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Lempert theorem for strongly linearly convex domains

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Abstract. In 1984 L. Lempert showed that the Lempert function and the Carathéodory distance coincide on non-planar bounded strongly linearly convex domains with real-analytic boundaries. Following his paper, we present a slightly modified and more detailed version of the proof. Moreover, the Lempert Theorem is proved for non-planar bounded strongly linearly convex domains.

The aim of this paper is to present a detailed version of the proof of the Lempert Theorem in the case of non-planar bounded strongly linearly convex domains. Lempert's original proof is presented only in the proceedings of a conference (see [6]) with a very limited access and at some places it was quite sketchy. We were encouraged by some colleagues to prepare an extended version of the proof, in which all doubts could be removed and some details of the proofs could be simplified. We hope to have done it below. Certainly, *the idea of the proof belongs entirely to Lempert*. The main differences we would like to draw attention to are:

- the results are obtained in the $\mathcal{C}^2$-smooth case;
- the notions of stationary mappings and E-mappings are separated;
- a geometry of domains is investigated only in neighborhoods of boundaries of stationary mappings (viewed as boundaries of analytic discs)—this allows us to obtain localization properties for stationary mappings.

Additional motivation for presenting the proof is the fact, showed recently in [7], that the so-called symmetrized bidisc may be exhausted by strongly linearly convex domains. On the other hand, it cannot be exhausted by domains biholomorphic to convex ones ([1]). Therefore, the equality of the Lempert function and the Carathéodory distance for strongly linearly convex domains does not follow directly from [5].

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1. Introduction and results. Let us recall the objects we will deal with. Throughout the paper, $\mathbb{D}$ denotes the unit open disc on the complex plane, $\mathbb{T}$ is the unit circle and p is the Poincaré distance on $\mathbb{D}$.

Let $D \subset \mathbb{C}^n$ be a domain and let $z, w \in D$, $v \in \mathbb{C}^n$. The *Lempert function* is defined as

$$(1)\quad \widetilde{k}_D(z,w) := \inf\{p(0,\xi) : \xi \in [0,1) \text{ and } \exists f \in \mathcal{O}(\mathbb{D}, D) : f(0) = z,\ f(\xi) = w\}.$$

The *Kobayashi–Royden (pseudo)metric* is

$$(2)\quad \kappa_D(z;v) := \inf\{\lambda^{-1} : \lambda > 0 \text{ and } \exists f \in \mathcal{O}(\mathbb{D}, D) : f(0) = z,\ f'(0) = \lambda v\}.$$

Note that

$$(3)\quad \widetilde{k}_D(z,w) = \inf\{p(\zeta,\xi) : \zeta, \xi \in \mathbb{D} \text{ and } \exists f \in \mathcal{O}(\mathbb{D}, D) : f(\zeta) = z,\ f(\xi) = w\},$$

$$(4)\quad \kappa_D(z;v) = \inf\{|\lambda|^{-1}/(1-|\zeta|^2) : \lambda \in \mathbb{C}_*,\ \zeta \in \mathbb{D} \text{ and } \exists f \in \mathcal{O}(\mathbb{D}, D) : f(\zeta) = z,\ f'(\zeta) = \lambda v\}.$$

If $z \neq w$ (resp. $v \neq 0$), a mapping f for which the infimum in (3) (resp. in (4)) is attained is called a $\widetilde{k}_D$-*extremal* (or a *Lempert extremal*) for z, w (resp. a κ_D-*extremal* for z, v). A mapping which is a $\widetilde{k}_D$-extremal or a κ_D-extremal will be called just an *extremal* or an *extremal mapping*.

We shall say that $f : \mathbb{D} \to D$ is a unique $\widetilde{k}_D$-extremal for z, w (resp. a unique κ_D-extremal for z, v) if any other $\widetilde{k}_D$-extremal $g : \mathbb{D} \to D$ for z, w (resp. κ_D-extremal for z, v) satisfies $g = f \circ a$ for some Möbius function a.

The next objects we are dealing with are the *Carathéodory (pseudo) distance*

$$c_D(z,w) := \sup\{p(F(z), F(w)) : F \in \mathcal{O}(D, \mathbb{D})\}$$

and the *Carathéodory–Reiffen (pseudo)metric*

$$\gamma_D(z;v) := \sup\{|F'(z)v| : F \in \mathcal{O}(D, \mathbb{D}),\ F(z) = 0\}.$$

A holomorphic mapping $f : \mathbb{D} \to D$ is said to be a *complex geodesic* if $c_D(f(\zeta), f(\xi)) = p(\zeta, \xi)$ for any $\zeta, \xi \in \mathbb{D}$.

Here is some more notation. Let $z_1, \ldots, z_n$ be the standard complex coordinates in $\mathbb{C}^n$ and $x_1, \ldots, x_{2n}$ the corresponding real coordinates in $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n \simeq \mathbb{R}^{2n}$. We use $T_D^{\mathbb{R}}(a)$, $T_D^{\mathbb{C}}(a)$ to denote the real and the complex tangent space to a $\mathcal{C}^1$-smooth domain D at a point $a \in \partial D$, i.e. the sets

$$T_D^{\mathbb{R}}(a) := \left\{X \in \mathbb{C}^n : \operatorname{Re} \sum_{j=1}^n \frac{\partial r}{\partial z_j}(a) X_j = 0\right\},$$

$$T_D^{\mathbb{C}}(a) := \left\{X \in \mathbb{C}^n : \sum_{j=1}^n \frac{\partial r}{\partial z_j}(a) X_j = 0\right\},$$

where r is a defining function of D. Let $\nu_D(a)$ be the outward unit normal vector to ∂D at a.

Let $\mathcal{C}^k(\overline{\mathbb{D}})$, where $k \in (0,\infty]$, denote the class of continuous functions on $\overline{\mathbb{D}}$ which are of class $\mathcal{C}^k$ on $\mathbb{D}$ and

- if $k \in \mathbb{N} \cup \{\infty\}$ then derivatives in $\mathbb{D}$ up to order k extend continuously on $\overline{\mathbb{D}}$;
- if $k - [k] =: c > 0$ then derivatives in $\mathbb{D}$ up to order $[k]$ extend to c-Hölder continuous functions on $\overline{\mathbb{D}}$.

We shall denote by $\mathcal{C}^\omega$ the class of real-analytic functions. Further, saying that f is of class $\mathcal{C}^k(\mathbb{T})$, $k \in (0,\infty] \cup \{\omega\}$, we mean that the function $t \mapsto f(e^{it})$, $t \in \mathbb{R}$, is in $\mathcal{C}^k(\mathbb{R})$. For a compact set $K \subset \mathbb{C}^n$ let $\mathcal{O}(K)$ denote the set of functions extending holomorphically on a neighborhood of K (we assume that all neighborhoods are open). In that case we shall sometimes say that a given function is of class $\mathcal{O}(K)$. Note that $\mathcal{C}^\omega(\mathbb{T}) = \mathcal{O}(\mathbb{T})$.

Let $|\cdot|$ denote the Euclidean norm in $\mathbb{C}^n$ and let $\operatorname{dist}(z,S) := \inf\{|z-s| : s \in S\}$ be the distance of the point $z \in \mathbb{C}^n$ to a set $S \subset \mathbb{C}^n$. Moreover, for $S \subset \mathbb{C}^n$ we define $S_* := S \setminus \{0\}$. Let $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| = 1\}$ be the unit ball and $B_n(a,r) := \{z \in \mathbb{C}^n : |z-a| < r\}$ the open ball with center $a \in \mathbb{C}^n$ and radius $r > 0$. Put

$$z \bullet w := \sum_{j=1}^{n} z_j w_j$$

for $z, w \in \mathbb{C}^n$ and let $\langle \cdot\,, - \rangle$ be the hermitian inner product on $\mathbb{C}^n$. The real inner product on $\mathbb{C}^n$ is denoted by $\langle \cdot\,, - \rangle_{\mathbb{R}} = \operatorname{Re}\langle \cdot\,, - \rangle$.

We use ∇ to denote the gradient $(\partial/\partial x_1, \ldots, \partial/\partial x_{2n})$. For real-valued functions the gradient is naturally identified with $2(\partial/\partial \overline{z}_1, \ldots, \partial/\partial \overline{z}_n)$. Recall that

$$\nu_D(a) = \frac{\nabla r(a)}{|\nabla r(a)|}.$$

Let $\mathcal{H}$ be the Hessian matrix

$$\left[\frac{\partial^2}{\partial x_j \partial x_k}\right]_{1 \le j,k \le 2n}.$$

Sometimes, for a $\mathcal{C}^2$-smooth function u and a vector $X \in \mathbb{R}^{2n}$ the Hessian

$$\sum_{j,k=1}^{2n} \frac{\partial^2 u}{\partial x_j \partial x_k}(a) X_j X_k = X^T \mathcal{H}u(a) X$$

will be denoted by $\mathcal{H}u(a;X)$. We denote by $\|\cdot\|$ the operator norm.

Definition 1.1. Let $D \subset \mathbb{C}^n$ be a domain. We say that D is *linearly convex* (resp. *weakly linearly convex*) if through any point $a \in \mathbb{C}^n \setminus D$ (resp. $a \in \partial D$) there passes an $(n-1)$-dimensional complex hyperplane disjoint from D.

