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Multiple values and uniqueness problem for meromorphic mappings sharing hyperplanes

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Abstract. The purpose of this article is to deal with multiple values and the uniqueness problem for meromorphic mappings from $\mathbb{C}^m$ into the complex projective space $\mathbb{P}^n(\mathbb{C})$ sharing hyperplanes. We obtain two uniqueness theorems which improve and extend some known results.

1. Introduction and main results. In 1926, R. Nevanlinna [12] proved the well-known five-value theorem that if two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$ have the same inverse images (ignoring multiplicities) for five distinct values in $\mathbb{P}^1(\mathbb{C})$, then $f(z) \equiv g(z)$. We know that five cannot be reduced to four: for example, $f(z) = e^z$ and $g(z) = e^{-z}$ share four values $0, 1, -1, \infty$ (ignoring multiplicities), but $f(z) \not\equiv g(z)$. There have been several improvements of Nevanlinna's five-value theorem. H. X. Yi ([21, Theorem 3.15]) adopted the method of dealing with multiple values due to L. Yang [19] and obtained a uniqueness theorem for meromorphic functions of one variable. Later, Hu, Li and Yang [11, Theorem 3.9] extended this result to meromorphic functions in several variables.

Theorem 1.1 ([11, Theorem 3.9]). *Let f and g be nonconstant meromorphic functions on $\mathbb{C}^m$, let a_j $(j = 1, \dots, q)$ be distinct complex elements in $\mathbb{P}^1(\mathbb{C})$ and suppose $m_j \in \mathbb{Z}^+ \cup \{\infty\}$ $(j = 1, \dots, q)$ satisfy $m_1 \geq \dots \geq m_q$ and $\nu^1_{f-a_j,\leq m_j} = \nu^1_{g-a_j,\leq m_j}$ $(j = 1, \dots, q)$. If $\sum_{j=3}^{q} \frac{m_j}{m_j+1} > 2$, then $f(z) \equiv g(z)$.*

Over the last few decades, there have been several generalizations of Nevanlinna's five-value theorem to the case of meromorphic mappings from $\mathbb{C}^m$ into the complex projective space $\mathbb{P}^n(\mathbb{C})$. Some of the first results in this direction are due to Fujimoto [8, 9].

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Let g be a nonconstant meromorphic mapping from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ such that the linear span of $g(\mathbb{C}^m)$ is of dimension l and $\operatorname{rank} g \geq \mu$, where μ is a positive integer. For a hyperplane H in $\mathbb{P}^n(\mathbb{C})$, we denote by $\nu_{(g,H)}$ the map from $\mathbb{C}^m$ into $\mathbb{Z}$ whose value $\nu_{(g,H)}(z)$ $(z \in \mathbb{C}^m)$ is the intersection multiplicity of the images of g and H at $g(z)$. Let $H_1, \dots, H_q$ be hyperplanes in general position such that $\dim g^{-1}(H_i \cap H_j) \leq m-2$ for $i \neq j$. Take $k_j \in \mathbb{Z}^+ \cup \{\infty\}$ $(j = 1, \dots, q)$ with $k_1 \geq \cdots \geq k_q \geq 1$. We denote by

$$\mathcal{G} = \mathcal{G}(g; \mu; l; k_j; \{H_j\}_{j=1}^q)$$

the set of all nonconstant meromorphic mappings $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ satisfying the following conditions:

(a) the linear span of $f(\mathbb{C}^m)$ is of dimension l and $\operatorname{rank} f \geq \mu$,
(b) $\min\{\nu_{(f,H_j),\leq k_j}, 1\} = \min\{\nu_{(g,H_j),\leq k_j}, 1\}$,
(c) $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z \in \mathbb{C}^m : 0 < \nu_{(g,H_j)} \leq k_j\}$.

For brevity we will omit "$\leq k_j$" if $k_j = \infty$. We also define a subfamily $\mathcal{G}_0$ of $\mathcal{G}$ by

$$\mathcal{G}_0 = \{f \in \mathcal{G} : \delta_g(H_j) \leq \delta_f(H_j) \text{ for all } 1 \leq j \leq q\}.$$

Set $\gamma = l - \mu + 1$ and let

$$\begin{aligned} C(\mu; l; \{k_j\}) &= q - n + l - \sum_{j=1}^q \frac{\gamma}{k_j + 1} - \frac{2\gamma k_1}{k_1 + 1} \\ &= q - \gamma q - n + l + \sum_{j=2}^q \frac{\gamma k_j}{k_j + 1} - \frac{\gamma k_1}{k_1 + 1}. \end{aligned}$$

In 2000, Aihara [1] obtained the following three theorems. The first one is a generalization of the uniqueness theorem due to Gopalakrishna and Bootsnuramath [10].

Theorem 1.2 ([1, Theorem 0.1]). *If $n+1 < C(\mu; l; \{k_j\})$, then $\mathcal{G} = \{g\}$.*

The following two theorems generalized Theorem 1 of [18] due to Ueda.

Theorem 1.3 ([1, Theorem 0.2]). *Suppose that $n + 1 = C(\mu; l; \{k_j\})$. If $\delta_g(H_j) > 0$ for at least one H_j $(1 \leq j \leq q)$, then $\mathcal{G} = \{g\}$.*

Theorem 1.4 ([1, Theorem 0.3]). *Suppose that*

$$n + 1 - C(\mu; l; \{k_j\}) < \frac{\gamma}{k_1 + 1} \sum_{j=1}^q \delta_g(H_j).$$

Then $\mathcal{G}_0 = \{g\}$.

