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On the variational calculus in fibered-fibered manifolds

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Abstract. In this paper we extend the variational calculus to fibered-fibered manifolds. Fibered-fibered manifolds are surjective fibered submersions $\pi : Y \to X$ between fibered manifolds. For natural numbers $s \geq r \leq q$ with $r \geq 1$ we define (r, s, q)th order Lagrangians on fibered-fibered manifolds $\pi : Y \to X$ as base-preserving morphisms $\lambda : J^{r,s,q}Y \to \bigwedge^{\dim X} T^*X$. Then similarly to the fibered manifold case we define critical fibered sections of Y. Setting $p = \max(q, s)$ we prove that there exists a canonical "Euler" morphism $\mathcal{E}(\lambda) : J^{r+s,2s,r+p}Y \to \mathcal{V}^*Y \otimes \bigwedge^{\dim X} T^*X$ of λ satisfying a decomposition property similar to the one in the fibered manifold case, and we deduce that critical fibered sections σ are exactly the solutions of the "Euler–Lagrange" equations $\mathcal{E}(\lambda) \circ j^{r+s,2s,r+p}\sigma = 0$. Next we study the naturality of the "Euler" morphism. We prove that any natural operator of the "Euler" morphism type is $c\mathcal{E}(\lambda)$, $c \in \mathbb{R}$, provided $\dim X \geq 2$.

0. Introduction. The most important problem in the variational calculus is to characterize critical values. It is known that critical sections of a fibered manifold $p : X \to X_0$ with respect to an rth order Lagrangian $\lambda : J^rX \to \bigwedge^{\dim X_0} T^*X_0$ can be characterized by means of the solutions of the so-called Euler–Lagrange equations. There exists a unique Euler map $E(\lambda) : J^{2r}X \to V^*X \otimes \bigwedge^{\dim X_0} T^*X_0$ satisfying some decomposition formula. Then the Euler–Lagrange equations are $E(\lambda) \circ j^{2r}\sigma = 0$ with unknown section σ (see [2]).

Fibered-fibered manifolds generalize fibered manifolds. They are surjective fibered submersions $\pi : Y \to X$ between fibered manifolds. They appear naturally in differential geometry if we consider transverse natural bundles over foliated manifolds in the sense of R. Wolak [4] (see [3]).

A simple example of a fibered-fibered manifold is the following trivial fibered-fibered manifold. We consider four manifolds X_1, X_2, X_3, X_4. Then the obvious projection $\pi : X_1 \times X_2 \times X_3 \times X_4 \to X_1 \times X_2$ is a fibered-fibered manifold (we consider $X_1 \times X_2 \times X_3 \times X_4$ as the trivial fibered manifold

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over $X_1 \times X_3$ and $X_1 \times X_2$ as the trivial fibered manifold over X_1). Taking X_1, X_2, X_3, X_4 compact we produce compact fibered-fibered manifolds.

In [3], for fibered-fibered manifolds, using the concept of (r, s, q)-jets on fibered manifolds [2], we extended the notion of r-jet prolongation bundle to the (r, s, q)-jet prolongation bundle $J^{r,s,q}Y$ for $r, s, q \in \mathbb{N} \setminus \{0\}$, $s \geq r \leq q$.

The purpose of the present paper is to construct the variational calculus in fibered-fibered manifolds.

In Section 2 we define (r, s, q)th order Lagrangians as base-preserving morphisms $\lambda : J^{r,s,q}Y \to \bigwedge^{\dim X} T^*X$. Then similarly to the fibered manifold case we define critical fibered sections of Y. Setting $p = \max(q, s)$ we prove that there exists a canonical "Euler" morphism $\mathcal{E}(\lambda) : J^{2p,2p,2p}Y \to \mathcal{V}^*Y \otimes \bigwedge^{\dim X} T^*X$ of λ satisfying a decomposition property similar to the one in the fibered manifold case, where $\mathcal{V}Y \subset TY$ is the vector subbundle of vectors vertical with respect to two obvious projections from Y (onto X and onto Y_0). Then we deduce that critical fibered sections σ are exactly the solutions of the "Euler–Lagrange" equations $\mathcal{E}(\lambda) \circ j^{2p,2p,2p}\sigma = 0$. Next we observe that $\mathcal{E}(\lambda)$ can be factorized through $J^{r+s,2s,r+p}Y$ and the "Euler–Lagrange" equations are in fact of the form $\mathcal{E}(\lambda) \circ j^{r+s,2s,r+p}\sigma = 0$.

Section 1 provides some background on the variational calculus in fibered manifolds.

In [1], I. Kolář studied the naturality of the Euler operator $E(\lambda)$ on fibered manifolds. He showed that any natural operator of the Euler operator type is of the form $cE(\lambda)$, $c \in \mathbb{R}$, provided $\dim X_0 \geq 2$.

In Section 3 we study the naturality of the "Euler" operator $\mathcal{E}(\lambda)$ on fibered-fibered manifolds. We prove that any natural operator of the "Euler" operator type is of the form $c\mathcal{E}(\lambda)$, $c \in \mathbb{R}$, provided $\dim X \geq 2$.

A 2-fibered manifold is a sequence of two surjective submersions $X \to X_1 \to X_0$. For example, given a fibered manifold $X \to M$ we have 2-fibered manifolds $TX \to X \to M$, $T^*X \to X \to M$, $J^rX \to X \to M$, etc. Every 2-fibered manifold $X \to X_1 \to X_0$ can be considered as the fibered-fibered manifold $X \to X_1$, where we consider X as the fibered manifold $X \to X_0$ and X_1 as the fibered manifold $X_1 \to X_0$. So, the results of the paper can be obviously applied to produce the variational calculus on 2-fibered manifolds.

A fibered manifold $X \to X_0$ can be considered as the 2-fibered manifold $X \to X_0 \to \mathrm{pt}$ with the one-point manifold pt. So, we recover the known variational calculus on fibered manifolds.

