is $\sum_{j=1}^{\infty} \lambda_j(t_0) p_j(t_0) y$. This means that $\Phi_t(\mathcal{D}_t) \subset \mathcal{D}_{t_0}$. The proof of the inverse inclusion is similar. ■

Remark 1. If, in Proposition 2, $\Phi_j(t) : H_j(t) \to H_j(t_0)$ is an isometry for all j and t, then Φ_t is also an isometry.

Theorem 3. *If the mappings*

$$I \ni t \mapsto p_j(t) \quad \textit{and} \quad I \ni t \mapsto \Phi_j(t) \quad \textit{for } j = 1, 2, \ldots$$

are continuous and there exist $M, m > 0$ such that

$$m\|x\| \le \|\Phi_j(t) x\| \le M\|x\| \quad \textit{for } j = 1, 2, \ldots,\ x \in X,$$

then for any compact set $K \subset I \times H$, the mapping $K \ni (t, x) \mapsto \Phi_t x$ is continuous.

Proof. By Dini's theorem, the sequence

$$S_\nu(t, x) = \sum_{j=1}^{\nu} \|p_j(t) x\|^2, \quad \nu = 1, 2, \ldots,$$

converges uniformly to $\|x\|^2$ on compact subsets of $I \times H$. Since

$$\Big\| \sum_{j=p}^{p+s} \Phi_j(t)(p_j(t)x) \Big\|^2 \le M^2 \sum_{j=p}^{p+s} \|p_j(t)x\|^2,$$

the Cauchy condition of uniform convergence is satisfied for the series $\sum_{j=1}^{\infty} \Phi_j(t)(p_j(t)x)$ and so Φ is continuous on K. ∎

To obtain a higher regularity for the family (Φ_t) we must assume a higher regularity for $\Phi_j(t)$ and some assumptions that guarantee differentiability of series term by term.

EXAMPLE 1. The mapping

$$\Phi_j(t) := p_j(t_0) \circ p_j(t) : H \to H_j(t_0) \subset H$$

is a bounded linear map. Assuming, for example, that

$$H_j^{\perp}(t_0) \cap H_j(t) = \{0\}$$

we see that $\Phi_j(t)$ is injective.

If additionally we assume that $\dim H_j(t) = k_j < \infty$ is independent of t then

$$\Phi_j(t)|_{H_j(t)} : H_j(t) \to H_j(t_0)$$

is an isomorphism. Moreover

$$\|\Phi_j(t)x\|^2 \le \|p_j(t)x\|^2 \quad \text{and} \quad \Phi_j(t_0) = p_j(t_0).$$

Therefore, using the same method as in the proof of Theorem 3, we may prove that the continuity of the mapping $I \times H \ni (t,x) \mapsto p_j(t)x \in H$ for $j = 1, 2, \ldots$ implies the continuity of Φ on compact subsets of $I \times H$.

3. Families of closed operators with bounded inverses. Let X, Y be Banach spaces over the field $\mathbb{K}$ of real or complex numbers. We endow the space $C(X,Y)$ of closed linear operators $A : X \to Y$ with the topology of generalized convergence [3, Ch. IV]. The domain of a given operator $A : X \to Y$ is denoted by $\mathcal{D}(A)$. The space of bounded linear operators $A : X \to Y$ is denoted by $\mathcal{B}(X,Y)$, and $\mathrm{Isom}(X,Y)$ is the subspace of $\mathcal{B}(X,Y)$ of bijective bounded linear operators with bounded inverses. The subspace of $C(X,Y)$ consisting of the invertible densely defined operators A such that $A^{-1} \in \mathcal{B}(Y,X)$ will be denoted by $\mathcal{R}(X,Y)$. If $X = Y$ we will write $C(X)$, $\mathcal{B}(X)$, $\mathrm{Aut}(X)$, $\mathcal{R}(X)$ instead of $C(X,X)$, $\mathcal{B}(X,X)$, $\mathrm{Isom}(X,X)$, $\mathcal{R}(X,X)$, respectively. Since $\mathcal{B}(X,Y) \subset C(X,Y)$, we may consider $\mathcal{B}(X,Y)$ with the induced topology, which by [3, Ch. IV, Theorem 2.23] is equivalent to the norm topology in $\mathcal{B}(X,Y)$. Let us also recall that by the same theorem, the convergence of A_n to A in $\mathcal{R}(X,Y)$ is equivalent to the convergence of A_n^{-1} to A^{-1} in $\mathcal{B}(Y,X)$, $\mathrm{Isom}(X,Y)$ is open in $C(X,Y)$ and $\mathrm{Aut}(X)$ is open in $C(X)$.

