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On approximation by special analytic polyhedral pairs

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Dedicated to Professor Józef Siciak
on the occasion of his seventieth birthday

Abstract. For bounded logarithmically convex Reinhardt pairs "compact set – domain" (K, D) we solve positively the problem on simultaneous approximation of such a pair by a pair of special analytic polyhedra, generated by the same polynomial mapping $f : D \to \mathbb{C}^n$, $n = \dim \Omega$. This problem is closely connected with the problem of approximation of the pluripotential $\omega(D, K; z)$ by pluripotentials with a finite set of isolated logarithmic singularities ([23, 24]). The latter problem has been solved recently for arbitrary pluriregular pairs "compact set – domain" (K, D) by Poletsky [12] and S. Nivoche [10, 11], while the first one is still open in the general case.

1. Introduction. The problem of approximation of the Green pluripotential $\omega(D, K; z)$ by pluripotentials with a finite set of isolated logarithmic singularities ([23, 24]), which is of great importance in Complex Potential Theory, has been solved recently (E. Poletsky [12], S. Nivoche [10, 11]) for general pluriregular pairs (K, D); in the particular case of Reinhardt pairs it was proved independently in [25] in a different way (see Corollary 4 below). The above problem is closely connected with the problem of simultaneous approximation of a pluriregular pair "open set – compact set" $D \supset K$ by a pair of special analytic polyhedra, generated by the same mapping $f : D \to \mathbb{C}^n$ (see Problem 2 below). In the one-dimensional case both problems were studied long ago: moreover, approximation of the Green potential by partial sums of its integral representation (as well as approximation of a pair (D, K) by a pair of lemniscates) is a powerful tool in analytic function theory (see, e.g., [20, 13, 19]).

In the present paper, which is a revised version of the preprint [25] updated in connection with the above mentioned results of Poletsky and Nivoche, we give a positive answer to the second problem for Reinhardt

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pairs in $\mathbb{C}^n$. This result is based on simultaneous reduction of the frames of analytic polyhedral pairs under certain quite special conditions, which hold for similar n-circular polyhedral pairs (Lemma 2). The construction used for that reduction is of independent interest: for example, it was applied, after some generalization, in [2] to strengthen the classical Lelong–Bremermann Lemma (see, e.g., [5, Q]) by proving that the number N of analytic fuctions involved is bounded by $2n+1$.

It is quite obvious that a positive answer to the second problem yields automatically a positive solution of the first one, while the converse conclusion, as far as I know, is still open in the general case.

We also consider some applications in approximation theory and discuss the connection of the above problems with an extremal problem in Complex Analysis (cf. [24, item 3.2.5]).

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2. Preliminaries. Let D be a bounded pseudoconvex domain on a Stein manifold Ω and K be a compact subset of D. The *Green pluripotential* $\omega(D,K;z)$ *of* K *with respect to* D was introduced by J. Siciak [16] (see also [17, 18, 22, 15]):

$$(1)\qquad \omega(z)=\omega(D,K;z)=\overline{\lim_{\zeta\to z}}\sup\{u(\zeta): u\in P(D),\, u\le 1,\, u|_K\le 0\},$$

where $P(D)$ stands for the set of all plurisubharmonic functions in D.

The pair (K,D) is said to be *pluriregular* if $\omega(z)\le 0$ on K and $\omega(z_k)\to 1$ for any sequence $\{z_k\}$ having no limit points in D; everywhere in this paper, when using the term "pluriregular pair" we will assume that the following two additional natural conditions are fulfilled: (a) $A(D)$ is dense in $A(K)$ (this is some sort of Runge condition); (b) D has no components disjoint from K (which is a dual Runge condition, impying that $A(K)^*$ is densely embedded into $A(D)^*$).

The *complex Monge–Ampère operator* $(dd^c)^n$ (see [3]) is defined for any bounded plurisubharmonic function u in D so that $(dd^cu)^n$ is a non-negative Borel measure on D (here $d=\partial+\bar\partial,\ d^c=i(\bar\partial-\partial)$). In particular, for a pluriregular pair we get a CPT analogue of the equilibrium measure $\mu_0(K,\Omega):=(dd^c\omega)^n$, supported by K (see [3]).

The *pluricapacity* $\tau(K,\Omega)$ of K with respect to Ω (or of the condenser (K,Ω)) is the number

$$(2)\qquad \tau(K,\Omega)=(2\pi)^{-n}\int_K (dd^c\omega(\Omega,K;z))^n,$$

which differs from the Bedford–Taylor pluricapacity ([3]) only by a constant factor.

The *multipolar Green pluripotential* for an open set $D \subset \Omega$ with a given sequence $\Lambda = (\lambda_1, \ldots, \lambda_m) \in D^m$ of logarithmic poles and distribution of measures $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^n_+$ is the following extremal plurisubharmonic function:

$$g_D(\Lambda, \alpha; z) := \limsup_{\zeta \to z} \sup \{u(\zeta) : u \in G(D, \Lambda, \alpha)\}, \tag{3}$$

where $G(D, \Lambda, \alpha)$ is the set of all functions $u \in P(D)$ such that $u \le 0$ in D and $u(\zeta) \le \alpha \ln |\zeta - \lambda_j| + \text{const}$ in a neighborhood of any point λ_j, $j = 1, \ldots, m$.