A domain D is said to be *strongly linearly convex* if

- D has $\mathcal{C}^2$-smooth boundary;
- there exists a defining function r of D such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(a) X_j \overline{X}_k > \left| \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(a) X_j X_k \right|, \quad a \in \partial D,\ X \in T_D^{\mathbb{C}}(a)_*. \tag{5}$$

More generally, any point $a \in \partial D$ for which there exists a defining function r satisfying (5) is called a *point of strong linear convexity* of D.

Furthermore, we say that a domain D has *real-analytic boundary* if it possesses a real-analytic defining function.

Note that the condition (5) does not depend on the choice of the defining function of D.

Remark 1.2. Let $D \subset \mathbb{C}^n$ be a strongly linearly convex domain. Then

- any $(n-1)$-dimensional complex tangent hyperplane intersects ∂D at precisely one point; in other words
$$\overline{D} \cap (a + T_D^{\mathbb{C}}(a)) = \{a\}, \qquad a \in \partial D;$$
- for $a \in \partial D$ the equation $\langle w - a, \nu_D(a) \rangle = 0$ describes the $(n-1)$-dimensional complex tangent hyperplane $a + T_D^{\mathbb{C}}(a)$, consequently
$$\langle z - a, \nu_D(a) \rangle \neq 0, \qquad z \in D,\ a \in \partial D.$$

The main aim of the paper is to present a detailed proof of

Theorem 1.3 (Lempert Theorem). *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strongly linearly convex domain. Then*

$$c_D = \widetilde{k}_D \quad \textit{and} \quad \gamma_D = \kappa_D.$$

Important roles will be played by strongly convex domains and strongly convex functions.

Definition 1.4. A domain $D \subset \mathbb{C}^n$ is called *strongly convex* if

- D has $\mathcal{C}^2$-smooth boundary;
- there exists a defining function r of D such that

$$\sum_{j,k=1}^{2n} \frac{\partial^2 r}{\partial x_j \partial x_k}(a) X_j X_k > 0, \qquad a \in \partial D,\ X \in T_D^{\mathbb{R}}(a)_*. \tag{6}$$

Generally, any point $a \in \partial D$ for which there exists a defining function r satisfying (6) is called a *point of strong convexity* of D.

Remark 1.5. A strongly convex domain $D \subset \mathbb{C}^n$ is convex and strongly linearly convex. Moreover, it is strictly convex, i.e. for any different points

$a, b \in \overline{D}$ the interior of the segment $[a, b] = \{ta + (1-t)b : t \in [0,1]\}$ is contained in D (i.e. $ta + (1-t)b \in D$ for any $t \in (0,1)$).

Observe also that any bounded convex domain with real-analytic boundary is strictly convex. Actually, if a domain D with real-analytic boundary were not strictly convex, then we would be able to find two distinct points $a, b \in \partial D$ such that the segment $[a, b]$ lies entirely in ∂D. On the other hand, the identity principle would imply that the set

$$\{t \in \mathbb{R} : \exists \varepsilon > 0 : sa + (1-s)b \in \partial D \text{ for } |s-t| < \varepsilon\}$$

is open-closed in $\mathbb{R}$. Therefore it has to be empty. This immediately gives a contradiction.

Remark 1.6. It is well-known that for any convex domain $D \subset \mathbb{C}^n$ there is a sequence $\{D_m\}$ of bounded strongly convex domains with real-analytic boundaries and such that $D_m \subset D_{m+1}$ and $\bigcup_m D_m = D$.

In particular, Theorem 1.3 holds for convex domains.

Definition 1.7. Let $U \subset \mathbb{C}^n$ be a domain. A function $u : U \to \mathbb{R}$ is called *strongly convex* if

- u is $\mathcal{C}^2$-smooth;
- $$\sum_{j,k=1}^{2n} \frac{\partial^2 u}{\partial x_j \partial x_k}(a) X_j X_k > 0, \qquad a \in U,\ X \in (\mathbb{R}^{2n})_*.$$

Definition 1.8. The degree of a continuous function (treated as a curve) $\mathbb{T} \to \mathbb{T}$ is called its *winding number*. The fundamental group is a homotopy invariant. Thus the definition of the winding number of a continuous function $\varphi : \mathbb{T} \to \mathbb{C}_*$ is the same. We denote it by $\operatorname{wind} \varphi$.

In the case of a $\mathcal{C}^1$-smooth function $\varphi : \mathbb{T} \to \mathbb{C}_*$, its winding number is just the index of φ at 0, i.e.

$$\operatorname{wind} \varphi = \frac{1}{2\pi i} \int\limits_{\varphi(\mathbb{T})} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int\limits_0^{2\pi} \frac{\frac{d}{dt}\varphi(e^{it})}{\varphi(e^{it})}\, dt.$$

Remark 1.9.

- If $\varphi \in \mathcal{C}(\mathbb{T}, \mathbb{C}_*)$ extends to a function $\widetilde{\varphi} \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ then $\operatorname{wind} \varphi$ is the number of zeroes of $\widetilde{\varphi}$ in $\mathbb{D}$ counted with multiplicities;
- $\operatorname{wind}(\varphi\psi) = \operatorname{wind} \varphi + \operatorname{wind} \psi$ for $\varphi, \psi \in \mathcal{C}(\mathbb{T}, \mathbb{C}_*)$;
- $\operatorname{wind} \varphi = 0$ if $\varphi \in \mathcal{C}(\mathbb{T})$ and $\operatorname{Re} \varphi > 0$.

Definition 1.10. The boundary of a domain D of $\mathbb{C}^n$ is *real-analytic in a neighborhood* U of the set $S \subset \partial D$ if there exists a function $r \in \mathcal{C}^\omega(U, \mathbb{R})$ such that $D \cap U = \{z \in U : r(z) < 0\}$ and ∇r does not vanish in U.

Definition 1.11. Let $D \subset \mathbb{C}^n$ be a domain with $\mathcal{C}^1$-smooth boundary. We call a holomorphic mapping $f : \mathbb{D} \to D$ a *stationary mapping* if

(1) f extends to a holomorphic mapping (denoted by the same letter) in a neighborhood of $\overline{\mathbb{D}}$;
(2) $f(\mathbb{T}) \subset \partial D$;
(3) there exists a real-analytic function $\rho : \mathbb{T} \to \mathbb{R}_{>0}$ such that the mapping $\mathbb{T} \ni \zeta \mapsto \zeta\rho(\zeta)\overline{\nu_D(f(\zeta))} \in \mathbb{C}^n$ extends to a mapping (denoted by $\widetilde{f}$) holomorphic in a neighborhood of $\overline{\mathbb{D}}$.

Furthermore, we call a holomorphic mapping $f : \mathbb{D} \to D$ a *weak stationary mapping* if

(1′) f extends to a $\mathcal{C}^{1/2}$-smooth mapping (denoted by the same letter) on $\overline{\mathbb{D}}$;
(2′) $f(\mathbb{T}) \subset \partial D$;
(3′) there exists a $\mathcal{C}^{1/2}$-smooth function $\rho : \mathbb{T} \to \mathbb{R}_{>0}$ such that the mapping $\mathbb{T} \ni \zeta \mapsto \zeta\rho(\zeta)\overline{\nu_D(f(\zeta))} \in \mathbb{C}^n$ extends to a mapping $\widetilde{f} \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}^{1/2}(\overline{\mathbb{D}})$.

The definition of a (weak) stationary mapping $f : \mathbb{D} \to D$ extends naturally to the case when ∂D is real-analytic in a neighborhood of $f(\mathbb{T})$.

Directly from the definition of a stationary mapping f, it follows that f and $\widetilde{f}$ extend holomorphically on some neighborhoods of $\overline{\mathbb{D}}$. We shall denote their intersection by $\mathbb{D}_f$.

Definition 1.12. Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strongly linearly convex domain. A holomorphic mapping $f : \mathbb{D} \to D$ is called a (*weak*) *E-mapping* if it is a (weak) stationary mapping and

(4) setting $\varphi_z(\zeta) := \langle z - f(\zeta), \nu_D(f(\zeta))\rangle$, $\zeta \in \mathbb{T}$, we have $\operatorname{wind} \varphi_z = 0$ for some $z \in D$.

Remark 1.13. The strong linear convexity of D implies $\varphi_z(\zeta) \neq 0$ for any $z \in D$ and $\zeta \in \mathbb{T}$. Therefore, $\operatorname{wind} \varphi_z$ vanishes for all $z \in D$ if it vanishes for some $z \in D$.

Additionally, any stationary mapping of a convex domain is an E-mapping (as $\operatorname{Re} \varphi_z < 0$).

We shall prove that in a class of non-planar bounded strongly linearly convex domains with real-analytic boundaries, weak stationary mappings are just stationary mappings, so there is no difference between E-mappings and weak E-mappings.

We have the following result describing extremal mappings.

THEOREM 1.14. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strongly linearly convex domain. Then a holomorphic mapping $f : \mathbb{D} \to D$ is an extremal if and only if it is a weak E-mapping.*

For a domain D with real-analytic boundary, a holomorphic mapping $f : \mathbb{D} \to D$ is an extremal if and only if it is an E-mapping.