All manifolds and maps are assumed to be of class $\mathcal{C}^\infty$.

1. Background: variational calculus in fibered manifolds

1.1. A *fibered manifold* is a surjective submersion $p : X \to X_0$ between manifolds. If $p' : X' \to X'_0$ is another fibered manifold then a map $f : X \to X'$

is called *fibered* if there exist a (unique) map $f_0 : X_0 \to X'_0$ such that $p' \circ f = f_0 \circ p$.

Denote the set of (local) sections of p by $\mathit{\Gamma} X$. The *r-jet prolongation*

$$J^r X = \{ j^r_{x_0}\sigma \mid \sigma \in \mathit{\Gamma} X,\, x_0 \in X_0 \}$$

of X is a fibered manifold over X_0 with respect to the source projection $p^r : J^r X \to X_0$. If $p' : X' \to X'_0$ is another fibered manifold and $f : X_0 \to X'$ is a fibered map covering a local diffeomorphism $f_0 : X_0 \to X'_0$ then we have $J^r f : J^r X \to J^r X'$ given by $J^r f(j^r_x\sigma) = j^r_{f_0(x)}(f \circ \sigma \circ f_0^{-1})$ for $j^r_x\sigma \in J^r X$.

1.2. Let $p : X \to X_0$ be as above. A vector field V on X is *projectable* if there exists a vector field V_0 on X_0 such that V is p-related to V_0. If V is projectable on X, then its flow $\operatorname{Exp} tV$ is formed by local fibered diffeomorphisms, and we can define a vector field

$$\mathcal{J}^r V = \frac{\partial}{\partial t}_{|t=0} J^r(\operatorname{Exp} tV)$$

on $J^r X$. If V is p-vertical (i.e. $V_0 = 0$), then $\mathcal{J}^r V$ is p^r-vertical.

1.3. An *rth order Lagrangian* on a fibered manifold $p : X \to X_0$ with $\dim X_0 = m$ is a base-preserving morphism

$$\lambda : J^r X \to \textstyle\bigwedge^m T^* X_0 .$$

Given a section $\sigma \in \mathit{\Gamma} X$ and a compact subset $K \subset \operatorname{dom}(\sigma)$ contained in a chart domain, the *action* is

$$S(\lambda, \sigma, K) = \int_K \lambda \circ j^r\sigma .$$

A section $\sigma \in \mathit{\Gamma} X$ is called *critical* if for any compact $K \subset \operatorname{dom}(\sigma)$ contained in a chart domain and any p-vertical vector field η on X with compact support in $p^{-1}(K)$ we have

$$\frac{d}{dt}_{|t=0} S(\lambda, \operatorname{Exp} t\eta \circ \sigma, K) = 0 .$$

By derivation inside the integral we see that σ is critical iff for any compact $K \subset \operatorname{dom}(\sigma)$ contained in a chart domain and any p-vertical vector field η on X with compact support in $p^{-1}(K)$ we have

$$\int_K \langle \delta\lambda, \mathcal{J}^r \eta \rangle \circ j^r\sigma = 0,$$

where $\delta\lambda : V J^r X \to \bigwedge^m T^* X_0$ is the p^r-vertical part of the differential of λ.

1.4. Given a base-preserving morphism $\varphi : J^q X \to \bigwedge^k T^* X_0$, its formal exterior differential $D\varphi : J^{q+1} X \to \bigwedge^{k+1} T^* X_0$ is defined by

$$D\varphi(j^{q+1}_{x_0}\sigma) = d(\varphi \circ j^q\sigma)(x_0)$$

for every local section σ of X, where d means the exterior differential at $x_0 \in X_0$ of the local k-form $\varphi \circ j^q\sigma$ on X_0.

1.5. In the following assertion we do not indicate explicitly the pull back to $J^{2r}X$.

PROPOSITION 1 ([2, Prop. 49.3]). *For every rth order Lagrangian $\lambda : J^rX \to \bigwedge^m T^*X_0$, $m = \dim X_0$, there exists a morphism $K(\lambda) : J^{2r-1}X \to V^*J^{r-1}X \otimes \bigwedge^{m-1} T^*X_0$ over the identity of $J^{r-1}X$ and a unique Euler morphism $E(\lambda) : J^{2r}X \to V^*X \otimes \bigwedge^m T^*X_0$ over the identity of X such that*

$$\langle \delta\lambda, \mathcal{J}^r\eta\rangle = D(\langle K(\lambda), \mathcal{J}^{r-1}\eta\rangle) + \langle E(\lambda), \eta\rangle \tag{1}$$

for any vertical vector field η on X.

REMARK 1. The morphism $E(\lambda)$ is called the *Euler morphism.* If $f : J^qX \to \mathbb{R}$ is a function, we have a coordinate decomposition

$$Df = (D_if)dx^i,$$

where

$$D_if = \frac{\partial f}{\partial x^i} + \sum_{|\alpha|\le q} \frac{\partial f}{\partial y^p_\alpha}\, y^p_{\alpha+1_i} : J^{q+1}X \to \mathbb{R}$$

is the so-called *formal* (or *total*) *derivative* of f and (x^i, y^k) are fiber coordinates on X and (x^i, y^k_α) are the induced coordinates on J^qX. The local coordinate form of $E(\lambda)$ is

$$E(\lambda) = \sum_{k=1}^{n} \sum_{|\alpha|\le r} (-1)^{|\alpha|} D_\alpha \frac{\partial L}{\partial y^k_\alpha}\, dy^k \otimes d^m x$$

(see the proof of Proposition 49.3 in [2]), where $d^mx = dx^1 \wedge \cdots \wedge dx^m$, $\lambda = L \otimes d^mx$ and D_α means the iterated formal derivative with respect to the multiindex α.