We will use the notation $D_s \Uparrow D$ to mean that D_s and D are open sets, D_s is relatively compact in D_{s+1}, $s \in \mathbb{N}$, and $D = \bigcup_{s=1}^{\infty} D_s$; $K_s \Downarrow K$ means that K_s and K are compact sets, $K_{s+1} \subset \operatorname{int} K_s$, and $K = \bigcap_{s=1}^{\infty} K_s$.

3. Two problems. We discuss two important problems of Complex Potential Theory.

Problem 1 ([23, 24]). Let (K, D) be a pluriregular pair. Do there exist $\Lambda^{(s)} \in K^{m_s}$ and $\alpha^{(s)} \in \mathbb{R}^{m_s}_+$, $m_s \in \mathbb{N}$, $s \in \mathbb{N}$, such that

$$g_D(\Lambda^{(s)}, \alpha^{(s)}; z) \to \omega(D, K; z) - 1 \tag{4}$$

uniformly on any compact subset of $\overline{D} \setminus K$?

For the one-dimensonal case this problem has an immediate positive answer: it is sufficient to take an appropriate sequence of integral sums of the integral representing the Green potential. The lack of such representation in the multidimensional case was for a long time a serious obstacle to attacking Problem 1, which has been solved only recently in [12, 10, 11] (the particular case of Reinhardt pairs was done independently, in a different way, in [25], see also Corollary 4 below).

Notice that the solution of this problem itself may be considered as an efficient substitution for an integral representation, especially in connection with many applications.

Let D be an open set on a Stein manifold Ω, $\dim \Omega = n$, and $f : D \to \mathbb{C}^m$ be an analytic mapping, $m \ge n$. We say that (L, G) is a *similar analytic polyhedral pair in D* (represented by the mapping f) if there exists an open subset U in D and multiradii $\mathbf{r}^{(\nu)} = (r_j^{(\nu)}) \in \mathbb{R}^m_+$, $\nu = 0, 1$, $r_j^{(0)} < r_j^{(1)}$, $j = 1, \ldots, m$, such that

$$\begin{aligned} G &= \{z \in U : |f_j(z)| < r_j^{(1)}, \ j = 1, \ldots, m\}, \\ L &= \{z \in G : |f_j(z)| \le r_j^{(0)}, \ j = 1, \ldots, m\}, \end{aligned} \tag{5}$$

and G is relatively compact in U. The quadruple $[U, f; \mathbf{r}^{(0)}, \mathbf{r}^{(1)}]$ is called the *frame* of the pair (L, G); to stress that the polyhedral pair is generated by m analytic functions we will speak about m-polyhedral pairs or m-frames. If $m = n$ we say that the polyhedral pair (L, G) is *special*; a special polyhedral pair is a natural multidimensional analogue of a lemniscate pair in $\mathbb{C}$, generated by a single analytic function.

The following statement is known as the Lelong–Bremermann Lemma ([9, 4], see also [5, Q]):

PROPOSITION 1. *Let u be a continuous plurisubharmonic function on a pseudoconvex domain D. Then for each compact subset A of D and any $\varepsilon > 0$ there exists an analytic mapping $f = (f_j) : D \to \mathbb{C}^N$ and numbers $\alpha_j > 0$ such that*

$$|\omega(z) - \max\{\alpha_j \ln |f_j(z)| : j = 1, \dots, N\}| < \varepsilon, \qquad z \in A. \tag{6}$$

A drawback of this result is that, in general, the number $N = N(\varepsilon, A)$ in (6) may increase without bound.

It is proved in [2], using the reduction suggested in the next section, that the constant N has a bound $\leq 2n+1$, $n = \dim D$. Since the plurisubharmonic function $\omega(z) = \omega(D, K; z)$ is continuous in D for any pluriregular pair (K, D) (see [22]), one can derive from this that for any pluriregular pair (K, D) there exists a sequence of similar analytic polyhedral pairs (L_s, G_s) generated by analytic mappings $f^{(s)} : D \to \mathbb{C}^{2n+1}$ such that $G_s \Uparrow D$ and $L_s \Downarrow K$. But it is well known that in the one-dimensional case any regular pair (K, L) can be approximated by pairs of analytic polyhedra (lemniscates) generated by single analytic functions $f_s : D_s \to \mathbb{C}$, with $D_s \uparrow D$ (see, e.g., [20, 8.7]; this is a natural development of Hilbert's result [6] about approximating a simple Jordan curve by polynomial lemniscates). So, for the multidimensional case the following problem arises naturally:

PROBLEM 2. Is it possible to approximate any pluriregular pair (K, D) simultaneously by *special* similar analytic polyhedral pairs?

This problem can be reformulated in the following equivalent form.

PROBLEM 2a. For any pluriregular pair (K, D), $\dim D = n$, find a sequence of analytic mappings $f^{(s)} = (f_j^{(s)}) : D \to \mathbb{C}^n$ and vectors $\alpha^{(s)} = (\alpha_j^{(s)}) \in \mathbb{R}_+^n$ such that the sequence

$$u^{(s)}(z) := \max\{\alpha_j^{(s)} \ln |f_j^{(s)}| : j = 1, \dots, n\} \tag{7}$$

converges to $\omega(D, K; z) - 1$ uniformly on any compact subset of $D \setminus K$.