Recall that in 1986, Yi [20] obtained a general theorem on multiple values and uniqueness of meromorphic functions in one variable which improved the results of [10, 18]. Thus it is natural to consider multiple values and uniqueness of meromorphic mappings by using a similar discussion to Yi [20]. The first main purpose of this paper is to obtain a general uniqueness theorem which improves and extends the above-mentioned results of Aihara [1] and Theorem 2.3 of [2]. We adopt the method of dealing with multiple values due to Yang [19].

THEOREM 1.5. *Let f and g be mappings in $\mathcal{G}$. Set*

$$B_1 = \frac{\delta_f(H_1)+\delta_f(H_2)}{k_3+1} + \sum_{j=3}^{q} \frac{k_j+\delta_f(H_j)}{k_j+1} - \frac{\gamma q + 2n + 1 - q - l}{\gamma},$$

$$B_2 = \frac{\delta_g(H_1)+\delta_g(H_2)}{k_3+1} + \sum_{j=3}^{q} \frac{k_j+\delta_g(H_j)}{k_j+1} - \frac{\gamma q + 2n + 1 - q - l}{\gamma}.$$

If $\min\{B_1, B_2\} \geq 0$ *and* $\max\{B_1, B_2\} > 0$, *then* $f(z) \equiv g(z)$.

Let

$$\begin{aligned} B(\mu; l; \{k_j\}) &= q - n + l - \sum_{j=1}^{q} \frac{\gamma}{k_j+1} - \sum_{j=1}^{2} \frac{\gamma k_j}{k_j+1} \\ &= q - \gamma q - n + l + \sum_{j=3}^{q} \frac{\gamma k_j}{k_j+1}. \end{aligned}$$

Noting that $1 \geq \frac{k_1}{k_1+1} \geq \cdots \geq \frac{k_q}{k_q+1} \geq \frac{1}{2}$, one can see that $B(\mu; l; \{k_j\}) \leq C(\mu; l; \{k_j\})$. From Theorem 1.5 we easily obtain the following corollaries which are improvements of Theorems 1.2–1.4 respectively.

COROLLARY 1.1. *If* $n+1 < B(\mu; l; \{k_j\})$, *then* $\mathcal{G} = \{g\}$.

COROLLARY 1.2. *Suppose that* $n+1 = B(\mu; l; \{k_j\})$. *If* $\delta_g(H_j) > 0$ *for at least one* H_j $(1 \leq j \leq q)$, *then* $\mathcal{G} = \{g\}$.

COROLLARY 1.3. *Suppose that*

$$n+1 - B(\mu; l; \{k_j\}) < \sum_{j=1}^{2} \frac{\gamma \delta_g(H_j)}{k_3+1} + \sum_{j=3}^{q} \frac{\gamma \delta_g(H_j)}{k_j+1}.$$

Then $\mathcal{G}_0 = \{g\}$.

Denote by $\mathcal{F}_{\leq m_j}(g, \{H_j\}_{j=1}^q, d)$ the set of all linearly nondegenerate (that is the special case of $\mathcal{G}$ where $l = n$, $\mu = 1$ and $\gamma = l - \mu + 1 = n$) meromorphic mappings $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ satisfying the conditions:

(a) $\nu^d_{(f,H_j),\leq m_j} = \nu^d_{(g,H_j),\leq m_j}$,
(b) $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z \in \mathbb{C}^n : 0 < \nu_{(g,H_j)} \leq m_j\}$.

For brevity we will omit "$\leq m_j$" if $m_j = \infty$. Denote by $\sharp S$ the cardinality of a set S.

Thus by Corollary 1.1 we can also get Theorem 1.4 of [3] which is an exact extension of Theorem 1.1 to linearly nondegenerate meromorphic mappings sharing $3n+2$ hyperplanes in general position. This yields $\sharp\mathcal{F}_{\leq k}(g, \{H_j\}_{j=1}^{3n+2}, 1) = 1$ for $k > n^2 + 2n - 1$, which is an improvement of Smiley's $3n+2$ hyperplanes uniqueness theorem [16].

Many authors have searched for the best number q of hyperplanes in general position. For example, Thai and Quang [17] considered $q < 3n+2$ and proved that if $n \geq 2$ then $\sharp\mathcal{F}(g, \{H_j\}_{j=1}^{3n+1}, 1) = 1$. In [6], Dethloff and Tan considered $q \geq 2n+3$ and obtained $\sharp\mathcal{F}(g, \{H_j\}_{j=1}^{2n+3}, n) = 1$. In [4], Chen and Yan improved the above results and obtained the best result available at present that $\sharp\mathcal{F}(g, \{H_j\}_{j=1}^{2n+3}, 1) = 1$. Recently, Cao and Yi [3] obtained the following result concerning multiple values and uniqueness by a similar method to [17, 4].

THEOREM 1.6 ([3]). *Let f and g be linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, and let $H_1, \ldots, H_q$ $(q \geq 2n)$ be hyperplanes in general position such that $\dim g^{-1}(H_i \cap H_j) \leq m-2$ for $i \neq j$. Let m_j $(j = 1, \ldots, q)$ be positive integers or ∞ such that $m_1 \geq \cdots \geq m_q \geq 1$,*

$$\nu^1_{(f,H_j),\leq m_j} = \nu^1_{(g,H_j),\leq m_j} \quad (j = 1, \ldots, q),$$

and $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z \in \mathbb{C}^n : 0 < \nu_{(g,H_j)} \leq m_j\}$. If

$$(1) \qquad \sum_{j=3}^{q} \frac{m_j}{m_j+1} > \frac{nq - q + n + 1}{n} - \frac{4n-4}{q+2n-2} + \left(\frac{1}{m_1+1} + \frac{1}{m_2+1}\right),$$

then $f(z) \equiv g(z)$.