Proposition 1 yields immediately the following well known fact.

PROPOSITION 2. *A section $\sigma \in \Gamma X$ is critical iff it satisfies the Euler–Lagrange equations $E(\lambda) \circ j^{2r}\sigma = 0$.*

2. The variational calculus in fibered-fibered manifolds

2.1. In [3], we generalized the concept of fibered manifolds as follows. A *fibered-fibered manifold* is a fibered surjective submersion $\pi : Y \to X$ between fibered manifolds $p^Y : Y \to Y_0$ and $p^X : X \to X_0$, i.e. a surjective submersion which sends fibers into fibers such that the restricted maps (between fibers) are submersions. If $\pi' : Y' \to X'$ is another fibered-fibered manifold then a fibered map $f : Y \to Y'$ is called *fibered-fibered* if there exists a (unique) fibered map $f_0 : X \to X'$ such that $\pi' \circ f = f_0 \circ \pi$.

Let $r, s, q \in \mathbb{N} \setminus \{0\}$, $s \geq r \leq q$.

Denote the set of local fibered maps $\sigma : X \to Y$ with $\pi \circ \sigma = \mathrm{id}_{\mathrm{dom}(\sigma)}$ (fibered sections) by $\mathit{\Gamma}_{\mathrm{fib}} Y$. By 12.19 in [1], $\sigma, \varrho \in \mathit{\Gamma}_{\mathrm{fib}} Y$ represent the same (r, s, q)-jet $j^{r,s,q}_x \sigma = j^{r,s,q}_x \varrho$ at a point $x \in X$ iff

$$j^r_x \sigma = j^r_x \varrho, \quad j^s_x(\sigma|X_{x_0}) = j^s_x(\varrho|X_{x_0}), \quad j^q_{x_0}\sigma_0 = j^q_{x_0}\varrho_0,$$

where X_0 and Y_0 are the bases of fibered manifolds X and Y, $x_0 \in X_0$ is the element under x, X_{x_0} is the fiber of X over x_0 and $\sigma_0, \varrho_0 : X_0 \to Y_0$ are the underlying maps of σ, ϱ. The (r, s, q)-*jet prolongation*

$$J^{r,s,q}Y = \{j^{r,s,q}_x \sigma \mid \sigma \in \mathit{\Gamma}_{\mathrm{fib}} Y,\ x \in X\}$$

of Y is a fibered manifold over X with respect to the source projection $\pi^{r,s,q}_X : J^{r,s,q}Y \to X$ (see [3]). We also have the target projection $\pi^{r,s,q}_Y : J^{r,s,q}Y \to Y$. If $\pi' : Y' \to X'$ is another fibered-fibered manifold and $f : Y \to Y'$ is a fibered-fibered map covering a local fibered diffeomorphism $f_0 : X \to X'$ then we have a map $J^{r,s,q}f : J^{r,s,q}Y \to J^{r,s,q}Y'$ given by $J^{r,s,q}f(j^{r,s,q}_x\sigma) = j^{r,s,q}_{f_0(x)}(f \circ \sigma \circ f_0^{-1})$ for any $j^{r,s,q}_x\sigma \in J^{r,s,q}Y$.

2.2. Let $\pi : Y \to X$ be a fibered-fibered manifold which is a fibered submersion between fibered manifolds $p^Y : Y \to Y_0$ and $p^X : X \to X_0$. A projectable vector field W on the fibered manifold Y is *projectable-projectable* if there exists a π-related (to W) projectable vector field $\underline{W}$ on X. If W is projectable-projectable on Y, then its flow $\mathrm{Exp}\, tW$ is formed by local fibered-fibered diffeomorphisms, and we define a vector field

$$\mathcal{J}^{r,s,q}W = \frac{\partial}{\partial t}_{|t=0} J^{r,s,q}(\mathrm{Exp}\, tW)$$

on $J^{r,s,q}Y$. If additionally W is π-*vertical* and p^Y-*vertical* (i.e. W is π-related and p^Y-related to zero vector fields), then $\mathcal{J}^{r,s,q}W$ is $\pi^{r,s,q}_X$-vertical and $p^Y \circ \pi^{r,s,q}_Y$-vertical.

2.3. Let r, s, q be as above. An (r, s, q)th *order Lagrangian* on a fibered-fibered manifold $\pi : Y \to X$ with $\dim X = m$ is a base-preserving (covering the identity of X) morphism

$$\lambda : J^{r,s,q}Y \to \bigwedge\nolimits^m T^*X.$$

Given a fibered section $\sigma \in \mathit{\Gamma}_{\mathrm{fib}} Y$ and a compact subset $K \subset \mathrm{dom}(\sigma) \subset X$ contained in a chart domain, the *action* is

$$S(\lambda, \sigma, K) = \int_K \lambda \circ j^{r,s,q}\sigma.$$

A fibered section $\sigma \in \mathit{\Gamma}_{\mathrm{fib}} Y$ is called *critical* (with respect to λ) if for any compact $K \subset \mathrm{dom}(\sigma)$ contained in a chart domain and any π-vertical and

p^Y-vertical vector field η on Y with compact support in $\pi^{-1}(K)$ we have

$$\frac{d}{dt}_{|t=0} S(\lambda, \operatorname{Exp} t\eta \circ \sigma, K) = 0.$$

By derivation inside the integral we see that σ is critical iff for any compact $K \subset dom(\sigma)$ contained in a chart domain and any π-vertical and p^Y-vertical vector field η on Y with compact support in $\pi^{-1}(K)$ we have

$$\int \langle \delta\lambda, \mathcal{J}^{r,s,q}\eta\rangle j^{r,s,q}\sigma = 0,$$

where $\delta\lambda : \mathcal{V}J^{r,s,q}X \to \bigwedge^m T^*X$ is the restriction of the differential of λ to the vector subbundle $\mathcal{V}J^{r,s,q} \subset TJ^{r,sq}Y$ of vectors vertical with respect to the projections from $J^{r,s,q}Y$ onto X and onto Y_0.