LEMMA 4. *Let $\mathcal{H}$ be a metric space, $A_h \in \mathcal{R}(X,Y)$ and $\Phi_h \in \mathrm{Aut}(X)$ for $h \in \mathcal{H}$. If the mappings*

$$\mathcal{H} \ni h \mapsto A_h \in C(X,Y) \quad \text{and} \quad \mathcal{H} \ni h \mapsto \Phi_h \in \mathcal{B}(X) \tag{14}$$

are continuous then the mapping

$$\mathcal{H} \ni h \mapsto A_h \circ \Phi_h \in \mathcal{R}(X,Y) \tag{15}$$

is also continuous.

Proof. Since $A_h \in \mathcal{R}(X,Y)$, the continuity of $\mathcal{H} \ni h \mapsto A_h \in C(X,Y)$ is equivalent to the continuity of $\mathcal{H} \ni h \mapsto A_h^{-1} \in \mathcal{B}(Y,X)$. Thus the mapping $\mathcal{H} \ni h \mapsto (A_h \circ \Phi_h)^{-1} = \Phi_h^{-1} \circ A_h^{-1} \in \mathcal{B}(Y,X)$ is continuous, and hence so is the mapping (15). ■

LEMMA 5. *Let $A_j \in \mathcal{R}(X,Y)$ and $R_j = A_j^{-1}$ for $j = 1,2$. If $R_j \circ A_i$ is bounded for $i,j = 1,2$, then $\mathcal{D}(A_1^*) = \mathcal{D}(A_2^*)$.*

Proof. By symmetry, it is enough to prove that $\mathcal{D}(A_1^*) \subset \mathcal{D}(A_2^*)$. Take $y^* \in \mathcal{D}(A_1^*)$. Since

$$\begin{aligned} |\langle A_2x, y^*\rangle| &= |\langle A_1 \circ (R_1 \circ A_2)x, y^*\rangle| = |\langle (R_1 \circ A_2)x, A_1^*y^*\rangle| \\ &\le \|A_1^*y^*\| \cdot \|R_1 \circ A_2\| \cdot \|x\| \quad \text{for } x \in \mathcal{D}(A_2), \end{aligned}$$

we have $y^* \in \mathcal{D}(A_2^*)$ and so $\mathcal{D}(A_1^*) \subset \mathcal{D}(A_2^*)$. ■

LEMMA 6. *If $A \in C(X,Y)$ and $\Phi \in \mathcal{B}(X)$ is such that $\mathcal{D}(A \circ \Phi)$ is dense in X then $\mathcal{D}(A^*) \subset \mathcal{D}((A \circ \Phi)^*)$. Moreover, if $\Phi \in \mathrm{Aut}(X)$ then $\mathcal{D}(A^*) = \mathcal{D}((A \circ \Phi)^*)$.*

Proof. Let $y^* \in \mathcal{D}(A^*)$. Since Φ is continuous, for $x \in \mathcal{D}(A \circ \Phi)$ we have

$$|\langle (A \circ \Phi)x, y^*\rangle| = |\langle \Phi x, A^*y^*\rangle| \le \|A^*y^*\| \cdot \|\Phi x\| \le \|A^*y^*\| \cdot \|\Phi\| \cdot \|x\|$$

and so $y^* \in \mathcal{D}((A \circ \Phi)^*)$.

If Φ is invertible then by the above $\mathcal{D}((A \circ \Phi)^*) \subset \mathcal{D}((A \circ \Phi \circ \Phi^{-1})^*) = \mathcal{D}(A^*)$. ■

THEOREM 7. *Let $(\mathcal{H}, \varrho)$ be a connected metric space and $(A_h)_{h \in \mathcal{H}}$ a family of linear operators $A_h \in \mathcal{R}(X,Y)$. If $R_k \circ A_h$ is closable for each $h,k \in \mathcal{H}$, and for each $k \in \mathcal{H}$ the mapping*

$$\mathcal{H} \ni h \mapsto \overline{R_k \circ A_h} \in C(X) \tag{16}$$

is continuous, then:

(i) $\overline{R_k \circ A_h} \in \mathrm{Aut}(X)$ *for each $h,k \in \mathcal{H}$,*
(ii) *for any $h,k \in \mathcal{H}$ there exist $m, M > 0$ such that*

$$m\|R_hy\| \le \|R_ky\| \le M\|R_hy\| \quad \text{for } y \in Y,$$

(iii) $\mathcal{D}(A_h^*) = \mathcal{D}^* = \mathrm{const}$,
(iv) $\mathcal{D}(A_h^* \circ R_k^*) = X^*$ *for all $h,k \in \mathcal{H}$.*