From Theorem 1.6, we get

$$\sharp\mathcal{F}_{\leq k}(g, \{H_j\}_{j=1}^{2n+3}, 1) = 1 \quad \text{for } k > \frac{8n^3 + 14n^2 - 2}{3n+2}.$$

The same year, Quang obtained a better estimate:

THEOREM 1.7 ([14]).

$$\sharp\mathcal{F}_{\leq k}(g, \{H_j\}_{j=1}^{2n+3}, 1) = 1 \quad \textit{for } k > \frac{4n^3 + 11n^2 + n - 2}{3n+2}.$$

For $n = 1$, condition (1) reduces to $\sum_{j=3}^q \frac{m_j}{m_j+1} > 2 + (\frac{1}{m_1+1} + \frac{1}{m_2+1})$. The conditions of Theorems 1.6 and 1.1 suggest that there may exist a better lower estimate than (1). Another main purpose of this paper is to improve (1) by proving the following theorem.

THEOREM 1.8. *Let f and g be linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, and let $H_1, \ldots, H_q$ $(q \geq 2n)$ be hyperplanes in* *general position such that* $\dim g^{-1}(H_i \cap H_j) \le n-2$ *for* $i \ne j$. *Let* m_j $(j=1,\dots,q)$ *be positive integers or* ∞ *such that* $m_1 \ge \dots \ge m_q \ge 1$,

$$\nu^1_{(f,H_j),\le m_j} = \nu^1_{(g,H_j),\le m_j} \quad (j=1,\dots,q),$$

and $f(z)=g(z)$ *on* $\bigcup_{j=1}^q \{z \in \mathbb{C}^m : 0 < \nu_{(g,H_j)} \le m_j\}$. *If*

$$\sum_{j=3}^q \frac{m_j}{m_j+1} > \frac{nq-q+n+1}{n} - \frac{4n-4}{q+2n-2} + \left(\frac{1}{m_1+1} + \frac{1}{m_2+1}\right) - \frac{2nq}{q+2n-2}\cdot\frac{1}{m_1+1},$$

then $f(z) \equiv g(z)$.

For $n=1$, the condition of Theorem 1.8 reduces to $\sum_{j=3}^q \frac{m_j}{m_j+1} > 2 + \left(\frac{1}{m_2+1} - \frac{1}{m_1+1}\right)$, which is very close to the condition $\sum_{j=3}^q \frac{m_j}{m_j+1} > 2$ in Theorem 1.1. Furthermore, from Theorem 1.8 one can deduce the following corollaries which improve the above-mentioned uniqueness theorems for meromorphic mappings sharing hyperplanes in general position [8, 16, 17, 6, 7, 4, 3, 14].

Corollary 1.4. *If* $q \ge 2n+3$, *then*

$$\sharp\mathcal{F}_{\le k}(g,\{H_j\}_{j=1}^q,1)=1 \quad \text{for } k > \frac{qn(q-2)}{(q+n-1)(q-2n-2)} - 1.$$

Corollary 1.5.

$$\sharp\mathcal{F}_{\le k}(g,\{H_j\}_{j=1}^{2n+3},1)=1 \quad \text{for } k > \frac{4n^3+8n^2-2}{3n+2}.$$

Corollary 1.6. *If* $n \ge 2$, *then*

$$\sharp\mathcal{F}_{\le k}(g,\{H_j\}_{j=1}^{3n+1},1)=1 \quad \text{for } k > \frac{9n^2-4n+3}{4(n-1)}.$$

Corollary 1.7. *If* $n \ge 3$, *then*

$$\sharp\mathcal{F}_{\le k}(g,\{H_j\}_{j=1}^{3n},1)=1 \quad \text{for } k > \frac{9n^3-10n^2+9n-2}{(4n-1)(n-2)}.$$

Corollary 1.8. *If* $n \ge 4$, *then*

$$\sharp\mathcal{F}_{\le k}(g,\{H_j\}_{j=1}^{3n-1},1)=1 \quad \text{for } k > \frac{9n^3-16n^2+17n-6}{(4n-2)(n-3)}.$$

However, we do not know whether the condition in Theorem 1.8 can be reduced to $\sum_{j=3}^q \frac{m_j}{m_j+1} > 2$ for $n=1$.

2. Preliminaries. We set $\|z\| = (\sum_{j=1}^{m} |z_j|^2)^{1/2}$ for $z = (z_1, \dots, z_m) \in \mathbb{C}^m$. For $r > 0$, define $B(r) = \{z \in \mathbb{C}^m : \|z\| < r\}$, $S(r) = \{z \in \mathbb{C}^m : \|z\| = r\}$, $d^c = (4\pi\sqrt{-1})^{-1}(\overline{\partial} - \partial)$,

$$\upsilon = (dd^c\|z\|^2)^{m-1} \quad \text{and} \quad \sigma = d^c \log \|z\|^2 \wedge (dd^c\|z\|^2)^{m-1}.$$

Let h be a nonzero entire function on $\mathbb{C}^m$. For $a \in \mathbb{C}^m$, we can write h as $h(z) = \sum_{j=0}^{\infty} P_j(z-a)$, where $P_j(z)$ is either identically zero or a homogeneous polynomial of degree j. The number $\nu_h(a) := \min\{j : P_j \neq 0\}$ is said to be the *zero-multiplicity* of h at a. Set $\operatorname{Supp} \nu_h := \overline{\{z \in \mathbb{C}^m : \nu_h(z) \neq 0\}}$.