2.4. Given a base-preserving morphism $\varphi : J^{\widetilde{p},\widetilde{p},\widetilde{p}}Y \to \bigwedge^k T^*X$, its formal exterior differential $\mathcal{D}\varphi : J^{\widetilde{p}+1,\widetilde{p}+1,\widetilde{p}+1}Y \to \bigwedge^{k+1} T^*X$ is defined by

$$\mathcal{D}\varphi(j_x^{\widetilde{p}+1,\widetilde{p}+1,\widetilde{p}+1}\sigma) = d(\varphi \circ j^{\widetilde{p},\widetilde{p},\widetilde{p}}\sigma)(x)$$

for every local fibered section σ of Y, where d means the exterior differential at $x \in X$ of the local k-form $\varphi \circ j^{\widetilde{p},\widetilde{p},\widetilde{p}}\sigma$ on X.

(We remark that if $\widetilde{s} > \widetilde{r} \leq \widetilde{q}$ then given a base-preserving morphism $\varphi : J^{\widetilde{r},\widetilde{s},\widetilde{q}}Y \to \bigwedge^k T^*X$ the value $d(\varphi \circ j^{\widetilde{r},\widetilde{s},\widetilde{q}}\sigma)(x)$ is usually not determined by $j_x^{\widetilde{r}+1,\widetilde{s}+1,\widetilde{q}+1}\sigma$. Then the corresponding formal exterior differential does not exist. One can see that the above-mentioned value depends on $j_x^{\widetilde{p}+1,\widetilde{p}+1,\widetilde{p}+1}\sigma$ for $\widetilde{p} = \max(\widetilde{s}, \widetilde{q})$, but the relevant formal exterior differential will not be used.)

2.5. In the following assertion we do not indicate explicitly the pull back to $J^{2p,2p,2p}Y$.

PROPOSITION 3. *Let r, s, q be natural numbers with $s \geq r \leq q$, $r \geq 1$, $p = \max(q, s)$. For every (r, s, q)th order Lagrangian $\lambda : J^{r,s,q}Y \to \bigwedge^m T^*X$, there are a morphism $\mathcal{K}(\lambda) : J^{2p-1,2p-1,2p-1}Y \to \mathcal{V}^*J^{p-1,p-1,p-1}Y \otimes \bigwedge^{m-1} T^*X$ over the identity of $J^{p-1,p-1,p-1}Y$ and a canonical "Euler" morphism $\mathcal{E}(\lambda) : J^{2p,2p,2p}Y \to \mathcal{V}^*Y \otimes \bigwedge^m T^*X$ over the identity of Y satisfying*

$$\langle \delta\lambda, \mathcal{J}^{r,s,q}\eta\rangle = \mathcal{D}(\langle \mathcal{K}(\lambda), \mathcal{J}^{p-1,p-1,p-1}\eta\rangle) + \langle \mathcal{E}(\lambda), \eta\rangle \tag{2}$$

for every π-vertical and p^Y-vertical vector field η on Y. Here $\mathcal{V}Y$ is the vector subbundle of TY of vectors that are π-vertical and p^Y-vertical, and $\mathcal{V}J^{p-1,p-1,p-1}Y$ is the vector subbundle of $TJ^{p-1,p-1,p-1}Y$ of vectors vertical with respect to the obvious projections from $J^{p-1,p-1,p-1}Y$ onto X and onto Y_0.

Proof. Let $\pi^{p,p,p}_{r,s,q} : J^{p,p,p}Y \to J^{r,s,q}Y$ be the jet projection and let $i_p : J^{p,p,p}Y \to J^pY$ be the canonical inclusion, where in J^pY we consider Y as a fibered manifold over X. Using a suitable partition of unity on X and local fibered-fibered coordinate arguments we produce a pth order Lagrangian $\Lambda : J^pY \to \bigwedge^m T^*X$ such that $\Lambda \circ i_p = \lambda \circ \pi^{p,p,p}_{r,s,q}$. Then by the decomposition formula (Proposition 1) there exists a morphism $K(\Lambda) : J^{2p-1}Y \to V^*J^{p-1}Y \otimes \bigwedge^{m-1} T^*X$ and the Euler morphism $E(\Lambda) : J^{2p}X \to V^*Y \otimes \bigwedge^m T^*X$ satisfying

$$\langle \delta\Lambda, \mathcal{J}^p\eta\rangle = D(\langle K(\Lambda), \mathcal{J}^{p-1}\eta\rangle) + \langle E(\Lambda), \eta\rangle$$

for every π-vertical vector field η on Y. Composing both sides of the last formula with i_{2p} and using the obvious equality $D(\varphi) \circ i_{2p} = \mathcal{D}(\varphi \circ i_{2p-1})$ for $\varphi : J^{2p-1}Y \to \bigwedge^k T^*X$ we easily obtain (2) for any π-vertical and p^Y-vertical vector field η on Y, provided we put $\mathcal{E}(\lambda) =$ the restriction of $E(\Lambda)\circ i_{2p}$ to $\mathcal{V}Y$ and $\mathcal{K}(\lambda) =$ the restriction of $K(\Lambda) \circ i_{2p-1}$ to $\mathcal{V}J^{p-1,p-1,p-1}Y \subset VJ^{p-1}Y$. Using Remark 1 it is easy to see (see Remark 2) that the definition of $\mathcal{E}(\lambda)$ does not depend on the choice of Λ. ■