In the present paper we solve this problem positively for Reinhardt pairs (K, D).

Notice that an approach to general pluriregular pairs (K,D) is given in [10, 11] (see also [12]). Namely, it is proved there that such a pair can be approximated by pairs (5) if in the expression for L the set G is replaced by some open set $V\supset K$. So, in that context, instead of the uniform convergence on compact subsets of $D\setminus K$ for the sequence (7), a weaker condition is proved: $\int_D (dd^c(u^{(s)}(z)-\omega(z)))^n\to 0$ as $s\to\infty$.

4. Reduction of analytic polyhedral pairs. Any similar analytic m-polyhedral pair with a frame $[D,f;\mathbf{r}^{(0)},\mathbf{r}^{(1)}]$ can be represented by its *normalized frame*:

$$[D,g;\mathbf{r}]:=[D,g;\mathbf{1},\mathbf{r}], \tag{8}$$

where $g=(f_j/r_j^{(0)})$, $\mathbf{1}:=(1,\dots,1)$, $\mathbf{r}:=(r_j^{(1)}/r_j^{(0)})$.

We say that an analytic polyhedral pair with the frame (8) is *equilateral* if

$$\mathbf{r}=r\cdot\mathbf{1},\qquad r\in\mathbb{R}_+. \tag{9}$$

Let (L,G) be an equilateral similar analytic polyhedral pair with the m-frame (8), (9). For any $J=\{j_1,\dots,j_k\}$ with $j_1<\dots<j_k$, $1\le k\le m$, we define

$$\sigma(J):=\{z\in\overline{G\setminus L}:|g_{j_1}(z)|=\dots=|g_{j_k}(z)|>|g_j(z)|,\ j\notin J\}$$

Suppose that there exists l with $n\le l<m$ such that

$$\sigma(J)=\emptyset \tag{10}$$

if $\#J>l$, and l is the smallest number satisfying this condition. Then we consider the sequence of analytic mappings

$$\begin{gathered} g^{(s)}=(g_1^{(s)},\dots,g_l^{(s)}):D\to\mathbb{C}^l,\\ g_k^{(s)}(z)=\Big(\sum_{j_1<\dots<j_k}(g_{j_1}(z))^s\dots(g_{j_k}(z))^s\Big)^{l!/k}, \end{gathered} \tag{11}$$

where $k=1,\dots,l$.

LEMMA 2. *Let (L,G) be an equilateral polyhedral pair with the frame* (8) *such that the condition* (10) *holds. Let*

$$\varphi(z):=\frac{1}{\ln r}\max\{\ln|g_j(z)|:j=1,\dots,m\}, \tag{12}$$

$$\varphi_s(z):=\frac{1}{sl!\ln r}\max\{\ln|g_k^{(s)}(z)|:k=1,\dots,l\}, \tag{13}$$

where $g_k^{(s)}(z)$ are defined in (11). *Then $\varphi_s(z)\to\varphi(z)$ uniformly on $\overline{G\setminus L}$.*

Proof. Since, by the construction,

$$|g_k^{(s)}(z)|\le 2^{ml!/k}(\max\{|g_j(z)|:j=1,\dots,m\})^{sl!},$$

we get an estimate from above:

$$\varphi_s(z) \le \varphi(z) + \frac{m \ln 2}{sk \ln r} \tag{14}$$

for each $z \in \overline{G \setminus L}$.

Now we deal with the estimate from below for the function (13). First, by the continuity of φ on $\overline{G \setminus L}$, for each $\varepsilon > 0$ we can find $\sigma(\varepsilon) > 0$ such that

$$|\varphi(z) - \varphi(\zeta)| < \varepsilon \quad \text{if } |\zeta - z| < \sigma(\varepsilon),\ z, \zeta \in \overline{G \setminus L}. \tag{15}$$

Now we fix any $\zeta \in \overline{G \setminus L}$. By the hypothesis, there is $J = J(\zeta) = \{j_1, \ldots, j_k\}$, $k = k(\zeta) \le l$, such that

$$|g_{j_1}(\zeta)| = \ldots = |g_{j_k}(\zeta)| > |g_i(\zeta)|, \quad i \notin J, \tag{16}$$

hence

$$d(\zeta) := \max\left\{\left|\frac{g_{i_1}(\zeta)\ldots g_{i_k}(\zeta)}{g_{j_1}(\zeta)\ldots g_{j_k}(\zeta)}\right| : I = \{i_1, \ldots, i_k\} \ne J\right\} < 1. \tag{17}$$