Proof. Since $\mathrm{Aut}(X)$ is open in $C(X)$ and $\overline{R_k \circ A_k} = \mathrm{Id}_X \in \mathrm{Aut}(X)$, there exists $\delta = \delta(k) > 0$ such that $\overline{R_k \circ A_h} \in \mathrm{Aut}(X)$ for any $h \in \mathcal{H}$ such that $\varrho(h, k) < \delta$. Thus, for a given $k \in \mathcal{H}$,

$$\mathfrak{M} = \{h \in \mathcal{H} : \overline{R_k \circ A_h} \in \mathrm{Aut}(X)\} \neq \emptyset.$$

To prove that $\mathfrak{M} = \mathcal{H}$ it is enough to prove that $\mathfrak{M}$ is open and closed. For given $h_0 \in \mathfrak{M}$, $h \in \mathcal{H}$ we have

$$\overline{R_k \circ A_h} = \overline{R_k \circ A_{h_0}} \circ \overline{R_{h_0} \circ A_h}.$$

Since $\overline{R_k \circ A_{h_0}} \in \mathrm{Aut}(X)$, and by the same argument as before there exists $\delta = \delta(h_0) > 0$ such that $\overline{R_{h_0} \circ A_h} \in \mathrm{Aut}(X)$ for any $h \in \mathcal{H}$ satisfying $\varrho(h, h_0) < \delta$, the set $\mathfrak{M}$ is open. Suppose now that $h_n \in \mathfrak{M}$ for $n = 1, 2, \ldots$ and $h_n \to h_0 \in \mathcal{H}$ as $n \to \infty$. Then there exists $n \in \mathbb{N}$ such that $\overline{R_{h_0} \circ A_{h_n}} \in \mathrm{Aut}(X)$, by the previous part of the proof. Since $\overline{R_k \circ A_{h_n}} = \overline{R_k \circ A_{h_0}} \circ \overline{R_{h_0} \circ A_{h_n}}$ and $\overline{R_k \circ A_{h_n}}$, $\overline{R_{h_0} \circ A_{h_n}}$ are automorphisms of X, it follows that $h_0 \in \mathfrak{M}$ and so $\mathfrak{M}$ is closed.

To prove (ii) fix $h, k \in \mathcal{H}$. Since $\overline{R_k \circ A_h} \in \mathrm{Aut}(X)$, there exist $m, M > 0$ such that

$$m\|x\| \leq \|(R_k \circ A_h)x\| \leq M\|x\| \quad \text{for } x \in \mathcal{D}(A_h).$$

Since A_h is onto, taking $y = A_h x$ we get

$$m\|R_h y\| \leq \|R_k y\| \leq M\|R_h y\| \quad \text{for } y \in Y.$$

To prove (iii) observe that for $k, h \in \mathcal{H}$ we have $A_h = A_k \circ \overline{R_k \circ A_h}$. Thus, by Lemma 6, $\mathcal{D}(A_h^*) = \mathcal{D}(A_k^*)$, because $\overline{R_k \circ A_h} \in \mathrm{Aut}(X)$.

(iv) is a consequence of the fact that $R_k^* \in \mathcal{B}(X^*, Y^*)$ is the inverse to A_k^* (see e.g. [3, Ch. III, Theorem 5.30]), which has the same domain as A_h^*, because of (3). ∎

Remark 2. Observe that for $h, k \in \mathcal{H}$, if $R_k \circ A_h$ and $R_h \circ A_k$ are closable then conditions (i)–(iv) of Theorem 7 are equivalent. If $\mathcal{D}(A_h^* \circ R_k^*)$ is dense in X^* in the weak* topology on X^* then $R_k \circ A_h$ is closable. If condition (iii) of Theorem 7 is satisfied, then $R_k \circ A_h$ is closable and (i), (ii), (iv) hold.

A sufficient condition for the assumptions of Theorem 7 to hold is presented in the following

Proposition 8. *If $\mathcal{H} = [0, T]$, all the operators of the family $(A_t^*)_{t \in [0,T]}$ have the same domain $\mathcal{D}^*$ and for every $y^* \in \mathcal{D}^*$ the mapping*

$$[0, T] \ni t \mapsto A_t^* y^* \in X^* \tag{17}$$

is of class $\mathcal{C}^1$ then the family $(A_t)_{t \in [0,T]}$ satisfies the assumptions of Theorem 7.

Proof. By [4, Ch. II, Lemma 1.5], the family $(A_t^* \circ R_s^*)_{s,t \in [0,T]}$ of bounded operators is continuous with respect to (s, t). Since also $A_t^* \circ R_s^* = (R_s \circ A_t)^*$

and $\mathcal{D}((R_s \circ A_t)^*) = X^*$, the mapping $R_s \circ A_t$ is closable, and by [3, Ch. IV, Theorem 2.23], the continuity of the family $(R_s \circ A_t)^*$ with respect to (s,t) implies the continuity of $(\overline{R_s \circ A_t})$. ∎

3.1. *Some remarks on the case of differential operators.* Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $\mathcal{H}$ a connected metric space. Let

$$A_h = \sum_{|\alpha|\le m} a_\alpha(x,h)D^\alpha \quad \text{for } h \in \mathcal{H} \tag{18}$$

be a family of differential operators of order m with coefficients a_α continuous in $\overline{\Omega} \times \mathcal{H}$. Closedness of A_h and continuity of the mapping $h \mapsto A_h$ depend on the domain $\mathcal{D}(A_h)$, the space X in which $\mathcal{D}(A_h)$ is contained, and the space Y of values of A_h.