Remark 2. Let (x^i, X^I, y^k, Y^K) for $i = 1, \ldots, m_1$, $I = 1, \ldots, m_2$, $k = 1, \ldots, n_1$ and $K = 1, \ldots, n_2$ be a fibered-fibered local coordinate system on a fibered-fibered manifold Y. If $f : J^{\widetilde{p},\widetilde{p},\widetilde{p}}Y \to \mathbb{R}$ is a function we have the decomposition

$$\mathcal{D}(f) = \mathcal{D}_i(f)dx^i + \mathcal{D}_I(f)dX^I,$$

where $\mathcal{D}_i(f) : J^{\widetilde{p}+1,\widetilde{p}+1,\widetilde{p}+1}Y \to \mathbb{R}$ and $\mathcal{D}_I(f) : J^{\widetilde{p}+1,\widetilde{p}+1,\widetilde{p}+1}Y \to \mathbb{R}$ are the "total" derivatives of f. Let $F : J^{\widetilde{p}}Y \to \mathbb{R}$ be such that $F \circ i_{\widetilde{p}} = f$. From the clear equality $D(F) \circ i_{\widetilde{p}+1} = \mathcal{D}(f)$ we easily deduce that

$$\mathcal{D}_i(f) = D_i(F) \circ i_{\widetilde{p}+1} \quad \text{and} \quad \mathcal{D}_I(f) = D_I(F) \circ i_{\widetilde{p}+1}.$$

In particular, since D_i and D_I, and $D_{i'}$ and $D_{I'}$, commute, so do $\mathcal{D}_i$ and $\mathcal{D}_I$, and $\mathcal{D}_{i'}$ and $\mathcal{D}_{I'}$. From the formulas for D_i and D_I (see Remark 1) and from the above formulas for $\mathcal{D}_i$ and $\mathcal{D}_I$ we easily see that in local coordinates

$$\mathcal{D}_i(f) = \frac{\partial f}{\partial x^i} + \sum_{k=1}^{n_1} \sum_{|\widetilde{\alpha}|\le\widetilde{p}} \frac{\partial f}{\partial y^k_{\widetilde{\alpha}}}\, y^k_{\widetilde{\alpha}+1_i} + \sum_{K=1}^{n_2} \sum_{|\widetilde{\beta}|+|\widetilde{\gamma}|\le\widetilde{p}} \frac{\partial f}{\partial Y^K_{(\widetilde{\beta},\widetilde{\gamma})}}\, Y^K_{(\widetilde{\beta}+1_i,\widetilde{\gamma})}$$

and

$$\mathcal{D}_I(f) = \frac{\partial f}{\partial X^I} + \sum_{K=1}^{n_2} \sum_{|\widetilde{\beta}|+|\widetilde{\gamma}|\le\widetilde{p}} \frac{\partial f}{\partial Y^K_{(\widetilde{\beta},\widetilde{\gamma})}}\, Y^K_{(\widetilde{\beta},\widetilde{\gamma}+1_I)},$$

where $(x^i, X^I, y^k_{\widetilde{\alpha}}, Y^K_{(\widetilde{\beta},\widetilde{\gamma})})$ is the induced coordinate system on $J^{\widetilde{p},\widetilde{p},\widetilde{p}}Y$, $\widetilde{\alpha} = (\widetilde{\alpha}^1, \ldots, \widetilde{\alpha}^{m_1})$, $\widetilde{\beta} = (\widetilde{\beta}^1, \ldots, \widetilde{\beta}^{m_1})$ and $\widetilde{\gamma} = (\widetilde{\gamma}^1, \ldots, \widetilde{\gamma}^{m_2})$.

Let $(x^i, X^I, y^k_\alpha, Y^K_{(\beta,\gamma)})$ be the induced coordinates on $J^{p,p,p}Y$. Then using the formula of Remark 1 it is easy to see that the local coordinate form of

$\mathcal{E}(\lambda)$ is

$$\mathcal{E}(\lambda)=\sum_{K=1}^{n_2}\sum_{|\beta|+|\gamma|\le p}(-1)^{|\beta|+|\gamma|}\mathcal{D}_{(\beta,\gamma)}\frac{\partial L}{\partial Y^K_{(\beta,\gamma)}}\,dY^K\otimes(d^{m_1}x\wedge d^{m_2}X),$$

where $d^{m_1}x=dx^1\wedge\cdots\wedge dx^{m_1}$, $d^{m_2}X=dX^1\wedge\cdots\wedge dX^{m_2}$, $\lambda\circ\pi^{p,p,p}_{r,s,q}=L\otimes(d^{m_1}\wedge d^{m_2}X)$ and $\mathcal{D}_{(\beta,\gamma)}$ denotes the iterated "total" derivative with index (β,γ), $\beta=(\beta^1,\dots,\beta^{m_1})$, $\gamma=(\gamma^1,\dots,\gamma^{m_2})$.

From the above local formula it follows that $\mathcal{E}(\lambda)$ can be factorized through $J^{r+s,2s,r+p}Y$.

Proposition 3 implies the following fact.

Proposition 4. *A fibered section $\sigma\in\Gamma_{\mathrm{fib}}Y$ is critical iff it satisfies the "Euler–Lagrange" equations $\mathcal{E}(\lambda)\circ j^{2p,2p,2p}\sigma=0$. In view of Remark 2 these equations are $\mathcal{E}(\lambda)\circ j^{r+s,2s,r+p}\sigma=0$.*

Remark 3. In the proof of Proposition 4 we use the fact that if η is a π-vertical and p_Y-vertical vector field on Y and $f:X\to\mathbb{R}$ is a map with compact support then $(f\circ\pi)\eta$ is π-vertical and p_Y-vertical. If η is only π-vertical projectable-projectable then $(f\circ\pi)\eta$ may not be projectable-projectable. That is why in the definition of critical fibered sections we consider the η's which are π-vertical and p_Y-vertical.