If ∂D is of class $\mathcal{C}^k$, $k = 3, 4, \ldots, \infty$, then any weak E-mapping $f : \mathbb{D} \to D$ and its associated mappings $\widetilde{f}, \rho$ are $\mathcal{C}^{k-1-\varepsilon}$-smooth for any $\varepsilon > 0$.

The idea of the proof of the Lempert Theorem is as follows. In the real-analytic case we shall show that E-mappings are complex geodesics (because they have left inverses). Then we shall prove that for any different points $z, w \in D$ (resp. for a point $z \in D$ and a vector $v \in (\mathbb{C}^n)_*$) there is an E-mapping passing through z, w (resp. such that $f(0) = z$ and $f'(0) = v$). This will give the equality between the Lempert function and the Carathéodory distance. In the general case, we exhaust a $\mathcal{C}^2$-smooth domain by strongly linearly convex domains with real-analytic boundaries.

To prove Theorem 1.14, we shall additionally observe that (weak) E-mappings are unique extremals.

In what follows and if not mentioned otherwise, $D \subset \mathbb{C}^n$, $n \geq 2$, is a *bounded strongly linearly convex domain with real-analytic boundary.*

2. Weak stationary mappings of strongly linearly convex domains with real-analytic boundaries are stationary mappings

PROPOSITION 2.1. *A weak stationary mapping of D is a stationary mapping of D with the same associated mappings.*

Proof. Let $f : \mathbb{D} \to D$ be a weak stationary mapping. Our aim is to prove that $f, \widetilde{f} \in \mathcal{O}(\overline{\mathbb{D}})$ and $\rho \in \mathcal{C}^\omega(\mathbb{T})$.

Choose $\zeta_0 \in \mathbb{T}$. Since $\widetilde{f}(\zeta_0) \neq 0$, we can assume that $\widetilde{f}_1 \neq 0$ in $\overline{\mathbb{D}} \cap U_0$, where U_0 is a neighborhood of ζ_0. This implies $\nu_{D,1}(f(\zeta_0)) \neq 0$, so $\nu_{D,1}$ does not vanish on some set $V_0 \subset \partial D$, relatively open in ∂D, containing the point $f(\zeta_0)$. Shrinking U_0, if necessary, we may assume that $f(\mathbb{T} \cap U_0) \subset V_0$.

Define $\psi : V_0 \to \mathbb{C}^{2n-1}$ by

$$\psi(z) = \left(z_1, \ldots, z_n, \overline{\left(\frac{\nu_{D,2}(z)}{\nu_{D,1}(z)} \right)}, \ldots, \overline{\left(\frac{\nu_{D,n}(z)}{\nu_{D,1}(z)} \right)} \right).$$

The set $M := \psi(V_0)$ is the graph of a $\mathcal{C}^\omega$ function defined on the local $\mathcal{C}^\omega$ submanifold V_0, so it is a local $\mathcal{C}^\omega$ submanifold in $\mathbb{C}^{2n-1}$ of real dimension $2n-1$. Direct calculations show that M is totally real.

Let

$$g(\zeta) := \left(f_1(\zeta), \ldots, f_n(\zeta), \frac{\widetilde{f}_2(\zeta)}{\widetilde{f}_1(\zeta)}, \ldots, \frac{\widetilde{f}_n(\zeta)}{\widetilde{f}_1(\zeta)} \right), \qquad \zeta \in \overline{\mathbb{D}} \cap U_0.$$

If $\zeta \in \mathbb{T} \cap U_0$ then $\widetilde{f}_k(\zeta)\widetilde{f}_1(\zeta)^{-1} = \overline{\nu_{D,k}(f(\zeta))}\,\overline{\nu_{D,1}(f(\zeta))}^{-1}$, so $g(\zeta) = \psi(f(\zeta))$. Therefore, $g(\mathbb{T} \cap U_0) \subset M$. Thanks to the Reflection Principle (see Appendix), g extends holomorphically past $\mathbb{T} \cap U_0$, so f extends holomorphically on a neighborhood of ζ_0.

The mapping $\overline{\nu_D \circ f}$ is real-analytic on $\mathbb{T}$, so it extends to a mapping h holomorphic in a neighborhood W of $\mathbb{T}$. For $\zeta \in \mathbb{T} \cap U_0$ we have

$$\frac{\zeta h_1(\zeta)}{\widetilde{f}_1(\zeta)} = \frac{1}{\rho(\zeta)}.$$

The function on the left side is holomorphic in $\mathbb{D} \cap U_0 \cap W$ and continuous in $\overline{\mathbb{D}} \cap U_0 \cap W$. Since it has real values on $\mathbb{T} \cap U_0$, the Reflection Principle implies that it is holomorphic in a neighborhood of $\mathbb{T} \cap U_0$. Hence ρ and $\widetilde{f}$ are holomorphic in a neighborhood of ζ_0. ∎

3. (Weak) E-mappings vs. extremal mappings and complex geodesics. In this section we will prove some important properties of (weak) E-mappings. In particular, we will show that they are complex geodesics and unique extremals.

3.1. Weak E-mappings are complex geodesics and unique extremals. The results of this subsection concern weak E-mappings of bounded strongly linearly convex domains $D \subset \mathbb{C}^n$, $n \geq 2$.

Let

$$G(z, \zeta) := (z - f(\zeta)) \bullet \widetilde{f}(\zeta), \qquad z \in \mathbb{C}^n,\ \zeta \in \mathbb{D}_f.$$

PROPOSITION 3.1.1. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strongly linearly convex domain and let $f : \mathbb{D} \to D$ be a weak E-mapping. Then there exist an open set $W \supset \overline{D} \setminus f(\mathbb{T})$ and a holomorphic mapping $F : W \to \mathbb{D}$ such that for any $z \in W$ the number $F(z)$ is a unique solution of the equation $G(z, \zeta) = 0$, $\zeta \in \mathbb{D}$. In particular, $F \circ f = \mathrm{id}_{\mathbb{D}}$.*

In what follows we will strengthen the above proposition for domains with real-analytic boundaries (see Proposition 3.2.4).

Proof of Proposition 3.1.1. Set $A := \overline{D} \setminus f(\mathbb{T})$. Since D is strongly linearly convex, φ_z does not vanish in $\mathbb{T}$ for any $z \in A$, so by a continuity argument the condition (4) of Definition 1.12 holds for every z in some open set $W \supset A$. For a fixed $z \in W$ we have

$$G(z, \zeta) = \zeta \rho(\zeta) \varphi_z(\zeta), \qquad \zeta \in \mathbb{T},$$

so $\operatorname{wind} G(z, \cdot\,) = 1$. Since $G(z, \cdot\,) \in \mathcal{O}(\mathbb{D})$, it has in $\mathbb{D}$ exactly one simple root $F(z)$. Hence $G(z, F(z)) = 0$ and $\frac{\partial G}{\partial \zeta}(z, F(z)) \neq 0$. By the Implicit Function Theorem, F is holomorphic in W. The equality $F(f(\zeta)) = \zeta$ for $\zeta \in \mathbb{D}$ is clear. ∎

From the proposition above we immediately get

COROLLARY 3.1.2. *Every weak E-mapping $f : \mathbb{D} \to D$ of a bounded strongly linearly convex domain $D \subset \mathbb{C}^n$, $n \geq 2$, is a complex geodesic. In particular,*

$$c_D(f(\zeta), f(\xi)) = \widetilde{k}_D(f(\zeta), f(\xi)) \quad \textit{and} \quad \gamma_D(f(\zeta); f'(\zeta)) = \kappa_D(f(\zeta); f'(\zeta))$$

for any $\zeta, \xi \in \mathbb{D}$.

Using left inverses of weak E-mappings we may prove the uniqueness of extremals.

PROPOSITION 3.1.3. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strongly linearly convex domain and let $f : \mathbb{D} \to D$ be a weak E-mapping. Then for any $\xi \in (0,1)$ the mapping f is a unique $\widetilde{k}_D$-extremal for $z = f(0)$, $w = f(\xi)$ (resp. a unique κ_D-extremal for $z = f(0)$, $v = f'(0)$).*

Proof. Suppose that g is a $\widetilde{k}_D$-extremal for z, w (resp. a κ_D-extremal for z, v) such that $g(0) = z$, $g(\xi) = w$ (resp. $g(0) = z$, $g'(0) = v$). Our aim is to show that $f = g$. Proposition 3.1.1 provides us with the mapping F, which is a left inverse for f. By the Schwarz Lemma, F is a left inverse for g as well, that is, $F \circ g = \mathrm{id}_{\mathbb{D}}$. We claim that $\lim_{\mathbb{D} \ni \zeta \to \zeta_0} g(\zeta) = f(\zeta_0)$ for any $\zeta_0 \in \mathbb{T}$ (in particular, we shall show that the limit exists).