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$. For each $a \in \mathbb{C}^m$, we choose nonzero holomorphic functions φ_0 and φ_1 on a neighborhood U of a such that $\varphi = \varphi_0/\varphi_1$ on U and $\dim(\varphi_0^{-1} \cap \varphi_1^{-1}(0)) \le m-2$, and we define $\nu_\varphi := \nu_{\varphi_0}$, $\nu_\varphi^\infty := \nu_{\varphi_1}$, which are independent of the choices of φ_0 and φ_1.

Let f be a nonconstant meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. We can choose holomorphic functions $f_0, f_1, \dots, f_n$ on $\mathbb{C}^m$ such that $I_f := \{z \in \mathbb{C}^m : f_0(z) = \dots = f_n(z) = 0\}$ is of dimension at most $m-2$ and $\tilde{f} = (f_0 : \dots : f_n)$. As usual, $(f_0 : \dots : f_n)$ is a reduced representation of f. The characteristic function of f is defined by

$$T_f(r) = \int_{S(r)} \log \|\tilde{f}\|\, \sigma - \int_{S(1)} \log \|\tilde{f}\|\, \sigma \quad (r > 1),$$

where $\|\tilde{f}\| = (\sum_{j=0}^{n} |f_j|^2)^{1/2}$. Note that $T_f(r)$ is independent of the choice of the reduced representation of f.

Let k, M be positive integers or $+\infty$. For a divisor ν on $\mathbb{C}^m$, we define the following counting functions:

$$\nu^M(z) = \min\{\nu(z), M\}, \quad \nu^M_{\le k}(z) = \begin{cases} 0 & \text{if } \nu(z) > k, \\ \nu^M(z) & \text{if } \nu(z) \le k, \end{cases}$$

$$\nu^M_{\ge k}(z) = \begin{cases} 0 & \text{if } \nu(z) < k, \\ \nu^M(z) & \text{if } \nu(z) \ge k, \end{cases} \qquad n(t) = \begin{cases} \int_{\operatorname{supp} \nu \cap B(t)} \nu(z)\, \upsilon & \text{if } m \ge 2, \\ \sum_{|z| \le t} \nu(z) & \text{if } m = 1. \end{cases}$$

Similarly, we define $n^M(t)$, $n^M_{\ge k}(t)$ and $n^M_{\le k}(t)$. We set

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2m-1}}\, dt \quad (r > 1).$$

Similarly, we define $N(r, \nu^M)$, $N(r, \nu^M_{\le k})$ and $N(r, \nu^M_{\ge k})$ and denote them by $N^M(r, \nu)$, $N^M_{\le k}(r, \nu)$ and $N^M_{\ge k}(r, \nu)$, respectively.

For a meromorphic function φ on $\mathbb{C}^m$, we denote

$$N_\varphi(r) = N(r, \nu_\varphi), \qquad N^M_\varphi(r) = N^M(r, \nu_\varphi),$$
$$N^M_{\varphi, \le k}(r) = N^M_{\le k}(r, \nu_\varphi), \quad N^M_{\varphi, \ge k}(r) = N^M_{\ge k}(r, \nu_\varphi).$$

For brevity we will omit the superscript M if $M = \infty$. We have the following Jensen's formula:

$$N_{\varphi}(r) - N_{1/\varphi}(r) = \int\limits_{S(r)} \log|\varphi|\,\sigma - \int\limits_{S(1)} \log|\varphi|\,\sigma.$$

For a closed subset A of a purely $(m-1)$-dimensional analytic subset of $\mathbb{C}^m$, we define

$$n_A^1(t) = \begin{cases} \int_{A\cap B(t)} \upsilon & \text{if } m \geq 2, \\ \sharp(A \cap B(t)) & \text{if } m = 1, \end{cases} \qquad N^1(r, A) = \int\limits_1^r \frac{n_A^1(t)}{t^{2m-1}}\,dt \quad (r > 1).$$

We now have the following Nevanlinna inequality:

Theorem 2.1. *Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Then*

$$N_{(f,H)}(r) \leq T_f(r) + O(1)$$

for a hyperplane H in $\mathbb{P}^n(\mathbb{C})$ with $f(\mathbb{C}^m) \not\subseteq H$, where $O(1)$ stands for a bounded term as $r \to \infty$.

Let f and H be as in Theorem 2.1. We define *Nevanlinna's deficiency* $\delta_f(H)$ by

$$\delta_f(H) = 1 - \limsup_{r\to\infty} \frac{N_{(f,H)}(r)}{T_f(r)}.$$

If $\delta_f(H) > 0$, then H is called a *deficient hyperplane* in the sense of Nevanlinna.

As usual, by writing "$\| P$" we mean the assertion P holds for all $r > 1$ excluding a Borel subset $E \subseteq [0, \infty)$ with finite Lebesgue measure. We have the following second main theorem for meromorphic mappings that may be linearly degenerate (see [13, p. 501]).