3. On naturality of the "Euler" operator. We say that a fibered manifold $p:X\to X_0$ is of *dimension* (m,n) if $\dim X_0=m$ and $\dim X=m+n$. All (m,n)-dimensional fibered manifolds and their local fibered diffeomorphisms form a category which we denote by $\mathcal{FM}_{m,n}$ and which is local and admissible in the sense of [2].

Similarly, we say that a fibered-fibered manifold $\pi:Y\to X$ is of *dimension* (m_1,m_2,n_1,n_2) if the fibered manifold X is of dimension (m_1,n_1) and the fibered manifold Y is of dimension (m_1+n_1,m_2+n_2). All (m_1,m_2,n_1,n_2)-dimensional fibered-fibered manifolds and their fibered-fibered local diffeomorphisms form a category which we denote by $\mathcal{FM}_{m_1,m_2,n_1,n_2}$ and which is local and admissible in the sense of [2]. The standard (m_1,m_2,n_1,n_2)-dimensional trivial fibered-fibered manifold $\pi:\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}$ will be denoted by $\mathbb{R}^{m_1,m_2,n_1,n_2}$. Any (m_1,m_2,n_1,n_2)-dimensional fibered-fibered manifold is locally $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-isomorphic to $\mathbb{R}^{m_1,m_2,n_1,n_2}$.

Given two fibered manifolds $Z_1\to M$ and $Z_2\to M$ over the same base M, we denote the space of all base-preserving fibered manifold morphisms of Z_1 into Z_2 by $\mathcal{C}^\infty_M(Z_1,Z_2)$. In [1], I. Kolář studied the rth order Euler morphism $E(\lambda)$ of the variational calculus on an (m,n)-dimensional fibered manifold $p:X\to X_0$ as the Euler operator

$$E:\mathcal{C}^\infty_{X_0}(J^rX,\textstyle\bigwedge^mT^*X_0)\to\mathcal{C}^\infty_X(J^{2r}X,V^*X\otimes\textstyle\bigwedge^mT^*X_0).$$

He deduced the following classification theorem:

THEOREM 1 ([1]). *Any $\mathcal{FM}_{m,n}$-natural operator* (*in the sense of* [2]) *of the type of the Euler operator is of the form cE, $c \in \mathbb{R}$, provided $m \geq 2$.*

In the present section we obtain a similar result in the fibered-fibered manifold case. Namely, we study the "Euler" morphism $\mathcal{E}(\lambda)$ of the variational calculus on an (m_1, m_2, n_1, n_2)-dimensional fibered-fibered manifold $\pi : Y \to X$ as the "Euler" operator

$$\mathcal{E} : \mathcal{C}^\infty_X(J^{r,s,q}Y, \bigwedge^m T^*X) \to \mathcal{C}^\infty_Y(J^{2p,2p,2p}Y, \mathcal{V}^*Y \otimes \bigwedge^m T^*X).$$

Here and from now on $s \geq r \leq q$ are natural numbers, $r \geq 1$, $p = \max(s, q)$ and $m = m_1 + m_2 = \dim X$. We prove the following classification theorem.

THEOREM 2. *Any $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-natural operator A* (*in the sense of* [2]) *of the type of the "Euler" operator is of the form $c\mathcal{E}$, $c \in \mathbb{R}$, provided $m \geq 2$.*

REMARK 4. The assumption of the last theorem means that for any $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-morphism $f : Y \to Y'$ and any (r, s, q)th order Lagrangians $\lambda \in \mathcal{C}^\infty_X(J^{r,s,q}Y, \bigwedge^m T^*X)$ and $\lambda' \in \mathcal{C}^\infty_{X'}(J^{r,s,q}Y', \bigwedge^m T^*X')$, if λ and λ' are f-related then so are $A(\lambda)$ and $A(\lambda')$. Moreover A is regular and local. The regularity means that A transforms any smoothly parametrized family of Lagrangians into a smoothly parametrized family of suitable type morphisms. The locality means that $A(\lambda)_u$ depends on the germ of λ at $\pi^{2p,2p,2p}_{r,s,q}(u)$.

Proof of Theorem 2. From now on let (x^i, X^I, y^k, Y^K), $i = 1, \ldots, m_1$, $I = 1, \ldots, m_2$, $k = 1, \ldots, n_1$, $K = 1, \ldots, n_2$, be the usual fibered-fibered coordinates on $\mathbb{R}^{m_1,m_2,n_1,n_2}$.

An $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-morphism $(x^i, X^I, y_k - \sigma^k(x^{i'}), Y^K - \Sigma^K(x^{i'}, X^{I'}))$ sends $j^{2p,2p,2p}_{(0,0)}(x^i, X^I, \sigma^k, \Sigma^K)$ into

$$\Theta = j^{2p,2p,2p}_{(0,0)}(x^i, X^I, 0, 0) \in (J^{2p,2p,2p}\mathbb{R}^{m_1,m_2,n_1,n_2})_{(0,0,0,0)}.$$

Then A is uniquely determined by the evaluations

$$\langle A(\lambda)_\Theta, v\rangle \in \bigwedge^m T^*_0\mathbb{R}^m$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}}(J^{r,s,q}\mathbb{R}^{m_1,m_2,n_1,n_2}, \bigwedge^m T^*\mathbb{R}^m)$ and $v \in T_0\mathbb{R}^{n_2} = \mathcal{V}_{(0,0,0,0)}\mathbb{R}^{m_1,m_2,n_1,n_2}$.

Using the invariance of A with respect to $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-morphisms of the form $\mathrm{id}_{\mathbb{R}^m} \times \mathrm{id}_{\mathbb{R}^{n_1}} \times \psi$ for linear $\psi : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ we see that A is uniquely determined by the evaluations

$$\left\langle A(\lambda)_\Theta, \frac{\partial}{\partial Y^1}_0\right\rangle \in \bigwedge^m T^*_0\mathbb{R}^m$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}}(J^{r,s,q}\mathbb{R}^{m_1,m_2,n_1,n_2}, \bigwedge^m T^*_0\mathbb{R}^m)$.