Assume the contrary. Then there are $\zeta_0 \in \mathbb{T}$ and a sequence $\{\zeta_m\} \subset \mathbb{D}$ convergent to ζ_0 such that the limit $Z := \lim_{m \to \infty} g(\zeta_m) \in \overline{D}$ exists and is not equal to $f(\zeta_0)$. We have $G(z, F(z)) = 0$, so putting $z = g(\zeta_m)$ we infer that

$$0 = (g(\zeta_m) - f(F(g(\zeta_m)))) \bullet \widetilde{f}(F(g(\zeta_m))) = (g(\zeta_m) - f(\zeta_m)) \bullet \widetilde{f}(\zeta_m).$$

Letting $m \to \infty$ we get

$$0 = (Z - f(\zeta_0)) \bullet \widetilde{f}(\zeta_0) = \zeta_0 \rho(\zeta_0) \langle Z - f(\zeta_0), \nu_D(f(\zeta_0)) \rangle.$$

This means that $Z - f(\zeta_0) \in T_D^{\mathbb{C}}(f(\zeta_0))$. Since D is strongly linearly convex, we deduce that $Z = f(\zeta_0)$, which is a contradiction.

Hence g extends continuously on $\overline{\mathbb{D}}$ and, by the maximum principle, $g = f$. ∎

PROPOSITION 3.1.4. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strongly linearly convex domain, let $f : \mathbb{D} \to D$ be a weak E-mapping and let a be an automorphism of $\mathbb{D}$. Then $f \circ a$ is a weak E-mapping of D.*

Proof. Set $g := f \circ a$.

Clearly, the conditions $(1')$ and $(2')$ of Definition 1.11 are satisfied by g.

To prove that g satisfies the condition (4) of Definition 1.12 fix a point $z \in D$. Let $\varphi_{z,f}$, $\varphi_{z,g}$ be the functions appearing in the condition (4) for f, g respectively. Then $\varphi_{z,g} = \varphi_{z,f} \circ a$. Since a maps $\mathbb{T}$ onto $\mathbb{T}$ diffeomorphically, we have $\operatorname{wind} \varphi_{z,g} = \pm \operatorname{wind} \varphi_{z,f} = 0$.

It remains to show that the condition $(3')$ of Definition 1.11 is also satisfied by g. Note that the function $\widetilde{a}(\zeta) := \zeta/a(\zeta)$ has a holomorphic branch of the logarithm in the neighborhood of $\mathbb{T}$. This follows from $\operatorname{wind}\widetilde{a} = 0$, but it may also be shown quite elementarily. Actually, it suffices to prove that $\widetilde{a}(\mathbb{T}) \neq \mathbb{T}$. Expand a as

$$a(\zeta) = e^{it}\frac{\zeta - b}{1 - \overline{b}\zeta}$$

with some $t \in \mathbb{R}$, $b \in \mathbb{D}$, and observe that $\widetilde{a}$ does not attain the value $-e^{-it}$. Indeed, if $\zeta/a(\zeta) = -e^{-it}$ for some $\zeta \in \mathbb{T}$, then

$$\frac{1 - \overline{b}\zeta}{1 - b\overline{\zeta}} = -1,$$

so $2 = 2\operatorname{Re}(b\overline{\zeta}) \leq 2|b|$, which is impossible.

Concluding, there exists a function v holomorphic in a neighborhood of $\mathbb{T}$ such that

$$\frac{\zeta}{a(\zeta)} = e^{iv(\zeta)}.$$

Note that $v(\mathbb{T}) \subset \mathbb{R}$. Expanding v in Laurent series

$$v(\zeta) = \sum_{k=-\infty}^{\infty} a_k\zeta^k, \qquad \zeta \text{ near } \mathbb{T},$$

we infer that $a_{-k} = \overline{a}_k$, $k \in \mathbb{Z}$. Therefore,

$$v(\zeta) = a_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}(a_k\zeta^k) = \operatorname{Re}\Big(a_0 + 2\sum_{k=1}^{\infty} a_k\zeta^k\Big), \qquad \zeta \in \mathbb{T}.$$

Hence, there is a function h holomorphic in a neighborhood of $\overline{\mathbb{D}}$ such that $v = \operatorname{Im} h$. Put $u := h - iv$. Then $u \in \mathcal{O}(\mathbb{T})$ and $u(\mathbb{T}) \subset \mathbb{R}$.

Take ρ as in the condition $(3')$ of Definition 1.11 for f and define

$$r(\zeta) := \rho(a(\zeta))e^{u(\zeta)}, \qquad \zeta \in \mathbb{T}.$$

We compute

$$\begin{aligned}\zeta r(\zeta)\overline{\nu_D(g(\zeta))} &= \zeta u^{u(\zeta)}\rho(a(\zeta))\overline{\nu_D(f(a(\zeta)))}\\ &= a(\zeta)h(\zeta)\rho(a(\zeta))\overline{\nu_D(f(a(\zeta)))} = h(\zeta)\widetilde{f}(a(\zeta)), \qquad \zeta \in \mathbb{T}.\end{aligned}$$

Thus $\mathbb{T} \ni \zeta \mapsto \zeta r(\zeta)\overline{\nu_D(g(\zeta))} \in \mathbb{C}^n$ extends to a mapping from $\mathcal{O}(\mathbb{D}) \cap \mathcal{C}^{1/2}(\overline{\mathbb{D}})$. ∎

Corollary 3.1.5. *Every weak E-mapping $f : \mathbb{D} \to D$ of a bounded strongly linearly convex domain $D \subset \mathbb{C}^n$, $n \geq 2$, is a unique $\widetilde{k}_D$-extremal for $f(\zeta), f(\xi)$ (resp. a unique κ_D-extremal for $f(\zeta), f'(\zeta)$), for any $\zeta, \xi \in \mathbb{D}$, $\zeta \neq \xi$.*

3.2. Generalization of Proposition 3.1.1. The results obtained in this subsection will play an important role later.

We start with

PROPOSITION 3.2.1. *Let $f : \mathbb{D} \to D$ be an E-mapping. Then the function $f' \bullet \widetilde{f}$ is a positive constant.*

Proof. Consider the curve

$$\mathbb{R} \ni t \mapsto f(e^{it}) \in \partial D.$$

The tangent vector $ie^{it} f'(e^{it})$ to this curve belongs to $T_D^{\mathbb{R}}(f(e^{it}))$, i.e.

$$\operatorname{Re} \langle ie^{it} f'(e^{it}), \nu_D(f(e^{it}))\rangle = 0.$$

Thus for $\zeta \in \mathbb{T}$,

$$0 = \rho(\zeta) \operatorname{Re} \langle i\zeta f'(\zeta), \nu_D(f(\zeta))\rangle = -\operatorname{Im} f'(\zeta) \bullet \widetilde{f}(\zeta),$$

so the holomorphic function $f' \bullet \widetilde{f}$ is a real constant C.

Considering the curve

$$[0, 1+\varepsilon) \ni t \mapsto f(t) \in \mathbb{C}^n$$

for small $\varepsilon > 0$ and noting that $f([0,1)) \subset D$, $f(1) \in \partial D$, we see that the derivative of $r \circ f$ at a point $t = 1$ is non-negative, where r is a defining function of D. Hence

$$0 \le \operatorname{Re} \langle f'(1), \nu_D(f(1))\rangle = \frac{1}{\rho(1)} \operatorname{Re}(f'(1) \bullet \widetilde{f}(1)) = \frac{C}{\rho(1)},$$

i.e. $C \ge 0$. For $\zeta \in \mathbb{T}$,

$$\frac{f(\zeta) - f(0)}{\zeta} \bullet \widetilde{f}(\zeta) = \rho(\zeta) \langle f(\zeta) - f(0), \nu_D(f(\zeta))\rangle.$$

This function has winding number 0. Therefore, the function

$$g(\zeta) := \frac{f(\zeta) - f(0)}{\zeta} \bullet \widetilde{f}(\zeta),$$

which is holomorphic in a neighborhood of $\overline{\mathbb{D}}$, does not vanish in $\mathbb{D}$. In particular, $C = g(0) \neq 0$. ∎

Since the function ρ is defined up to a constant factor, we can choose it so that $f' \bullet \widetilde{f} \equiv 1$, i.e.

$$\rho(\zeta)^{-1} = \langle \zeta f'(\zeta), \nu_D(f(\zeta))\rangle, \quad \zeta \in \mathbb{T}. \tag{7}$$

In that way $\widetilde{f}$ and ρ are uniquely determined by f.