Thus, using the continuity of all the functions involved, we can find a neighborhood $U(\zeta) = \{z \in \overline{G \setminus L} : |z - \zeta| < \varepsilon\}$ with $\delta(\zeta) < \sigma(\varepsilon)$ such that

$$\left|\frac{g_{i_1}(z)\ldots g_{i_k}(z)}{g_{j_1}(z)\ldots g_{j_k}(z)}\right| \le q(\zeta) := \frac{1 + d(\zeta)}{2} < 1, \qquad I = \{i_1, \ldots, i_k\} \ne J, \tag{18}$$

and

$$|g_j(z)| \ge r^{-\varepsilon} |g_j(\zeta)|, \qquad j \in J, \tag{19}$$

for all $z \in U(\zeta)$. Applying (11), (18), and (19) we obtain the estimate

$$\begin{aligned} |g_k^{(s)}(z)| &\ge |g_{j_1}(z)\ldots g_{j_k}(z)|^{sl!/k}\left(1 - \sum_{I \ne J}\left|\frac{g_{i_1}(z)\ldots g_{i_k}(z)}{g_{j_1}(z)\ldots g_{j_k}(z)}\right|^s\right)^{l!/k} \\ &\ge r^{-\varepsilon s l!} |g_{j_1}(\zeta)\ldots g_{j_k}(\zeta)|^{sl!/k} (1 - 2^m q(\zeta)^s)^{l!/k} \end{aligned}$$

for every $z \in U(\zeta)$. Hence taking into account (12), (13), and (15), we deduce that

$$\begin{aligned} \varphi_s(z) &\ge \frac{\ln |g_k^{(s)}(z)|}{sl! \ln r} \ge \frac{\ln |g_{j_1}(\zeta)|}{\ln r} + \frac{\ln(1 - 2^m q(\zeta))}{k \ln r} - \varepsilon \\ &\ge \varphi(\zeta) - 2\varepsilon \ge \varphi(z) - 3\varepsilon \end{aligned}$$

for $z \in U(\zeta)$ and $s \ge s_0(\zeta, \varepsilon)$.

Now, choosing a finite covering: $\overline{G \setminus L} \subset \bigcup_{i=1}^N U(\zeta^{(i)})$, we conclude that

$$\varphi^{(s)}(z) \ge \varphi(z) - 3\varepsilon \tag{20}$$

for all $z \in \overline{G \setminus L}$ if $s \ge s_0(\varepsilon) := \max\{s_0(\zeta^{(i)}, \varepsilon) : i = 1, \ldots, N\}$.

The estimates (14) and (20) imply that $\varphi_s(z)$ converges to $\varphi(z)$ uniformly on $\overline{G \setminus L}$.

5. Pluripotentials for Reinhardt pairs. Let D be a logarithmically convex bounded complete n-circular domain in $\mathbb{C}^n$. Its *characteristic function*

$$h_D(\theta) = \sup\Big\{\sum \theta_k \ln|z_k| : z \in D\Big\}, \quad \theta \in \overline{\mathbb{R}^n_+},$$

is convex and homogeneous. The domain D can be recovered from its characteristic function as follows:

$$D = \Big\{z \in \mathbb{C}^n : \sum \theta_k \ln|z_k| < h_D(\theta),\, \theta \in \Sigma\Big\},$$

where $\Sigma := \{\theta = (\theta_1, \ldots, \theta_n) \in \overline{\mathbb{R}^n_+} : \sum_{k=1}^n \theta_k = 1\}$.

Let D_0, D_1 be a pair of bounded logarithmically convex complete Reinhardt (= n-circular) domains such that $\overline{D}_0 \subset D_1$. The following formula for the pluripotential of $\overline{D}_0$ with respect to D_1 was presented in [23, Proposition 1.4.3]:

$$\omega(z) = \omega(D_1, \overline{D}_0; z) = \sup\{\gamma(\theta, z) : \theta \in \Sigma\} \tag{21}$$

for $z \in \overline{D}_1 \setminus D_0$, where

$$\gamma(\theta, z) := \frac{\sum_{\nu=1}^n \theta_\nu \ln|z_\nu| - h_{D_0}(\theta)}{h_{D_1}(\theta) - h_{D_0}(\theta)}. \tag{22}$$

The formula (21) is extended onto ∂D_1 by setting $\omega(z) \equiv 1$ there.

We consider the following level sets of the function ω:

$$D_\alpha := \{z \in D_1 : \omega(z) < \alpha\}, \quad \Gamma_\alpha := \{z \in \overline{D}_1 : \omega(z) = \alpha\} \tag{23}$$

with $\alpha \in [0, 1]$.

It is easy to see that the representation (21) leads to the following geometric description of the level sets (23).

LEMMA 3. *Let $0 < \alpha < 1$. Then Γ_α is the boundary of the domain D_α and $z = (z_\nu) \in D_1$ belongs to Γ_α if and only if there exist $\theta \in \Sigma$, $z^{(0)} = (z_\nu^{(0)}) \in \Gamma_0$, $z^{(1)} = (z_\nu^{(1)}) \in \Gamma_1$ such that $|z_\nu| = |z_\nu^{(0)}|^{1-\alpha} \cdot |z_\nu^{(1)}|^\alpha$ and $\gamma(\theta, z^{(0)}) = 0$, $\gamma(\theta, z^{(1)}) = 1$.*

Using these facts and Lemma 2 we are going to prove our main result, which gives a positive solution of Problem 2 in the case considered.