Theorem 2.2 (Second Main Theorem). *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a nonconstant meromorphic mapping with rank μ, and let $H_1, \ldots, H_q$ be hyperplanes in general position. Let l be the dimension of the smallest linear subspace of $\mathbb{P}^n(\mathbb{C})$ containing $f(\mathbb{C}^m)$. Then*

$$\| \ (q - 2n + l - 1)T_f(r) \leq \sum_{j=1}^{q} N_{(f,H_j)}^{l-\mu+1}(r) + o(T_f(r)).$$

3. Proof of Theorem 1.5. Let f be an arbitrary mapping in $\mathcal{G}$. By the Second Main Theorem we have

$$\| \ (q-2n+l-1)T_f(r) \le \sum_{j=1}^{q} N^{\gamma}_{(f,H_j)}(r) + o(T_f(r))$$
$$\le \gamma \sum_{j=1}^{q} N^{1}_{(f,H_j)}(r) + o(T_f(r)).$$

The following lemma is proved by using the method due to L. Yang [19] (see also Lemma 4.7 in [17]).

LEMMA 3.1. *Let f be a nonconstant meromorphic mapping from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, H be a hyperplane in general position, and k ($\ge s \ge 1$) be a positive integer. Then*

$$N^{s}_{(f,H)}(r) \le s\Big(1-\frac{s}{k+1}\Big)N^{1}_{(f,H),\le k}(r) + \frac{s}{k+1}N_{(f,H)}(r),$$

and

$$N^{s}_{(f,H)}(r) \le s\Big(1-\frac{s}{k+1}\Big)N^{1}_{(f,H),\le k}(r) + \frac{s}{k+1}(1-\delta_f(H))T_f(r) + o(T_f(r)).$$

Proof. From

$$N^{s}_{(f,H)}(r) = N^{s}_{(f,H),\le k}(r) + N^{s}_{(f,H),\ge k+1}(r)$$

and

$$N^{s}_{(f,H),\ge k+1}(r) \le \frac{s}{k+1}N_{(f,H),\ge k+1}(r) \le \frac{s}{k+1}(N_{(f,H)}(r) - N^{s}_{(f,H),\le k}(r)),$$

we deduce that

$$N^{s}_{(f,H)}(r) \le \Big(1-\frac{s}{k+1}\Big)N^{s}_{(f,H),\le k}(r) + \frac{s}{k+1}N_{(f,H)}(r)$$
$$\le s\Big(1-\frac{s}{k+1}\Big)N^{1}_{(f,H),\le k}(r) + \frac{s}{k+1}N_{(f,H)}(r).$$

This proves the first inequality of the lemma. The second follows immediately because $N_{(f,H)}(r) \le (1-\delta_f(H))T_f(r) + o(T_f(r))$. ∎

By Lemma 3.1, we have

$$N^{1}_{(f,H_j)}(r) \le \frac{k_j}{k_j+1}N^{1}_{(f,H_j),\le k_j}(r) + \frac{1}{k_j+1}(1-\delta_f(H_j))T_f(r) + o(T_f(r)).$$

The above inequality yields

$$\| \ (q-2n+l-1)T_f(r) \le \gamma \sum_{j=1}^{q} \frac{k_j}{k_j+1}N^{1}_{(f,H_j),\le k_j}(r) + o(T_f(r))$$
$$+ \gamma \sum_{j=1}^{q} \frac{1}{k_j+1}(1-\delta_f(H_j))T_f(r).$$

Noting that $1 \geq \frac{k_1}{k_1+1} \geq \cdots \geq \frac{k_q}{k_q+1} \geq \frac{1}{2}$, we have

$$\begin{aligned}
\sum_{j=1}^{q} & \frac{k_j}{k_j+1} N^1_{(f,H_j),\leq k_j}(r) \\
&= \sum_{j=1}^{2} \frac{k_j}{k_j+1} N^1_{(f,H_j),\leq k_j}(r) + \sum_{j=3}^{q} \frac{k_j}{k_j+1} N^1_{(f,H_j),\leq k_j}(r) \\
&\leq \sum_{j=1}^{2} \frac{k_j}{k_j+1} N^1_{(f,H_j),\leq k_j}(r) + \sum_{j=3}^{q} \frac{k_3}{k_3+1} N^1_{(f,H_j),\leq k_j}(r) \\
&\leq \sum_{j=1}^{2} \left(\frac{k_j}{k_j+1} - \frac{k_3}{k_3+1}\right)(1-\delta_f(H_j))T_f(r) \\
&\quad + \sum_{j=1}^{q} \frac{k_3}{k_3+1} N^1_{(f,H_j),\leq k_j}(r) + o(T_f(r)).
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
\Big\| \ & \frac{q-2n+l-1}{\gamma} T_f(r) \\
&\leq \sum_{j=1}^{q} \frac{k_3}{k_3+1} N^1_{(f,H_j),\leq k_j}(r) + \sum_{j=1}^{2} \left(\frac{k_j}{k_j+1} - \frac{k_3}{k_3+1}\right)(1-\delta_f(H_j))T_f(r) \\
&\quad + \sum_{j=1}^{q} \frac{1-\delta_f(H_j)}{k_j+1} T_f(r) + o(T_f(r)).
\end{aligned}$$