PROPOSITION 3.2.2. *Every E-mapping $f : \mathbb{D} \to D$ is injective in $\overline{\mathbb{D}}$.*

Proof. The function f has a left inverse in $\mathbb{D}$, so it suffices to check the injectivity on $\mathbb{T}$. Suppose that $f(\zeta_1) = f(\zeta_2)$ for some $\zeta_1, \zeta_2 \in \mathbb{T}$, $\zeta_1 \neq \zeta_2$,

and consider the curves

$$\gamma_j : [0,1] \ni t \mapsto f(t\zeta_j) \in \overline{D}, \qquad j = 1,2.$$

Since

$$\operatorname{Re}\langle \gamma_j'(1), \nu_D(f(\zeta_j))\rangle = \operatorname{Re}\langle \zeta_j f'(\zeta_j), \nu_D(f(\zeta_j))\rangle = \rho(\zeta_j)^{-1} \neq 0,$$

the curves γ_j hit ∂D transversally at their common point $f(\zeta_1)$. We claim that there exists $C > 0$ such that for $t \in (0,1)$ close to 1 there is $s_t \in (0,1)$ satisfying $\widetilde{k}_D(f(t\zeta_1), f(s_t\zeta_2)) < C$. This will finish the proof since

$$\widetilde{k}_D(f(t\zeta_1), f(s_t\zeta_2)) = p(t\zeta_1, s_t\zeta_2) \to \infty, \qquad t \to 1.$$

We may assume that $f(\zeta_1) = 0$ and $\nu_D(0) = (1,0,\dots,0) =: e_1$. There exists a ball $B \subset D$ tangent to ∂D at 0. Using a homothety if necessary, one can assume that $B = \mathbb{B}_n - e_1$. From the transversality of γ_1, γ_2 to ∂D there exists a cone

$$A := \{z \in \mathbb{C}^n : -\operatorname{Re} z_1 > k|z|\}, \qquad k > 0,$$

such that $\gamma_1(t), \gamma_2(t) \in A \cap B$ if $t \in (0,1)$ is sufficiently close to 1. For $z \in A$ let $k_z > k$ be the positive number satisfying

$$|z| = \frac{-\operatorname{Re} z_1}{k_z}.$$

Note that for any $a \in \gamma_1((0,1))$ sufficiently close to 0 one may find $b \in \gamma_2((0,1)) \cap A \cap B$ such that $\operatorname{Re} b_1 = \operatorname{Re} a_1$. To get a contradiction it suffices to show that $\widetilde{k}_D(a,b)$ is bounded from above by a constant independent of a and b.

We have the estimate

$$\begin{aligned}\widetilde{k}_D(a,b) &\le \widetilde{k}_{\mathbb{B}_n - e_1}(a,b) = \widetilde{k}_{\mathbb{B}_n}(a+e_1, b+e_1)\\ &= \tanh^{-1}\sqrt{1 - \frac{(1-|a+e_1|^2)(1-|b+e_1|^2)}{|1-\langle a+e_1, b+e_1\rangle|^2}}.\end{aligned}$$

The last expression is bounded from above if and only if

$$\frac{(1-|a+e_1|^2)(1-|b+e_1|^2)}{|1-\langle a+e_1, b+e_1\rangle|^2}$$

is bounded from below by some positive constant. We estimate

$$\begin{aligned}\frac{(1-|a+e_1|^2)(1-|b+e_1|^2)}{|1-\langle a+e_1, b+e_1\rangle|^2} &= \frac{(2\operatorname{Re} a_1 + |a|^2)(2\operatorname{Re} b_1 + |b|^2)}{|\langle a,b\rangle + a_1 + \overline{b}_1|^2}\\ &= \frac{\big(2\operatorname{Re} a_1 + \frac{(\operatorname{Re} a_1)^2}{k_a^2}\big)\big(2\operatorname{Re} a_1 + \frac{(\operatorname{Re} a_1)^2}{k_b^2}\big)}{|\langle a,b\rangle + 2\operatorname{Re} a_1 + i\operatorname{Im} a_1 - i\operatorname{Im} b_1|^2}\\ &\ge \frac{(\operatorname{Re} a_1)^2\big(2 + \frac{\operatorname{Re} a_1}{k_a^2}\big)\big(2 + \frac{\operatorname{Re} a_1}{k_b^2}\big)}{2|\langle a,b\rangle + i\operatorname{Im} a_1 - i\operatorname{Im} b_1|^2 + 2|2\operatorname{Re} a_1|^2}\end{aligned}$$

$$\geq \frac{(\operatorname{Re} a_1)^2\big(2+\frac{\operatorname{Re} a_1}{k_a^2}\big)\big(2+\frac{\operatorname{Re} a_1}{k_b^2}\big)}{2(|a|\,|b|+|a|+|b|)^2+8(\operatorname{Re} a_1)^2}$$
$$= \frac{(\operatorname{Re} a_1)^2\big(2+\frac{\operatorname{Re} a_1}{k_a^2}\big)\big(2+\frac{\operatorname{Re} a_1}{k_b^2}\big)}{2\big(\frac{(-\operatorname{Re} a_1)^2}{k_a^2k_b^2}-\frac{\operatorname{Re} a_1}{k_a}-\frac{\operatorname{Re} a_1}{k_b}\big)^2+8(\operatorname{Re} a_1)^2}$$
$$= \frac{\big(2+\frac{\operatorname{Re} a_1}{k_a^2}\big)\big(2+\frac{\operatorname{Re} a_1}{k_b^2}\big)}{2\big(\frac{-\operatorname{Re} a_1}{k_a^2k_b^2}+\frac{1}{k_a}+\frac{1}{k_b}\big)^2+8} > \frac{1}{2(1+2/k)^2+8}. \ \blacksquare$$

Assume that we are in the setting of Proposition 3.1.1 and D has real-analytic boundary. Our aim is to replace W with a neighborhood of $\overline{D}$.

Remark 3.2.3. For $\zeta_0 \in \mathbb{D}_f$ we have $G(f(\zeta_0),\zeta_0)=0$ and $\frac{\partial G}{\partial \zeta}(f(\zeta_0),\zeta_0) = -1$. By the Implicit Function Theorem there exist neighborhoods U_{ζ_0}, V_{ζ_0} of $f(\zeta_0),\zeta_0$ respectively and a holomorphic function $F_{\zeta_0}: U_{\zeta_0} \to V_{\zeta_0}$ such that for any $z \in U_{\zeta_0}$ the point $F_{\zeta_0}(\zeta)$ is the unique solution of the equation $G(z,\zeta)=0$, $\zeta \in V_{\zeta_0}$.

In particular, if $\zeta_0 \in \mathbb{D}$ then $F_{\zeta_0} = F$ near $f(\zeta_0)$.

Proposition 3.2.4. *Let $f:\mathbb{D}\to D$ be an E-mapping. Then there exist arbitrarily small neighborhoods U,V of $\overline{D},\overline{\mathbb{D}}$ respectively such that for any $z\in U$ the equation $G(z,\zeta)=0$, $\zeta\in V$, has exactly one solution.*

Proof. In view of Proposition 3.1.1 and Remark 3.2.3, it suffices to prove that there exist neighborhoods U,V of $\overline{D},\overline{\mathbb{D}}$ respectively such that for any $z\in U$ the equation $G(z,\cdot\,)=0$ has at most one solution $\zeta\in V$.

Assume the contrary. Then for any neighborhoods U of $\overline{D}$ and V of $\overline{\mathbb{D}}$ there are $z\in U$, $\zeta_1,\zeta_2\in V$, $\zeta_1\neq\zeta_2$ such that $G(z,\zeta_1)=G(z,\zeta_2)=0$. For $m\in\mathbb{N}$ put

$$U_m := \{z\in\mathbb{C}^n : \operatorname{dist}(z,D)<1/m\}, \qquad V_m := \{\zeta\in\mathbb{C} : \operatorname{dist}(\zeta,\mathbb{D})<1/m\}.$$

There exist $z_m\in U_m$, $\zeta_{m,1},\zeta_{m,2}\in V_m$, $\zeta_{m,1}\neq\zeta_{m,2}$ such that $G(z_m,\zeta_{m,1}) = G(z_m,\zeta_{m,2})=0$. Passing to a subsequence we may assume that $z_m\to z_0 \in \overline{D}$. Analogously we may assume $\zeta_{m,1}\to\zeta_1\in\overline{\mathbb{D}}$ and $\zeta_{m,2}\to\zeta_2\in\overline{\mathbb{D}}$. Clearly, $G(z_0,\zeta_1)=G(z_0,\zeta_2)=0$. Let us consider the possible cases.

1) If $\zeta_1,\zeta_2\in\mathbb{T}$ then $G(z_0,\zeta_j)=0$ is equivalent to

$$\langle z_0 - f(\zeta_j), \nu_D(f(\zeta_j))\rangle = 0, \qquad j=1,2,$$

consequently $z_0 - f(\zeta_j)\in T_D^{\mathbb{C}}(f(\zeta_j))$. By the strong linear convexity of D we get $z_0 = f(\zeta_j)$. But f is injective in $\overline{\mathbb{D}}$, so $\zeta_1=\zeta_2=:\zeta_0$. It follows from Remark 3.2.3 that in a sufficiently small neighborhood of (z_0,ζ_0) all solutions of the equation $G(z,\zeta)=0$ are of the form $(z,F_{\zeta_0}(z))$. The points $(z_m,\zeta_{m,1})$ and $(z_m,\zeta_{m,2})$ belong to this neighborhood for large m, which gives a contradiction.

2) If $\zeta_1 \in \mathbb{T}$ and $\zeta_2 \in \mathbb{D}$, then analogously to the above we deduce that $z_0 = f(\zeta_1)$. Let us take an arbitrary sequence $\{\eta_m\} \subset \mathbb{D}$ convergent to ζ_1. Then $f(\eta_m) \in D$ and $f(\eta_m) \to z_0$, so the sequence $G(f(\eta_m), \cdot\,)$ converges to $G(z_0, \cdot\,)$ uniformly on $\mathbb{D}$. Since $G(z_0, \cdot\,) \not\equiv 0$, $G(z_0, \zeta_2) = 0$ and $\zeta_2 \in \mathbb{D}$, we deduce from the Hurwitz Theorem that for large m the functions $G(f(\eta_m), \cdot\,)$ have roots $\theta_m \in \mathbb{D}$ such that $\theta_m \to \zeta_2$. Hence $G(f(\eta_m), \theta_m) = 0$ and from the uniqueness of solutions in $D \times \mathbb{D}$ (Proposition 3.1.1) we have

$$\theta_m = F(f(\eta_m)) = \eta_m.$$

This is a contradiction, because the left side tends to ζ_2 and the right side to ζ_1, as $m \to \infty$.