THEOREM 4. *Let D_0, D_1, and $\omega(z)$ be as above. Then there exist a sequence of polynomial mappings $f^{(\nu)} = (f_j^{(\nu)}) : \mathbb{C}^n \to \mathbb{C}^n$, a sequence $\alpha_\nu > 0$, and a sequence of open sets $G^{(\nu)} \Uparrow D_1$ such that*

$$\omega(z) = \lim_{\nu \to \infty} \alpha_\nu \max\{\ln|f_j^{(\nu)}(z)| : j = 1, \ldots, n\} \tag{24}$$

uniformly on any compact subset of $D_1 \setminus \overline{D}_0$, and a sequence of special polynomial polyhedral pairs $(M^{(\nu)}, H^{(\nu)})$ determined by the normalized equilateral *frames*

$$[G^{(\nu)}, f^{(\nu)}; \exp 1/\alpha_\nu] \tag{25}$$

approximates the pair $(\overline{D}_0, D_1)$ *so that* $H_\nu \Uparrow D_1$ *and* $M_\nu \Downarrow \overline{D}_0$.

As a corollary we get another proof of Problem 1 in the case considered (cf. [12, 10, 11]).

Corollary 5. *Let* $K = \overline{D}_0$, $D = D_1$. *Then in the setting of the previous theorem, for each* ν *the set* $\Lambda^{(\nu)} := \{\zeta \in G^{(\nu)} : f^{(\nu)}(\zeta) = 0\}$ *is finite and consists only of simple roots, and the relation* (4), *with* $\alpha^{(\nu)} := (\alpha_\nu, \dots, \alpha_\nu) \in \mathbb{R}^n$, *holds uniformly on any compact subset of* $D \setminus K$.

Before proving these statements we consider the following

Lemma 6. *Let* D_0, D_1, *and* ω *be as above. Then for each* $\varepsilon > 0$ *there exists a finite set of multi-indices* $k(j) = (k_i(j)) \in \mathbb{Z}^n_+$, *a natural number* q, *and real numbers* c_j, $j = 1, \dots, m$, *such that the maximum*

$$v(z) := \frac{1}{q} \max\Big\{\sum_{i=1}^{n} k_i(j) \ln |z_i| - c_j : j = 1, \dots, m\Big\} \tag{26}$$

is attained for no more than n *values of* j *at any point* z *satisfying the estimates* $0 \le v(z) \le 1$, *and*

$$|\omega(z) - v(z)| < \varepsilon, \qquad z \in \overline{D_1 \setminus D_0}. \tag{27}$$

Proof. First we notice that for each $\zeta = (\zeta_i) \in \overline{D_1 \setminus D_0}$ there is $\theta = \theta(\zeta)$ such that

$$\omega(\zeta) = \gamma(\theta(\zeta), \zeta). \tag{28}$$

Indeed, denoting by $\Sigma(\zeta)$ the set of all $\theta = (\theta_i) \in \Sigma$ such that $\theta_i = 0$ whenever $\zeta_i = 0$, we have

$$\omega(\zeta) = \sup\{\gamma(\theta, \zeta) : \theta \in \Sigma(\zeta)\},$$

and the function $\gamma(\theta, \zeta)$ is continuous in θ on the compact set $\Sigma(\zeta)$, so (28) is valid with some $\theta = \theta(\zeta) \in \Sigma(\zeta)$.

Now, since ω is continuous on $\overline{D}_1$, while the function $\gamma(\theta(\zeta), z)$ is continuous in some neighborhood of ζ, we can find for any $\varepsilon > 0$ some open neighborhood U_ζ of ζ such that

$$0 \le \omega(z) - \gamma(\theta(\zeta), z) < \varepsilon/2, \quad z \in U_\zeta.$$

Hence, using the covering theorem, we deduce that for each $\varepsilon > 0$ there is a finite set $\{\zeta^{(j)} : j = 1, \dots, m\}$ such that

$$|\omega(z) - u(z)| < \varepsilon/2, \quad z \in \overline{D_1 \setminus D_0}, \tag{29}$$

where

$$u(z) := \sup\Big\{\sum_{i=1}^{m} a_{i,j} \ln|z_i| - b_j : j = 1,\dots,m\Big\} \tag{30}$$

and

$$\begin{aligned} a_{i,j} &:= \frac{\theta(\zeta^{(j)})}{h_{D_1}(\theta(\zeta^{(j)})) - h_{D_0}(\theta(\zeta^{(j)}))},\\ b_j &:= \frac{h_{D_0}(\theta(\zeta^{(j)}))}{h_{D_1}(\theta(\zeta^{(j)})) - h_{D_0}(\theta(\zeta^{(j)}))}. \end{aligned} \tag{31}$$

Recall that, by the construction, the coefficients $a_{i,j}$ satisfy the condition

$$a_{i,j} = 0 \quad \text{if } \zeta_i^{(j)} = 0. \tag{32}$$

Now we are going to replace the coefficients (31) by some close values $\widetilde{a}_{i,j}$, $\widetilde{b}_j$, respectively, aiming at two targets:

(a) to afford the approximation of $\omega(z)$ by the new function

$$\widetilde{u}(z) := \sup\Big\{\sum_{i=1}^{m} \widetilde{a}_{i,j} \ln|z_i| - \widetilde{b}_j : j = 1,\dots,m\Big\} \tag{33}$$

so that

$$|\omega(z) - \widetilde{u}(z)| < \varepsilon, \quad z \in \overline{D_1 \setminus D_0}; \tag{34}$$

(b) to provide the condition of Lemma 2, namely, that for each z such that $0 \le \widetilde{u}(z) \le 1$ the maximum in (33) will be attained for no more than n values of j.