- If $\mathcal{D}(A_h) = X = H^m(\Omega)$ and $Y = \mathcal{L}^2(\Omega)$ then A_h is bounded and the mapping $\mathcal{H} \ni h \mapsto A_h \in \mathcal{B}(X,Y)$ is continuous.
- Let $X = Y = \mathcal{L}^2(\Omega)$ and let D be a closed subspace of $H^m(\Omega)$ such that D is dense in $\mathcal{L}^2(\Omega)$, and the mapping $A_h : D \to \mathcal{L}^2(\Omega)$ is one-to-one and onto for $h \in \mathcal{H}$. Then $R_h = A_h^{-1} \in \mathcal{B}(Y,X)$ and the mapping $\mathcal{H} \ni h \mapsto A_h \in C(X,Y)$ is continuous. This situation often occurs when considering strongly elliptic operators A_h with boundary operators independent of h,

$$B_j = \sum_{|\alpha|\le m_j} b_{j\alpha}(x)D^\alpha, \qquad 1 \le j \le m/2, \tag{19}$$

which cover A_h for each $h \in \mathcal{H}$. If additionally we know that $\mathcal{D}(A_h^*) = \mathcal{D}^*$ is independent of h then $R_k \circ A_h$ is closable for each h,k.

Now we show an example of a family $(\widetilde{A}_t)_{t\in\mathcal{T}}$ of elliptic operators with pairwise different domains for which the corresponding family $(\mathcal{D}_t)_{t\in\mathcal{T}}$ of domains is of class ${\mathcal{C}_a}^k$ and the family of domains of the conjugate operators is independent of t.

Keep the notation of Section 2.1 and assume that $a(x,t) = t$. The sets $\mathcal{D}_t$ given by (3) are dense linear subspaces of $\mathcal{L}^2(\Omega)$ such that $\mathcal{D}_t \ne \mathcal{D}_\tau$ for $t \ne \tau \in [0,T]$ and $\mathcal{D}_0 = \mathcal{D}$, where $\mathcal{D}$ is given by (4). The operator

$$A = -\Delta + \lambda I \tag{20}$$

is well defined on $H^2(\Omega)$; when considered as defined only on $\mathcal{D}_t$, it is closed, and for λ large enough, it is onto and one-to-one. By the closed graph theorem its inverse is bounded. Let A_t denote the operator given by (20) with domain $\mathcal{D}_t$.

Example 2. The family

$$\widetilde{A}_t = A_0 \circ \Phi_t : \mathcal{D}_t \to \mathcal{L}^2(\Omega)$$

parametrized by $t \in [0,T]$ is a continuous (with respect to t) family of closed densely defined linear differential operators with pairwise different domains. Indeed, since $(B_t = A_t \circ \Psi_t)_{t\in[0,T]}$ is a family of closed differential operators of order two with coefficients continuous with respect to both x and t, and with domains independent of t, the mapping $[0,T] \ni t \mapsto B_t \in C(\mathcal{L}^2(\Omega))$ is continuous and, by Lemma 4, the mapping $[0,T] \ni t \mapsto A_t = B_t \circ \Phi_t \in C(\mathcal{L}^2(\Omega))$ is also continuous.

By Lemma 6, the domain $\mathcal{D}(\widetilde{A}_t^*) = \mathcal{D}(A^*)$ is the same for all $t \in [0,T]$.

The next example show that in Theorem 7 the assumption of continuity of the mapping (16) cannot be replaced by the continuity of the family $(A_h)_{h\in\mathcal{H}}$.

Example 3. Let $(A_t)_{t\in[0,T]}$ be a family of self-adjoint operators with pairwise different domains, and with the same property for the family A_t^*. Since $\mathcal{C}_0^\infty(\Omega) \subset \bigcap_{t\in[0,T]} \mathcal{D}_t$, $\mathcal{C}_0^\infty$ is dense in $\mathcal{L}^2(\Omega)$, $(R_\tau \circ A_t)u = (R_\tau \circ A_\tau)u = u$ for $u \in \mathcal{C}_0^\infty(\Omega)$ and $(R_\tau \circ A_t)u \neq u$ for $u \in D_t \setminus D_\tau$, it follows that the operator $R_\tau \circ A_t$ is not closable for $t \neq \tau$. Thus, the mapping (16) is even not well defined.

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