3) We are left with the case $\zeta_1, \zeta_2 \in \mathbb{D}$. If $z_0 \in \overline{D} \setminus f(\mathbb{T})$ then $z_0 \in W$. In $W \times \mathbb{D}$ all solutions of the equation $G = 0$ are of the form $(z, F(z))$, $z \in W$. But for large m the points $(z_m, \zeta_{m,1})$, $(z_m, \zeta_{m,2})$ belong to $W \times \mathbb{D}$, which is a contradiction with the uniqueness.

If $z_0 \in f(\mathbb{T})$, then $z_0 = f(\zeta_0)$ for some $\zeta_0 \in \mathbb{T}$. Clearly, $G(f(\zeta_0), \zeta_0) = 0$, whence $G(z_0, \zeta_0) = G(z_0, \zeta_1) = 0$ and $\zeta_0 \in \mathbb{T}$, $\zeta_1 \in \mathbb{D}$. This is just case 2), considered above. ∎

COROLLARY 3.2.5. *There are neighborhoods U, V of $\overline{D}, \overline{\mathbb{D}}$ respectively with $V \Subset \mathbb{D}_f$ such that the function F extends holomorphically on U and all solutions of the equation $G|_{U \times V} = 0$ are of the form $(z, F(z))$, $z \in U$. In particular, $F \circ f = \mathrm{id}_V$.*

4. Hölder estimates

DEFINITION 4.1. For a given $c > 0$ let the family $\mathcal{D}(c)$ consist of all pairs (D, z), where $D \subset \mathbb{C}^n$, $n \geq 2$, is a bounded strongly pseudoconvex domain with real-analytic boundary and $z \in D$, satisfying

(1) $\mathrm{dist}(z, \partial D) \geq 1/c$;
(2) the diameter of D is not greater than c and D satisfies the interior ball condition with a radius $1/c$;
(3) for any $x, y \in D$ there exist $m \leq 8c^2$ and open balls $B_0, \ldots, B_m \subset D$ of radius $1/(2c)$ such that $x \in B_0$, $y \in B_m$ and the distance between the centers of the balls B_j, B_{j+1} is not greater than $1/(4c)$ for $j = 0, \ldots, m-1$;
(4) for any open ball $B \subset \mathbb{C}^n$ of radius not greater than $1/c$, intersecting non-emptily with ∂D, there exists a mapping $\Phi \in \mathcal{O}(\overline{D}, \mathbb{C}^n)$ such that
    (a) for any $w \in \Phi(B \cap \partial D)$ there is a ball of a radius c containing $\Phi(D)$ and tangent to $\partial\Phi(D)$ at w (let us call this the "exterior ball condition" with radius c);

(b) Φ is biholomorphic in a neighborhood of $\overline{B}$ and $\Phi^{-1}(\Phi(B)) = B$;
(c) the entries of all matrices Φ' on $B \cap \overline{D}$ and $(\Phi^{-1})'$ on $\Phi(B \cap \overline{D})$ are bounded in modulus by c;
(d) $\operatorname{dist}(\Phi(z), \partial\Phi(D)) \geq 1/c$;

(5) the normal vector ν_D is Lipschitz with constant $2c$, that is,

$$|\nu_D(a) - \nu_D(b)| \leq 2c|a - b|, \qquad a, b \in \partial D;$$

(6) the ε-hull of D, i.e. the domain $D_\varepsilon := \{w \in \mathbb{C}^n : \operatorname{dist}(w, D) < \varepsilon\}$, is strongly pseudoconvex for any $\varepsilon \in (0, 1/c]$.

Recall that the *interior ball condition* with radius $r > 0$ means that for any point $a \in \partial D$ there is $a' \in D$ and a ball $B_n(a', r) \subset D$ tangent to ∂D at a. Equivalently

$$D = \bigcup_{a' \in D'} B_n(a', r)$$

for some set $D' \subset D$.

It may be shown that (2) and (5) may be expressed in terms of boundedness of the normal curvature, boundedness of the domain and the condition (3). This however lies beyond the scope of this paper and needs some very technical arguments so we omit the proof of this fact. The reason to use (2) in such a form is its connection with (3) (this allows us to simplify the proof in some points).

Remark 4.2. Note that any convex domain satisfying conditions (1)–(4) of Definition 4.1 satisfies conditions (5) and (6), as well.

Actually, it follows from (2) that for any $a \in \partial D$ there exists a ball $B_n(a', 1/c) \subset D$ tangent to ∂D at a. Then

$$\nu_D(a) = \frac{a' - a}{|a' - a|} = c(a' - a).$$

Hence

$$|\nu_D(a) - \nu_D(b)| = c|a' - a - b' + b| = c|a' - b' - (a - b)| \leq c|a' - b'| + c|a - b|.$$

Since D is convex, we have $|a' - b'| \leq |a - b|$, which gives (5).

The condition (6) is also clear—for any $\varepsilon > 0$ the ε-hull of a strongly convex domain is strongly convex.

Remark 4.3. For a convex domain D the condition (3) of Definition 4.1 amounts to the condition (2).

Indeed, for two points $x, y \in D$ take two balls of radius $1/(2c)$ containing x, y and contained in D. Then divide the interval between the centers of the balls into $[4c^2] + 1$ equal parts and take balls of radius $1/(2c)$ with centers at the points of the partition.

Note also that if a strongly convex domain D satisfies the interior ball condition with radius $1/c$ and the exterior ball condition with radius c, then one can take $\Phi := \mathrm{id}_{\mathbb{C}^n}$.

Remark 4.4. For a strongly pseudoconvex domain D and $c' > 0$ and for any $z \in D$ such that $\mathrm{dist}(z, \partial D) > 1/c'$ there exists $c = c(c') > 0$ satisfying $(D, z) \in \mathcal{D}(c)$.

Indeed, the conditions (1)–(3) and (5)–(6) are clear. Only (4) is nontrivial. The construction of the mapping Φ amounts to constructing Fornæss peak functions. Actually, we can apply directly Proposition 1 from [2] to any boundary point of ∂D (obviously D has a Stein neighborhood basis). This gives a covering of ∂D with a finite number of balls B_j, mappings $\Phi_j \in \mathcal{O}(\overline{D}, \mathbb{C}^n)$ and strongly convex $\mathcal{C}^\infty$-smooth domains C_j, $j = 1, \ldots, N$, such that

- $\Phi_j(D) \subset C_j$;
- $\Phi_j(\overline{D}) \subset \overline{C}_j$;
- $\Phi_j(B_j \setminus \overline{D}) \subset \mathbb{C}^n \setminus \overline{C}_j$;
- $\Phi_j^{-1}(\Phi_j(B_j)) = B_j$;
- $\Phi_j|_{B_j} : B_j \to \Phi_j(B_j)$ is biholomorphic.

Therefore, one may choose $c > 0$ such that every C_j satisfies the exterior ball condition with c, i.e. for any $x \in \partial C_j$ there is a ball of radius c containing C_j and tangent to ∂C_j at x, every ball of radius $1/c$ intersecting ∂D is contained in some B_j (here one may use a standard argument invoking the Lebesgue number) and the conditions (c), (d) are also satisfied (with $\Phi := \Phi_j$).

In this section we use the words 'uniform', 'uniformly' if $(D, z) \in \mathcal{D}(c)$. This means that estimates will depend only on c and will be independent of D and z if $(D, z) \in \mathcal{D}(c)$, and of E-mappings of D mapping 0 to z (which we denote by $f : (\mathbb{D}, 0) \to (D, z)$).

Now we go back to the situation when $D \subset \mathbb{C}^n$, $n \geq 2$, is a bounded strongly linearly convex domain with real-analytic boundary.

Proposition 4.5. *Let $f : (\mathbb{D}, 0) \to (D, z)$ be an E-mapping. Then*

$$\mathrm{dist}(f(\zeta), \partial D) \leq C(1 - |\zeta|), \qquad \zeta \in \overline{\mathbb{D}},$$

with $C > 0$ uniform if $(D, z) \in \mathcal{D}(c)$.