To guarantee (a) we must retain the nullity of the new coefficients where the old coefficients vanish (see (32)); indeed, if some of the new coefficients $\widetilde{a}_{i,j}$ were non-zero, while $\zeta_{i,j} = 0$, then $\sum_{i=1}^{m} \widetilde{a}_{i,j} \ln|\zeta_i^{(j)}| - \widetilde{b}_j = -\infty$ so the closeness to the function ω would be violated.

To reach both purposes we use the following quite standard algebraic considerations. Given a set N of indices (i,j), we consider the set of all matrices

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{i,1} & \dots & a_{n,1} & b_1 \\ \multicolumn{6}{c}{\dotfill} \\ a_{1,j} & \dots & a_{i,j} & \dots & a_{n,j} & b_j \\ \multicolumn{6}{c}{\dotfill} \\ a_{1,m} & \dots & a_{i,m} & \dots & a_{n,m} & b_m \end{pmatrix} \tag{35}$$

such that $a_{i,j} = 0$ when $(i,j) \in N$. We identify this set of matrices with the space $\mathbb{R}^d$, where $d = m(n+1) - \#N$ (writing, for example, the matrix terms row-by-row and dropping those which are the prescribed zeros). Each minor M of a matrix A of order r may then be considered as a homogeneous polynomial $M(A)$ of degree $\le r$ in $\mathbb{R}^d$. Denote by $\mathcal{M}$ the set of all non-trivial minors M, i.e. such that $M(A) \not\equiv 0$ on $\mathbb{R}^d$. Then the set $\mathcal{A}_0$ of all matrices

$A \in \mathbb{R}^d$ such that $M(A) \neq 0$ for all $M \in \mathcal{M}$ is an open dense set in $\mathbb{R}^d$, since it can be obtained by removing some algebraic set from $\mathbb{R}^d$. It is clear that, by the construction, each matrix $A \in \mathcal{A}_0$ satisfies the condition: for every set $J = \{j_1, \ldots, j_{n+1}\}$ with $j_1 < \ldots < j_{n+1}$ the system

$$a_{1,j}x_1 + \ldots + a_{n,j}x_n = b_j, \quad j \in J, \tag{36}$$

has no solution. Analogously, we can show that there exists an open dense subset $\mathcal{A}_1$ obtained by removing some algebraic set from $\mathbb{R}^d$ and such that each matrix $A \in \mathcal{A}_1$ satisfies the condition: each system

$$a_{1,j}x_1 + \ldots + a_{n,j}x_n = 1 + b_j, \quad j \in J, \tag{37}$$

has no solution when $\#J = n + 1$. Thus for any matrix $A \in \mathcal{A}_0 \cap \mathcal{A}_1$ each of the systems (36) and (37) has no solution if $\#J = n + 1$.

Applying the above considerations to the matrix (35) defined by (31) (with N determined by the condition (32)), we can choose the coefficients $\widetilde{a}_{i,j}$ and $\widetilde{b}_j$ to be rational and such that the condition (34) holds and the condition (b) is valid for all z such that $\widetilde{u}(z) = 1$ or $\widetilde{u}(z) = 0$. Let us show that (b) is also true for all z such that $0 < \widetilde{u}(z) < 1$. Supposing the contrary, we find that there is z such that $\widetilde{u}(z) = \alpha$, $0 < \alpha < 1$, and

$$\sum_{i=1}^{n} \widetilde{a}_{i,j} \ln |z_i| = \alpha, \quad j \in J, \tag{38}$$

for some J with $\#J > n$. Then, by Lemma 3, $|z_i| = |z_i^{(0)}|^{1-\alpha} \cdot |z_i^{(1)}|^{\alpha}$ for some $z^{(0)}$ and $z^{(1)}$ such that $\widetilde{u}(z^{(0)}) = 0$ and $\widetilde{u}(z^{(1)}) = 1$. Hence, by what is proved above, there is $j_0 \in J$ such that $\sum_{i=1}^{n} \widetilde{a}_{i,j_0} \ln |z_i^{(1)}| < 1$, while $\sum_{i=1}^{n} \widetilde{a}_{i,j_0} \ln |z_i^{(0)}| \leq 0$. The last two estimates contradict (38) if $j = j_0$. Thus, the condition (b) is proved for all z such that $0 \leq \widetilde{u}(z) \leq 1$.

Since the numbers $\widetilde{a}_{i,j}$ are rational, there exist natural numbers $k_i(j)$ and q such that $\widetilde{a}_{i,j} = k_i(j)/q$, $j = 1, \ldots, m$. It is easy to check that the numbers $k_i(j)$, q, $c_j := q\widetilde{b}_j$, and the function $v(z) := \widetilde{u}(z)$ satisfy all the conditions of the lemma. Thus the proof is complete.