Note that $q = \sum_{j=1}^{q} \frac{k_j+1}{k_j+1}$. The above inequality implies that

$$\Big\| \ \left(\frac{2k_3}{k_3+1} + B_1\right) T_f(r) \leq \sum_{j=1}^{q} \frac{k_3}{k_3+1} N^1_{(f,H_j),\leq k_j}(r) + o(T_f(r)),$$

where

$$B_1 = \frac{\sum_{j=1}^{2} \delta_f(H_j)}{k_3+1} + \sum_{j=3}^{q} \frac{k_j+\delta_f(H_j)}{k_j+1} - \frac{\gamma q + 2n + 1 - q - l}{\gamma}.$$

For another meromorphic mapping $g \in \mathcal{G}$, we also have

$$\Big\| \ \left(\frac{2k_3}{k_3+1} + B_2\right) T_g(r) \leq \sum_{j=1}^{q} \frac{k_3}{k_3+1} N^1_{(g,H_j),\leq k_j}(r) + o(T_g(r)),$$

where

$$B_2 = \frac{\sum_{j=1}^{2} \delta_g(H_j)}{k_3+1} + \sum_{j=3}^{q} \frac{k_j + \delta_g(H_j)}{k_j+1} - \frac{\gamma q + 2n + 1 - q - l}{\gamma}.$$

Together with the above inequalities, we have

$$\Big\| \ \Big(\frac{2k_3}{k_3+1} + B_1\Big) T_f(r) + \Big(\frac{2k_3}{k_3+1} + B_2\Big) T_g(r) \\ \le \frac{k_3}{k_3+1} \sum_{j=1}^{q} (N^1_{(f,H_j),\le k_j}(r) + N^1_{(g,H_j),\le k_j}(r)) + o(T(r)),$$

where $T(r) = T_f(r) + T_g(r)$.

Assume that $f(z) \not\equiv g(z)$. Since $\nu^1_{(f,H_j),\le k_j} = \nu^1_{(g,H_j),\le k_j}$ and $f(z) = g(z)$ on $\bigcup_{j=1}^{q} \{z \in \mathbb{C}^m : 0 < \nu_{(g,H_j)} \le k_j\}$, we have

$$\sum_{j=1}^{q} (N^1_{(f,H_j),\le k_j}(r) + N^1_{(g,H_j),\le k_j}(r)) \le 2N_{f-g}(r) \le 2T(r) + O(1).$$

Therefore,

$$\| \ B_1 T_f(r) + B_2 T_g(r) \le o(T(r)).$$

This contradicts the assumption that $\max\{B_1, B_2\} > 0$ and $\min\{B_1, B_2\} \ge 0$.

4. Proof of Theorem 1.8. We shall use the technique of [6, 4] (see also [5, 14, 3]). For brevity we denote $T(r,f) + T(r,g)$ by $T(r)$. Suppose that $f(z) \not\equiv g(z)$. Then by changing indices if necessary, we may assume that

$$\underbrace{\frac{(f,H_1)}{(g,H_1)} \equiv \cdots \frac{(f,H_{k_1})}{(g,H_{k_1})}}_{\text{group 1}} \not\equiv \underbrace{\frac{(f,H_{k_1+1})}{(g,H_{k_1+1})} \equiv \cdots \equiv \frac{(f,H_{k_2})}{(g,H_{k_2})}}_{\text{group 2}} \\ \not\equiv \cdots \not\equiv \underbrace{\frac{(f,H_{k_{s-1}+1})}{(g,H_{k_{s-1}+1})} \equiv \cdots \equiv \frac{(f,H_{k_s})}{(g,H_{k_s})}}_{\text{group } s},$$

where $k_s = q$. Then the number of elements of every group is at most n because $f(z) \not\equiv g(z)$.

Define $\tau : \{1, \ldots, q\} \to \{1, \ldots, q\}$ by

$$\tau(i) = \begin{cases} i + n & \text{if } i + n \le q, \\ i + n - q & \text{if } i + n > q. \end{cases}$$

Obviously, τ is bijective. Since $q \ge 2n$, we have $|\tau(i) - i| \ge n$. Thus, $(f, H_i)/(g, H_i)$ and $(f, H_{\tau(i)})/(g, H_{\tau(i)})$ belong to distinct groups, and so $(f, H_i)/(g, H_i) \not\equiv (f, H_{\tau(i)})/(g, H_{\tau(i)})$.

Set $P_i := (f,H_i)(g,H_{\tau(i)}) - (f,H_{\tau(i)})(g,H_i) \not\equiv 0$, where $1 \le i \le q$. By the assumption and the definition of P_i we see that for $k \in \{i,\tau(i)\}$ every element z_0 of $\{z \in \mathbb{C}^m : 1 \le \nu_{(f,H_k)} \le m_k\}$ $(= \{z \in \mathbb{C}^m : 1 \le \nu_{(g,H_k)} \le m_k\})$ is a zero of P_i with

$$\nu_{P_i}(z_0) \ge \min\{\nu_{(f,H_k)}(z_0), \nu_{(g,H_k)}(z_0)\}$$

outside an analytic set of codimension ≥ 2. On the other hand, since $\nu^1_{(f,H_k),\le m_k} = \nu^1_{(g,H_k),\le m_k}$ we have

$$\begin{aligned}&\min\{\nu_{(f,H_k)}(z_0), \nu_{(g,H_k)}(z_0)\}\\ &\qquad \ge \nu^n_{(f,H_k),\le m_k}(z_0) + \nu^n_{(g,H_k),\le m_k}(z_0) - n\nu^1_{(f,H_k),\le m_k}(z_0).\end{aligned}$$