Proof. There exists a uniform C_1 such that

$$\text{if } \mathrm{dist}(w, \partial D) \geq 1/c \text{ then } k_D(w, z) < C_1.$$

Indeed, let $\mathrm{dist}(w, \partial D) \geq 1/c$ and let balls $B_0, \ldots, B_m$ with centers $b_0, \ldots, b_m$

be chosen for the points w, z as in the condition (3) of Definition 4.1. Then

$$\begin{aligned}
k_D(w,z) &\le k_D(w,b_0) + \sum_{j=0}^{m-1} k_D(b_j,b_{j+1}) + k_D(b_m,z)\\
&\le k_{B_n(w,1/c)}(w,b_0) + \sum_{j=0}^{m-1} k_{B_j}(b_j,b_{j+1}) + k_{B_n(z,1/c)}(b_m,z)\\
&= p\Big(0,\frac{|w-b_0|}{1/c}\Big) + \sum_{j=0}^{m-1} p\Big(0,\frac{|b_j-b_{j+1}|}{1/(2c)}\Big) + p\Big(0,\frac{|b_m-z|}{1/c}\Big)\\
&\le (m+2)p(0,1/2) \le (8c^2+2)p(0,1/2) =: C_1.
\end{aligned}$$

If $\zeta\in\mathbb{D}$ is such that $\operatorname{dist}(f(\zeta),\partial D)\ge 1/c$ then

$$k_D(f(0),f(\zeta)) \le C_2 - \frac{1}{2}\log\operatorname{dist}(f(\zeta),\partial D)$$

with a uniform $C_2 := C_1 + \frac{1}{2}\log c$.

In the other case, i.e. if $\operatorname{dist}(f(\zeta),\partial D) < 1/c$, denote by η the nearest point to $f(\zeta)$ lying on ∂D. Let $w\in D$ be the center of a ball B of radius $1/c$ tangent to ∂D at η. By the condition (2) of Definition 4.1 we have $B\subset D$. Hence

$$\begin{aligned}
&k_D(f(0),f(\zeta)) \le k_D(f(0),w) + k_D(w,f(\zeta))\\
&\le C_1 + k_B(w,f(\zeta)) \le C_1 + \frac{1}{2}\log 2 - \frac{1}{2}\log\Big(1-\frac{|f(\zeta)-w|}{1/c}\Big)\\
&= C_1 + \frac{1}{2}\log 2 - \frac{1}{2}\log(c\operatorname{dist}(f(\zeta),\partial B)) = C_3 - \frac{1}{2}\log\operatorname{dist}(f(\zeta),\partial D)
\end{aligned}$$

with a uniform $C_3 := C_1 + \frac{1}{2}\log(2/c)$.

We have obtained the same type estimates in both cases. On the other hand, by Corollary 3.1.2,

$$k_D(f(0),f(\zeta)) = p(0,\zeta) \ge -\frac{1}{2}\log(1-|\zeta|),$$

which finishes the proof. ∎

Recall that we have assumed that ρ is of the form (7).

PROPOSITION 4.6. *Let $f:(\mathbb{D},0)\to(D,z)$ be an E-mapping. Then*

$$C_1 < \rho(\zeta)^{-1} < C_2, \qquad \zeta\in\mathbb{T},$$

where C_1, C_2 are uniform if $(D,z)\in\mathcal{D}(c)$.

Proof. For the upper estimate fix $\zeta_0\in\mathbb{T}$. Set $B := B_n(f(\zeta_0),1/c)$ and let $\Phi\in\mathcal{O}(\overline{D},\mathbb{C}^n)$ be as in the condition (4) of Definition 4.1 for B. One can assume that $f(\zeta_0)=\Phi(f(\zeta_0))=0$ and $\nu_D(0)=\nu_{\Phi(D)}(0)=(1,0,\ldots,0)$.

Then $\Phi(D)$ is contained in the half-space $\{w \in \mathbb{C}^n : \operatorname{Re} w_1 < 0\}$. Putting $h := \Phi \circ f$ we have

$$h_1(\mathbb{D}) \subset \{w_1 \in \mathbb{C} : \operatorname{Re} w_1 < 0\}.$$

By the Schwarz Lemma on the half-plane,

$$|h_1'(t\zeta_0)| \le \frac{-2 \operatorname{Re} h_1(t\zeta_0)}{1 - |t\zeta_0|^2}. \tag{8}$$

Let δ be the signed boundary distance of $\Phi(D)$, i.e.

$$\delta(x) := \begin{cases} -\operatorname{dist}(x, \partial\Phi(D)), & x \in \Phi(D), \\ \operatorname{dist}(x, \partial\Phi(D)), & x \notin \Phi(D). \end{cases}$$

It is a defining function of $\Phi(D)$ in a neighborhood of 0 (recall that $\Phi^{-1}(\Phi(B)) = B$). Observe that

$$\delta(x) = \delta(0) + \operatorname{Re}\langle \nabla\delta(0), x\rangle + O(|x|^2) = \operatorname{Re} x_1 + O(|x|^2).$$

If $x \in \Phi(D)$ tends transversally to 0, then the angle between the vector x and the hyperplane $\{w \in \mathbb{C}^n : \operatorname{Re} w_1 = 0\}$ is separated from 0, i.e. its sine is $(-\operatorname{Re} x_1)/|x| > \varepsilon$ for some $\varepsilon > 0$ independent of x. Thus

$$\frac{\delta(x)}{\operatorname{Re} x_1} = 1 + O(|x|) \quad \text{as } x \to 0 \text{ transversally.}$$

Consequently,

$$-\operatorname{Re} x_1 \le 2 \operatorname{dist}(x, \partial\Phi(D)) \quad \text{as } x \to 0 \text{ transversally.} \tag{9}$$

We know that $t \mapsto f(t\zeta_0)$ hits ∂D transversally. Therefore, $t \mapsto h(t\zeta_0)$ hits $\partial\Phi(D)$ transversally as well. Indeed, we have

$$\begin{aligned} \left\langle \frac{d}{dt} h(t\zeta_0)\Big|_{t=1}, \nu_{\Phi(D)}(h(\zeta_0)) \right\rangle &= \left\langle \Phi'(0) f'(\zeta_0)\zeta_0, \frac{(\Phi^{-1})'(0)^* \nabla r(0)}{|(\Phi^{-1})'(0)^* \nabla r(0)|} \right\rangle \\ &= \frac{\langle \zeta_0 f'(\zeta_0), \nabla r(0)\rangle}{|(\Phi'(0)^{-1})^* \overline{\nabla r(0)}|} = \frac{\langle \zeta_0 f'(\zeta_0), \nu_D(f(\zeta_0))|\nabla r(0)|\rangle}{|(\Phi'(0)^{-1})^* \overline{\nabla r(0)}|}, \end{aligned} \tag{10}$$

where r is a defining function of D. In particular,

$$\operatorname{Re}\left\langle \frac{d}{dt} h(t\zeta_0)\Big|_{t=1}, \nu_{\Phi(D)}(h(\zeta_0)) \right\rangle = \frac{\rho(\zeta_0)^{-1}|\nabla r(0)|}{|(\Phi'(0)^{-1})^* \overline{\nabla r(0)}|} \ne 0.$$

This proves that $t \mapsto h(t\zeta_0)$ hits $\partial\Phi(D)$ transversally.

Consequently, we may put $x = h(t\zeta_0)$ into (9) to get

$$\frac{-2 \operatorname{Re} h_1(t\zeta_0)}{1 - |t\zeta_0|^2} \le \frac{4 \operatorname{dist}(h(t\zeta_0), \partial\Phi(D))}{1 - |t\zeta_0|^2}, \quad t \to 1. \tag{11}$$

But Φ is a biholomorphism near 0, so

$$\frac{4\,\mathrm{dist}(h(t\zeta_0),\partial\Phi(D))}{1-|t\zeta_0|^2}\le C_3\frac{\mathrm{dist}(f(t\zeta_0),\partial D)}{1-|t\zeta_0|},\qquad t\to 1, \tag{12}$$

where C_3 is uniform (thanks to the condition (4)(c) of Definition 4.1). By Proposition 4.5, the term on the right side of (12) does not exceed some uniform constant.

It follows from (10) that

$$\begin{aligned}\rho(\zeta_0)^{-1}&=|\langle f'(\zeta_0)\zeta_0,\nu_D(f(\zeta_0))\rangle|\\&\le C_4|\langle h'(\zeta_0),\nu_{\Phi(D)}(h(\zeta_0))\rangle|=C_4|h_1'(\zeta_0)|=\lim_{t\to 1}C_4|h_1'(t\zeta_0)|\end{aligned}$$

with a uniform C_4 (here we use the condition (4)(c) of Definition 4.1 again). Combining (8), (11) and (12) we get the upper estimate for $\rho(\zeta_0)^{-1}$.

Now we prove the lower estimate. Let r be the signed boundary distance to ∂D. For $\varepsilon=1/c$ the function

$$\varrho(w):=-\log(\varepsilon-r(w))+\log\varepsilon,\qquad w\in D_\varepsilon,$$

where D_ε is the ε-hull of D, is plurisubharmonic and defining for D. Indeed, we have

$$-\log(\varepsilon-r(w))=-\log\mathrm{dist}(w,\partial D_\varepsilon),\qquad w\in D_\varepsilon,$$

and D_ε is pseudoconvex.