Proof of Theorem 4. We shall use the notation

$$D_\alpha := \{z \in D_1 : \omega(z) < \alpha\}, \quad K_\alpha := \{z \in D_1 : \omega(z) \leq \alpha\} \tag{39}$$

with $0 < \alpha < 1$. Take a sequence (ε_ν) such that

$$5\varepsilon_\nu < \varepsilon_{\nu-1}, \quad \nu = 2, 3, \ldots, \quad \varepsilon_1 < 1/2. \tag{40}$$

Now, by Lemma 6, for each ν we can find $k(j) = k(j,\nu) = (k_i(j,\nu)) \in \mathbb{Z}_+^n$, $q = q_\nu \in \mathbb{N}$ and real numbers $c_j = c(j,\nu)$, $j = 1, \ldots, m = m_\nu$, such that the estimate (27) holds for the function $v(z) = v_\nu(z)$ with $\varepsilon = \varepsilon_{\nu+1}$ and the maximum in (26) is attained for no more than n values of j.

Now we can see that all the conditions of Lemma 2 are fulfilled with $l = n$ for the polynomial polyhedral pair $(L, G) = (L^{(\nu)}, G^{(\nu)})$ determined by the normalized equilateral frame $[D_1, g; r^{(\nu)}]$ with $g = (g_j) = (g_{j,\nu}) : \mathbb{C}^n \to \mathbb{C}^{m_\nu}$, where

$$g_{j,\nu}(z) := \frac{z^{k(j,\nu)}}{\exp\left(2\varepsilon_\nu q_\nu - c(j,\nu)\right)}, \qquad r^{(\nu)} := \exp\left(1 - 4\varepsilon_\nu\right) q_\nu. \tag{41}$$

This pair can be expressed in the form

$$G^{(\nu)} = \{z \in D_1 : v_\nu(z) < 1 - 2\varepsilon_\nu\}, \quad L^{(\nu)} = \{z \in D_1 : v_\nu(z) \leq 2\varepsilon_\nu\}.$$

It is easy to check that the embeddings

$$D_{1-3\varepsilon_\nu} \Subset G^{(\nu)} \Subset D_{1-\varepsilon_\nu}, \qquad K_{\varepsilon_\nu} \Subset L^{(\nu)} \Subset K_{3\varepsilon_\nu} \tag{42}$$

hold for all $\nu \in \mathbb{N}$. Thus, applying Lemma 2 to the above polynomial polyhedral pair, for any $\nu \in \mathbb{N}$ we get, by the construction (11), polynomial mappings $(g_{i,\nu}^{(s)})_{i=1}^n : \mathbb{C}^n \to \mathbb{C}^n$ such that the sequence

$$\varphi_\nu^{(s)}(z) := \frac{1}{sn! \ln r^{(\nu)}} \max\{\ln|g_{i,\nu}^{(s)}(z)| : i = 1, \dots, n\}$$

converges uniformly on $\overline{G^{(\nu)} \setminus L^{(\nu)}}$ to the function

$$\varphi_\nu(z) := \frac{v_\nu(z) - 2\varepsilon_\nu}{1 - 2\varepsilon_\nu}$$

as $s \to \infty$. Choose $s = s_\nu$ so large that

$$|\varphi_\nu(z) - \varphi_\nu^{(s_\nu)}(z)| < \varepsilon_{\nu+1}, \quad z \in \overline{G^{(\nu)} \setminus L^{(\nu)}}.$$

Hence for the function $u_\nu(z) := \varphi_\nu^{(s_\nu)}(z)(1 - 2\varepsilon) + 2\varepsilon$ we get

$$|u_\nu(z) - \omega(z)| < 2\varepsilon_{\nu+1}, \quad z \in \overline{G^{(\nu)} \setminus L^{(\nu)}}. \tag{43}$$

Then the sequence of polyhedral pairs $(M^{(\nu)}, H^{(\nu)})$ defined by

$$\begin{aligned} M^{(\nu)} &:= \{z \in G^{(\nu)} : u_\nu(z) \leq 4\varepsilon_\nu\}, \\ H^{(\nu)} &:= \{z \in G^{(\nu)} : u_\nu(z) < 1 - 4\varepsilon_\nu\} \end{aligned} \tag{44}$$

is as desired. First, it is easy to check that, due to (43) and (40), the embeddings

$$\begin{aligned} &G^{(\nu-1)} \Subset D_{1-\varepsilon_{\nu-1}} \Subset H^{(\nu)} \Subset D_{1-3\varepsilon_\nu} \Subset G^{(\nu)}, \\ &L^{(\nu)} \Subset K_{3\varepsilon_\nu} \Subset M^{(\nu)} \Subset K_{\varepsilon_{\nu-1}} \Subset L^{(\nu-1)} \end{aligned} \tag{45}$$

hold for $\nu = 2, 3, \dots$ So, $H^{(\nu)} \Uparrow D_1$ and $M^{(\nu)} \Downarrow \overline{D}_0$. Second, these pairs are determined by the normalized equilateral frames (25) with the polynomial mapping $f^{(\nu)} = (f_i^{(\nu)}) : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$f_i^{(\nu)}(z) := \frac{g_{i,\nu}^{(s_\nu)}(z)}{\exp\left(\dfrac{2\varepsilon_\nu(1-4\varepsilon_\nu)q_\nu s_\nu n!}{1-\varepsilon_\nu}\right)}, \qquad \alpha_\nu := \frac{1-2\varepsilon_\nu}{(1-4\varepsilon_\nu)(1-8\varepsilon_\nu)q_\nu s_\nu n!}.$$

Finally, by the construction, the formula (24) holds uniformly on any compact subset of $D_1 \setminus \overline{D}_0$.