Consider an arbitrary non-vanishing $f : \mathbb{R}^m \to \mathbb{R}$. There is $F : \mathbb{R}^m \to \mathbb{R}$ such that $\partial F/\partial X^1 = f$ and $F(0) = 0$. Then the $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-map

$$(x^1, \dots, x^{m_1}, F, X^2, \dots, X^{m_2}, y^1, \dots, y^{n_1}, Y^1, \dots, Y^{n_2})$$

preserves Θ, $\frac{\partial}{\partial Y^1}{}_0$ and sends $\mathrm{germ}_0(f d^{m_1}x \wedge d^{m_2}X)$ into $\mathrm{germ}_0(d^{m_1}x \wedge d^{m_2}X)$, where $d^{m_1}x$ and $d^{m_2}X$ are as in Remark 2. Then by naturality A is uniquely determined by the evaluations

$$\left\langle A(\lambda + b d^{m_1}x \wedge d^{m_2}X)_\Theta, \frac{\partial}{\partial Y^1}{}_0 \right\rangle \in \bigwedge^m T_0^* \mathbb{R}^m$$

for all $\lambda \in \mathcal{C}^\infty_{\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}}(J^{r,s,q}\mathbb{R}^{m_1,m_2,n_2,n_2}, \bigwedge^m T^*\mathbb{R}^m)$ satisfying the condition $\lambda(j^{r,s,q}_{(x_0,X_0)}(x^i, X^I, 0)) = 0$ for any $(x_0, X_0) \in \mathbb{R}^{m_1,m_2}$ and all $b \in \mathbb{R}$.

Let λ and b be as above. Using the invariance of A with respect to $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-maps $\psi_{\tau,\mathcal{T}} = \left(x^i, X^I, \frac{1}{\tau^k} y^k, \frac{1}{\mathcal{T}^K} Y^K\right)$ for $\tau^k > 0$ and $\mathcal{T}^K > 0$ we get the homogeneity condition

$$\left\langle A((\psi_{\tau,\mathcal{T}})_*\lambda + b d^{m_1}x \wedge d^{m_2}X)_\Theta, \frac{\partial}{\partial Y^1}{}_0 \right\rangle = \mathcal{T}^1 \left\langle A(\lambda + b d^{m_1}x \wedge d^{m_2}X)_\Theta, \frac{\partial}{\partial Y^1}{}_0 \right\rangle.$$

By Corollary 19.8 in [2] of the non-linear Peetre theorem we can assume that λ is a polynomial. The regularity of A implies that $\langle A(\lambda + b d^{m_1}x \wedge d^{m_2}X)_\Theta, \frac{\partial}{\partial Y^1}{}_0\rangle$ is smooth in the coordinates of λ and b. Then by the homogeneous function theorem (and the above type of homogeneity) we deduce that $\langle A(\lambda + b d^{m_1}x \wedge d^{m_2}X)_\Theta, \frac{\partial}{\partial Y^1}{}_0\rangle$ is a linear combination of the coordinates of λ on all $x^{\widetilde{\beta}} X^{\widetilde{\gamma}} Y^1_{(\beta,\gamma)} d^{m_1}x \wedge d^{m_2}X$ and $x^{\widetilde{\beta}} X^{\widetilde{\gamma}} Y^1_{((0),\varrho)} d^{m_1}x \wedge d^{m_2}X$ with coefficients being smooth functions in b, where $(x^i, X^I, y^k_\alpha, Y^K_{(\beta,\gamma)}, Y^K_{((0),\varrho)})$ is the induced coordinate system on $J^{r,s,q}\mathbb{R}^{m_1,m_2,n_1,n_2}$. (Here and from now on, α and β are m_1-tuples, and γ and ϱ are m_2-tuples with $|\alpha| \le q$, $|\beta| + |\gamma| \le r$ and $r + 1 \le |\varrho| \le s$.) In other words, A is determined by the values

$$\left\langle A((a x^{\widetilde{\beta}} X^{\widetilde{\gamma}} Y^1_{(\beta,\gamma)} + b) d^{m_1}x \wedge d^{m_2}X)_\Theta, \frac{\partial}{\partial Y^1}{}_0 \right\rangle = a f^{\widetilde{\beta},\widetilde{\gamma}}_{\beta,\gamma}(b) d^{m_1}x \wedge d^{m_2}X$$

and

$$\left\langle A((a x^{\widetilde{\beta}} X^{\widetilde{\gamma}} Y^1_{((0),\varrho)} + b) d^{m_1}x \wedge d^{m_2}X)_\Theta, \frac{\partial}{\partial Y^1}{}_0 \right\rangle = a f^{\widetilde{\beta},\widetilde{\gamma}}_{\varrho}(b) d^{m_1}x \wedge d^{m_2}X$$

for all $a, b \in \mathbb{R}$, all m_1-tuples $\widetilde{\beta}$, all m_2-tuples $\widetilde{\gamma}$ and all β, γ, ϱ as above.

By the invariance of A with respect to $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-maps of the form $(\tau^i x^i, \mathcal{T}^I X^I, y^k, Y^K)$ for $\tau^i \neq 0$ and $\mathcal{T}^I \neq 0$ we get the homogeneity conditions

$$\tau^{\widetilde{\beta}} \mathcal{T}^{\widetilde{\gamma}} \tau^{-\beta} \mathcal{T}^{-\gamma} f^{\widetilde{\beta},\widetilde{\gamma}}_{\beta,\gamma}(\tau^{(1,\dots,1)} \mathcal{T}^{(1,\dots,1)} b) = f^{\widetilde{\beta},\widetilde{\gamma}}_{\beta,\gamma}(b)$$

and

$$\tau^{\tilde{\beta}}\mathcal{T}^{\tilde{\gamma}}\mathcal{T}^{-\varrho}f_{\varrho}^{\tilde{\beta},\tilde{\gamma}}(\tau^{(1,\dots,1)}\mathcal{T}^{(1,\dots,1)}b) = f_{\varrho}^{\tilde{\beta},\tilde{\gamma}}f(b).$$

By the homogeneous function theorem these types of homogeneity imply that

(+) $f_{\beta,\gamma}^{\beta,\gamma}$ are constant, $f_{\varrho}^{(0),\varrho}$ are constant, $f_{\beta+(a,\dots,a),\gamma+(a,\dots,a)}^{\beta,\gamma}$ may possibly be not zero for natural numbers $a \geq 1$ with $|\beta|+|\gamma|+ma \leq r$, and all other f's are zero.