We also see that for any $j \in \{1,\dots,q\} \setminus \{i,\tau(i)\}$, any zero of (f,H_j) is a zero of P_i outside an analytic set of codimension ≥ 2. Thus

$$\begin{aligned}\nu_{P_i} \ge{}& \nu^n_{(f,H_i),\le m_i} + \nu^n_{(f,H_{\tau(i)}),\le m_{\tau(i)}} + \nu^n_{(g,H_i),\le m_i} + \nu^n_{(g,H_{\tau(i)}),\le m_{\tau(i)}}\\ &- n\nu^1_{(f,H_i),\le m_i} - n\nu^1_{(f,H_{\tau(i)}),\le m_{\tau(i)}} + \sum_{j=1,\, j\ne i,\tau(i)}^{q} \nu^1_{(f,H_j),\le m_j}\end{aligned}$$

outside an analytic set of codimension ≥ 2. Hence, for all $i \in \{1,\dots,q\}$,

$$\begin{aligned}N_{P_i} \ge{}& N^n_{(f,H_i),\le m_i}(r) + N^n_{(f,H_{\tau(i)}),\le m_{\tau(i)}}(r) + N^n_{(g,H_i),\le m_i}(r)\\ &+ N^n_{(g,H_{\tau(i)}),\le m_{\tau(i)}}(r) - nN^1_{(f,H_i),\le m_i}(r) - nN^1_{(f,H_{\tau(i)}),\le m_{\tau(i)}}(r)\\ &+ \sum_{j=1,\, j\ne i,\tau(i)}^{q} N^1_{(f,H_j),\le m_j}(r).\end{aligned}$$

On the other hand, by Jensen's formula we have

$$\begin{aligned}N_{P_i}(r) &= \int_{S(r)} \log|P_i|\sigma + O(1)\\ &\le \int_{S(r)} \log(|(f,H_i)|^2 + |(f,H_{\tau(i)})|^2)^{1/2}\,\sigma\\ &\quad + \int_{S(r)} \log(|(g,H_i)|^2 + |(g,H_{\tau(i)})|^2)^{1/2}\,\sigma + O(1)\\ &\le T(r) + O(1).\end{aligned}$$

Therefore, for all $i \in \{1,\dots,q\}$,

$$\begin{aligned}&T(r) + O(1)\\ &\ge N^n_{(f,H_i),\le m_i}(r) + N^n_{(f,H_{\tau(i)}),\le m_{\tau(i)}}(r) + N^n_{(g,H_i),\le m_i}(r) + N^n_{(g,H_{\tau(i)}),\le m_{\tau(i)}}(r)\\ &\quad - nN^1_{(f,H_i),\le m_i}(r) - nN^1_{(f,H_{\tau(i)}),\le m_{\tau(i)}}(r) + \sum_{j=1,j\ne i,\tau(i)}^{q} N^1_{(f,H_j),\le m_j}(r).\end{aligned}$$

Note that τ is bijective. Summing the above inequality over $1 \le i \le q$, we have

$$(q-2n-2)\sum_{j=1}^{q} N^{1}_{(f,H_j),\le m_j}(r) + 2\sum_{j=1}^{q}(N^{n}_{(f,H_j),\le m_j}(r) + N^{n}_{(g,H_j),\le m_j}(r)) \\ \le qT(r) + O(1).$$

By a similar discussion for g, we have

$$(q-2n-2)\sum_{j=1}^{q} N^{1}_{(g,H_j),\le m_j}(r) + 2\sum_{j=1}^{q}(N^{n}_{(g,H_j),\le m_j}(r) + N^{n}_{(g,H_j),\le m_j}(r)) \\ \le qT(r) + O(1).$$

Noting that $(1/n)N^{n}_{(f,H_j),\le m_j}(r) \le N^{1}_{(f,H_j),\le m_j}(r)$, from the above inequalities we get

$$\frac{q+2n-2}{2n}\sum_{j=1}^{q}(N^{n}_{(f,H_j),\le m_j}(r) + N^{n}_{(g,H_j),\le m_j}(r)) \le qT(r) + O(1).$$

Now by a similar discussion as in the proof of Lemma 3.1, the Second Main Theorem yields

$$\begin{aligned} \| \ (q-n-1)T(r) &\le \sum_{i=1}^{q}(N^{n}_{(f,H_i),\le m_i}(r) + N^{n}_{(g,H_i),\le m_i}(r)) \\ &\quad + \sum_{i=1}^{q}(N^{n}_{(f,H_i),\ge m_i+1}(r) + N^{n}_{(g,H_i),\ge m_i+1}(r)) + o(T(r)) \\ &\le \sum_{i=1}^{q}\left(1-\frac{n}{m_i+1}\right)(N^{n}_{(f,H_i),\le m_i}(r) + N^{n}_{(g,H_i),\le m_i}(r)) \\ &\quad + \sum_{i=1}^{q}\frac{n}{m_i+1}(N_{(f,H_i)}(r) + N_{(g,H_i)}(r)) + o(T(r)) \\ &\le \sum_{i=1}^{q}\left(1-\frac{n}{m_i+1}\right)(N^{n}_{(f,H_i),\le m_i}(r) + N^{n}_{(g,H_i),\le m_i}(r)) \\ &\quad + \sum_{i=1}^{q}\frac{n}{m_i+1}(T(r)) + o(T(r)) \\ &\le \left(1-\frac{n}{m_1+1}\right)\sum_{i=1}^{q}(N^{n}_{(f,H_i),\le m_i}(r) + N^{n}_{(g,H_i),\le m_i}(r)) \\ &\quad + \sum_{i=1}^{q}\frac{n}{m_i+1}(T(r)) + o(T(r)). \end{aligned}$$