Proof of Corollary 5. First, we can assume that the set $\Lambda^{(\nu)} := \{z \in G^{(\nu)} : f^{(\nu)}(z) = 0\}$ consists only of simple roots (otherwise we can change the polynomial map $f^{(\nu)}$ a little to provide this, preserving all other properties in Theorem 4). Setting $\alpha^{(\nu)} := (\alpha_\nu, \ldots, \alpha_\nu)$, we get

$$\omega(H^{(\nu)}, M^{(\nu)}; z) - 1 = g_{H^{(\nu)}}(\Lambda^\nu, \alpha^{(\nu)}; z), \qquad z \in H^{(\nu)} \setminus M^{(\nu)}.$$

Then, since

$$g_{D_1}(\Lambda^{(\nu)}, \alpha^{(\nu)}; z) \le g_{H^{(\nu)}}(\Lambda^{(\nu)}, \alpha^{(\nu)}; z) \le g_{D_1}(\Lambda^{(\nu)}, \alpha^{(\nu)}; z) + \varepsilon_{\nu-1}$$

everywhere in $H^{(\nu)}$ and $\omega(H^{(\nu)}, M^{(\nu)}; z)$ converges to $\omega(D, K; z)$ uniformly on compact subsets of $D \setminus K$, we get the relation (4) uniformly on any compact subset of $D \setminus K$.

6. Application to width asymptotics. The *Kolmogorov widths* of a compact set A in a Banach space X are the numbers

$$d_s(A) = d_s(A, X) := \inf_L \sup_{x \in A} \inf \{\|x - y\|_X : y \in L\}, \qquad s \in \mathbb{Z}_+,$$

where L runs through the set of all s-dimensional subspaces of X.

Let K be a compact subset in an open set $D \subset \mathbb{C}^n$ and A_K^D the subset of the Banach space $C(K)$ consisting of all analytic functions in D whose moduli do not exceed 1 there.

In [23] we conjectured the strong asymptotics

$$\ln d_s(A_K^D) \sim -\sigma s^{1/n} \tag{46}$$

with the constant

$$\sigma = \left(\frac{n!}{\tau(K, D)}\right)^{1/n} \tag{47}$$

(see (2)). For the one-dimensional case, this conjecture is equivalent to Kolmogorov's conjecture about the asymptotics of the ε-entropy of the compact set A_K^D; see, e.g., [24] for the history of the problem in that case. The conjecture was proved in [1] for Reinhardt pairs (K, D), using the Rauch–Taylor result about the computation of the real Monge–Ampère operator from convex functions.

Now we obtain this fact as a simple consequence of Corollary 5 and the results from [24, items 3.1.2, 3.2.5]. Namely it was shown in [24, Proposition

3.1.4] that under the conditions which are obviously fulfilled in our case, the asymptotics (46) with the constant (47) holds whenever Problem 1 is answered positively for a pair (K, D). So, by Corollary 5 we get

THEOREM 7. *Let* $K = \overline{D}_0$, $D = D_1$, $K \subset D$, D_ν *be bounded logarithmically convex domains*, $\nu = 1, 2$. *Then the asymptotics* (46) *holds with the constant* (47).

Notice that recently ([12, 10, 11]) a more general result was obtained in a similar way from the complete solution of Problem 1 considered there.

7. Connection with some extremal problem. For any pluriregular pair (K, D) and $m \in \mathbb{N}$ we consider the characteristic

$$\tau_m^+(K, D) := \min\Big\{\sum_{\mu=1}^{m} (\alpha_\mu)^n\Big\}, \tag{48}$$

where the minimum is taken over all $\alpha = (\alpha_\mu) \in \mathbb{R}_+^m$ for which there is $\Lambda = (\zeta_1, \ldots, \zeta_m) \in K^m$ such that $g_D(\Lambda, \alpha; z) \leq -1$, $z \in K$. It is easy to show that for every $m \in \mathbb{N}$ there exist $\Lambda = \overline{\Lambda}^{(m)}$ and $\alpha = \overline{\alpha}^{(m)}$ for which the minimum in (48) is attained.

It is obvious that the sequence (48) is non-decreasing, so the limit

$$\lim_{m\to\infty} \tau_m(K, D) =: \tau^+(K, D) \tag{49}$$

exists. The last characteristic was introduced in [23, 24] and used there to estimate the Kolmogorov widths $d_s(A_K^D)$ (see the previous section); it was also shown there that a positive solution of Problem 1 for a given pluriregular pair (K, D) leads to the equality

$$\tau^+(K, D) = \tau(K, D). \tag{50}$$

It follows from [12, Lemma 4.2] that the extremal sequence of pluripotentials $g_D(\overline{\Lambda}^{(m)}, \overline{\alpha}^{(m)}; z)$ converges to $\omega(D, K; z) - 1$ uniformly on any compact subset of $D \setminus K$. Therefore Problem 2 is equivalent to the following

PROBLEM 3. Is the relation (50) true for any pluriregular pair (K, D)?

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