Hence A is determined by the values

$$\left\langle A(x^{\beta}X^{\gamma}Y^1_{(\beta,\gamma)}d^{m_1}x \wedge d^{m_2}X)_{\Theta}, \frac{\partial}{\partial Y^1}_0\right\rangle, \tag{$*$}$$

$$\left\langle A(X^{\varrho}Y^1_{((0),\varrho)}d^{m_1}x \wedge dX^{m_2}X)_{\Theta}, \frac{\partial}{\partial Y^1}_0\right\rangle, \tag{$**$}$$

$$\left\langle A((x^{\beta}X^{\gamma}Y^1_{(\beta+(a,\dots,a),\gamma+(a,\dots,a))}+1)d^{m_1}x \wedge d^{m_2}X)_{\Theta}, \frac{\partial}{\partial Y^1}_0\right\rangle \tag{$***$}$$

for all β,γ,ϱ as above and natural numbers $a \geq 1$ with $|\beta|+|\gamma|+ma \leq r$, or (equivalently) if the above values are zero then $A=0$.

Let $\beta_{i_0} \neq 0$ for some i_0. We are going to use the invariance of A with respect to the locally defined $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-map

$$\psi^{i_0} = (x^i, X^I, y^k, Y^1 + x^{i_0}Y^1, Y^2, \dots, Y^{n_2})^{-1}$$

preserving x^i, X^I, Θ, $\frac{\partial}{\partial Y^1}_0$ and sending $Y^1_{(\beta,\gamma)}$ into $Y^1_{(\beta,\gamma)} + x^{i_0}Y^1_{(\beta,\gamma)} + Y^1_{(\beta-1_{i_0},\gamma)}$ (because we have

$$\begin{aligned} Y^1_{(\beta,\gamma)} &\circ J^{r,s,q}((\psi^{i_0})^{-1})(j^{r,s,q}_{(x_0^i,X_0^I)}(x^i,X^I,\sigma^k,\Sigma^K)) \\ &= \partial_{(\beta,\gamma)}(\Sigma^1 + x^{i_0}\Sigma^1)(x_0^i,X_0^I) \\ &= \partial_{(\beta,\gamma)}\Sigma^1(x_0^i,X_0^I) + x^{i_0}\partial_{(\beta,\gamma)}\Sigma^1(x_0^i,X_0^I) + \partial_{(\beta-1_{i_0},\gamma)}\Sigma^1(x_0^i,X_0^I) \\ &= (Y^1_{(\beta,\gamma)} + x^{i_0}Y^1_{(\beta,\gamma)} + Y^1_{(\beta-1_{i_0},\gamma)})(j^{r,s,q}_{(x_0^i,X_0^I)}(x^i,X^I,\sigma^k,\Sigma^K)), \end{aligned}$$

where $\partial_{(\beta,\gamma)}$ is the iterated partial derivative as indicated multiplied by $\frac{1}{\beta!\gamma!}$). Using this invariance, from

$$\left\langle A(x^{\beta-1_{i_0}}X^{\gamma}Y^1_{(\beta,\gamma)})_{\Theta}, \frac{\partial}{\partial Y^1}_0\right\rangle = 0$$

(see (+)) it follows that ($*$) is zero if it is zero for $\beta - 1_i$ in place of β. Continuing this procedure and a similar procedure with the $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-morphism

$$\Psi^{I_0} = (x^i, X^I, y^k, Y^1 + X^{I_0}Y^1, Y^2, \dots, Y^{n_2})^{-1}$$

in place of ψ^{i_0} we see that ($*$) is zero if it is zero for $\beta=(0)$ and $\gamma=(0)$.

Similarly, $(**)$ is zero if it is zero for $\varrho = (0)$. Similarly, $(***)$ is zero if it is zero for $\beta = (0)$ and $\gamma = (0)$.

Applying $\langle A((Y^1_{(2a,\dots,a,0)} + 1)d^{m_1}x \wedge d^{m_2}X)_\Theta, \frac{\partial}{\partial Y^1}{}_0\rangle$ to the $\mathcal{FM}_{m_1,m_2,n_2}$-map $\mathrm{id} + (0,\dots,0,x^1,0,\dots,0)$, where x^1 is in the mth position and where $(2a,a,\dots,a,0) \in (\mathbb{N}\cup\{0\})^{m_1} \times (\mathbb{N}\cup\{0\})^{m_2}$, and using $(+)$, since $m \geq 2$, we see that the values $(***)$ for $\beta = (0)$ and $\gamma = (0)$ are zero.

Hence A is uniquely determined by the value

$$\left\langle A(Y^1_{((0),(0))}d^{m_1}x \wedge d^{m_2}X)_\Theta, \frac{\partial}{\partial Y^1}{}_0 \right\rangle \in \bigwedge^m T^*_0\mathbb{R}^m.$$

Therefore the vector space of all the A in question is 1-dimensional. This ends the proof of Theorem 2. ■

Remark 5. In view of Remark 2 we note that Theorem 2 holds for $(r+s, 2s, r+p)$ in place of $(2p, 2p, 2p)$.

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