Therefore, removing the term $\sum_{i=1}^{q}(N^{n}_{(f,H_i),\le m_i}(r)+N^{n}_{(g,H_i),\le m_i}(r))$ from the above inequalities we have

$$\Big\|\ \Big(\frac{(q+2n-2)(q-n-1)}{2n}-q+\frac{nq}{m_1+1}-\frac{q+2n-2}{2}\sum_{j=1}^{q}\frac{1}{m_j+1}\Big)T(r)\le o(T(r)).$$

Hence,

$$\Big\|\ \Big(\frac{q-n-1}{n}-\frac{2q}{q+2n-2}+\frac{2nq}{q+2n-2}\cdot\frac{1}{m_1+1}-\sum_{j=1}^{q}\frac{1}{m_j+1}\Big)T(r)\le o(T(r)).$$

Noting that $q=\sum_{j=1}^{q}\frac{m_j+1}{m_j+1}$, we deduce from the above inequality that

$$\Big\|\ \Big(\sum_{j=3}^{q}\frac{m_j}{m_j+1}-\frac{nq-q+n+1}{n}+\frac{4n-4}{q+2n-2}\Big)T(r)\le\Big(\frac{1}{m_1+1}+\frac{1}{m_2+1}-\frac{2nq}{q+2n-2}\cdot\frac{1}{m_1+1}\Big)T(r)+o(T(r)).$$

This is a contradiction.

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References

[1] Y. Aihara, *Unicity theorems for meromorphic mappings with deficiencies*, Complex Variables 42 (2000), 259–268.

[2] V. H. An and T. D. Duc, *Uniqueness theorems and uniqueness polynomials for holomorphic curves*, Complex Var. Elliptic Equations 56 (2011), 253–262.

[3] T. B. Cao and H. X. Yi, *Uniqueness theorems for meromorphic mappings sharing hyperplanes in general position*, Sci. Sinica Math. 41 (2011), 135–144 (in Chinese).

[4] Z. H. Chen and Q. M. Yan, *Uniqueness theorem of meromorphic mappings into* $\mathbb{P}^N(\mathbb{C})$ *sharing* $2N+3$ *hyperplanes regardless of multiplicities*, Int. J. Math. 20 (2009), 717–726.

[5] G. Dethloff, S. D. Quang and T. V. Tan, *A uniqueness theorem for meromorphic mappings with two families of hyperplanes*, Proc. Amer. Math. Soc. 140 (2012), 189–197.

[6] G. Dethloff and T. V. Tan, *Uniqueness theorems for meromorphic mappings with few hyperplanes*, Bull. Sci. Math. 133 (2009), 501–514.

[7] G. Dethloff and T. V. Tan, *An extension of uniqueness theorems for meromorphic mappings*, Vietnam J. Math. 34 (2006), 71–94.

[8] H. Fujimoto, *The uniqueness problem of meromorphic maps into the complex projective spaces*, Nagoya Math. J. 58 (1975), 1–23.

[9] H. Fujimoto, *The uniqueness theorem for algebraically non-degenerate meromorphic maps into* $\mathbb{P}^N(\mathbb{C})$, Nagoya Math. J. 64 (1976), 117–147.

[10] H. S. Gopalakrishna and S. S. Bootsnuramath, *Uniqueness theorems for meromorphic functions*, Math. Scand. 39 (1976), 125–130.

[11] P. C. Hu, P. Li and C. C. Yang, *Unicity of Meromorphic Mappings*, Kluwer, 2003.

[12] R. Nevanlinna, *Eindeutigkeitssätze in der Theorie der meromorphen Funktionen*, Acta Math. 48 (1926), 367–391.

[13] J. Noguchi, *On Nevanlinna's second main theorem*, in: Geometric Complex Analysis, (Hayama, 1995), J. Noguchi et al. (eds.), Word Sci., Singapore, 1996, 489–503.

[14] S. D. Quang, *Unicity of meromorphic mappings sharing few hyperplanes*, Ann. Polon. Math. 102 (2011), 255–270.

[15] M. Ru, *Nevanlinna Theory and its Relation to Diophantine Approximation*, World Sci., Singapore, 2001.

[16] L. Smiley, *Geometric conditions for unicity of holomorphic curves*, in: Contemp. Math. 25, Amer. Math. Soc., 1983, 149–154.

[17] D. D. Thai and S. D. Quang, *Uniqueness problem with truncated multiplicities of meromorphic mappings in several complex variables*, Int. J. Math. 17 (2006), 1223–1257.

[18] H. Ueda, *Unicity theorems for entire or meromorphic functions*, Kodai Math. J. 3 (1980), 457–471.

[19] L. Yang, *Multiple values of meromorphic functions and functions combination*, Acta Math. Sinica Chinese Ser. 14 (1964), 428–437 (in Chinese).

[20] H. X. Yi, *On the multiple values and uniqueness of meromorphic functions*, Chinese Ann. Math. Ser. A 10 (1989), 421–427 (in Chinese).

[21] H. X. Yi and C. C. Yang, *Uniqueness Theory of Meromorphic Functions*, Science Press, 1995, and Kluwer, 2003.

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