Therefore, the function

$$v:=\varrho\circ f:\overline{\mathbb{D}}\to(-\infty,0]$$

is subharmonic on $\mathbb{D}$. Moreover, since f maps $\mathbb{T}$ in ∂D, we infer that $v=0$ on $\mathbb{T}$. Further, since $|f(\lambda)-z|<c$ for $\lambda\in\mathbb{D}$, we have

$$|f(\lambda)-z|<\frac{1}{2c}\quad\text{if } |\lambda|\le\frac{1}{2c^2}.$$

Therefore, for a fixed $\zeta_0\in\mathbb{T}$,

$$M_{\zeta_0}(x):=\max_{t\in[0,2\pi]}v(\zeta_0e^{x+it})\le-\log\left(1+\frac{1}{2c\varepsilon}\right)=:-C_5\quad\text{if } x\le-\log(2c^2).$$

Since M_{ζ_0} is convex for $x\le 0$ and $M_{\zeta_0}(0)=0$, we get

$$v(\zeta_0e^x)\le M_{\zeta_0}(x)\le\frac{C_5x}{\log(2c^2)}\quad\text{for } -\log(2c^2)\le x\le 0.$$

Hence (remember that $v(\zeta_0)=0$)

$$\begin{aligned}\frac{C_5}{\log(2c^2)}&\le\frac{d}{dx}v(\zeta_0e^x)\Big|_{x=0}=\sum_{j=1}^n\frac{\partial\varrho}{\partial z_j}(f(\zeta_0))f_j'(\zeta_0)\zeta_0\\&=\langle\zeta_0f'(\zeta_0),\nabla\varrho(f(\zeta_0))\rangle=\rho(\zeta_0)^{-1}|\nabla\varrho(f(\zeta_0))|.\end{aligned}\tag{13}$$

Moreover,

$$|\nabla\varrho(f(\zeta_0))| = \left\langle \nabla\varrho(f(\zeta_0)), \frac{\nabla\varrho(f(\zeta_0))}{|\nabla\varrho(f(\zeta_0))|}\right\rangle_{\mathbb{R}} = \langle \nabla\varrho(f(\zeta_0)), \nu_D(f(\zeta_0))\rangle_{\mathbb{R}}$$
$$= \frac{\partial\varrho}{\partial\nu_D}(f(\zeta_0)) = \lim_{t\to 0}\frac{\varrho(f(\zeta_0)+t\nu_D(f(\zeta_0))) - \varrho(f(\zeta_0))}{t} = \frac{1}{\varepsilon} = c,$$

as $r(a+t\nu(a)) = t$ if $a \in \partial D$ and $t \in \mathbb{R}$ is small enough. This, together with (13), finishes the proof of the lower estimate. ∎

Proposition 4.7. *Let $f : (\mathbb{D}, 0) \to (D, z)$ be an E-mapping. Then*

$$|f(\zeta_1) - f(\zeta_2)| \le C\sqrt{|\zeta_1 - \zeta_2|}, \qquad \zeta_1, \zeta_2 \in \overline{\mathbb{D}},$$

where C is uniform if $(D, z) \in \mathcal{D}(c)$.

Proof. Let $\zeta_0 \in \mathbb{D}$ be such that $1 - |\zeta_0| < 1/(cC)$, where C is as in Proposition 4.5. Then $B := B_n(f(\zeta_0), 1/c)$ intersects ∂D. Take Φ for the ball B from the condition (4) of Definition 4.1. Let w denote the nearest point to $\Phi(f(\zeta_0))$ lying on $\partial\Phi(D)$. From the conditions (4)(b)–(c) of Definition 4.1 we find that there is a uniform constant $r < 1$ such that the point w belongs to $\Phi(B \cap \partial D)$ provided that $|\zeta_0| \ge r$.

From the condition (4)(a) of Definition 4.1 we deduce that there is w_0 such that $\Phi(D) \subset B_n(w_0, c)$ and the ball $B_n(w_0, c)$ is tangent to $\partial\Phi(D)$ at w. Let

$$h(\zeta) := (\Phi \circ f)\left(\frac{\zeta_0 - \zeta}{1 - \overline{\zeta_0}\zeta}\right), \qquad \zeta \in \mathbb{D}.$$

Then h is holomorphic, $h(\mathbb{D}) \subset B_n(w_0, c)$ and $h(0) = \Phi(f(\zeta_0))$. Using the Schwarz Lemma for the unit ball, we get

$$\begin{aligned}|h'(0)| &\le \sqrt{c^2 - |h(0) - w_0|^2}\\ &\le \sqrt{2c(c - |\Phi(f(\zeta_0)) - w_0|)} = \sqrt{2c(|w_0 - w| - |\Phi(f(\zeta_0)) - w_0|)}\\ &\le \sqrt{2c}\sqrt{|\Phi(f(\zeta_0)) - w|} = \sqrt{2c}\sqrt{\operatorname{dist}(\Phi(f(\zeta_0)), \partial\Phi(D))}.\end{aligned}$$

Since

$$h'(0) = \Phi'(f(\zeta_0))f'(\zeta_0)\frac{d}{d\zeta}\frac{\zeta_0 - \zeta}{1 - \overline{\zeta_0}\zeta}\bigg|_{\zeta=0},$$

we deduce by the condition (4)(c) of Definition 4.1 that

$$|h'(0)| \ge C_1|f'(\zeta_0)|(1 - |\zeta_0|^2)$$

with a uniform C_1. Further,

$$\begin{aligned}|f'(\zeta_0)| &\le \frac{|h'(0)|}{C_1(1 - |\zeta_0|^2)} \le \frac{\sqrt{2c}}{C_1}\frac{\sqrt{\operatorname{dist}(\Phi(f(\zeta_0)), \partial\Phi(D))}}{1 - |\zeta_0|^2}\\ &\le C_2\frac{\sqrt{\operatorname{dist}(f(\zeta_0), \partial D)}}{1 - |\zeta_0|^2},\end{aligned}$$

where C_2 is uniform. Combining this with Proposition 4.5 we find that

$$|f'(\zeta_0)| \le C_3 \frac{\sqrt{1-|\zeta_0|}}{1-|\zeta_0|^2} = \frac{C_3}{\sqrt{1-|\zeta_0|}}, \tag{14}$$

where C_3 is a uniform constant.

We have shown that (14) holds for $r \le |\zeta_0| < 1$ with a uniform $r < 1$. For $|\zeta_0| < r$ we estimate in the following way:

$$|f'(\zeta_0)| \le \max_{|\zeta|=r} |f'(\zeta)| \le \frac{C_3}{\sqrt{1-r}} \le \frac{C_4}{\sqrt{1-|\zeta_0|}}$$

with a uniform $C_4 := C_3/\sqrt{1-r}$.

Using Theorems 8.2.1 and 8.2.2 with $\alpha = 1/2$, we finish the proof. ∎

PROPOSITION 4.8. *Let $f : (\mathbb{D}, 0) \to (D, z)$ be an E-mapping. Then*

$$|\rho(\zeta_1) - \rho(\zeta_2)| \le C\sqrt{|\zeta_1 - \zeta_2|}, \qquad \zeta_1, \zeta_2 \in \mathbb{T},$$

where C is uniform if $(D, z) \in \mathcal{D}(c)$.

Proof. It suffices to prove that there exist uniform $C, C_1 > 0$ such that

$$|\rho(\zeta_1) - \rho(\zeta_2)| \le C\sqrt{|\zeta_1 - \zeta_2|}, \qquad \zeta_1, \zeta_2 \in \mathbb{T},\ |\zeta_1 - \zeta_2| < C_1.$$

Fix $\zeta_1 \in \mathbb{T}$. Without loss of generality we may assume that $\nu_{D,1}(f(\zeta_1)) = 1$. Let $0 < C_1 \le 1/4$ be uniform and such that

$$|\nu_{D,1}(f(\zeta)) - 1| < 1/2, \qquad \zeta \in \mathbb{T} \cap B_n(\zeta_1, 3C_1).$$

This is possible, since by Proposition 4.7,

$$|\nu_D(f(\zeta)) - \nu_D(f(\zeta'))| \le 2c|f(\zeta) - f(\zeta')| \le C'\sqrt{|\zeta - \zeta'|}, \qquad \zeta, \zeta' \in \mathbb{T},$$

with a uniform $C' > 0$. There exists a function $\psi \in \mathcal{C}^1(\mathbb{T}, [0,1])$ such that $\psi = 1$ on $\mathbb{T} \cap B_n(\zeta_1, 2C_1)$ and $\psi = 0$ on $\mathbb{T} \setminus B_n(\zeta_1, 3C_1)$. Then the function $\varphi : \mathbb{T} \to \mathbb{C}$ defined by

$$\varphi := (\overline{\nu_{D,1} \circ f} - 1)\psi + 1$$

satisfies

- $\varphi(\zeta) = \overline{\nu_{D,1}(f(\zeta))}$, $\zeta \in \mathbb{T} \cap B_n(\zeta_1, 2C_1)$;
- $|\varphi(\zeta) - 1| < 1/2$, $\zeta \in \mathbb{T}$;
- φ is uniformly 1/2-Hölder continuous on $\mathbb{T}$, i.e. it is 1/2-Hölder continuous with a uniform constant (remember that ψ was chosen uniformly).

First observe that $\log \varphi$ is well-defined. Using properties listed above, we deduce that $\log \varphi$ and $\operatorname{Im} \log \varphi$ are uniformly 1/2-Hölder continuous on $\mathbb{T}$ as well. The function $\operatorname{Im} \log \varphi$ extends continuously to a function $v : \overline{\mathbb{D}} \to \mathbb{R}$, harmonic in $\mathbb{D}$. There is a function $h \in \mathcal{O}(\mathbb{D})$ such that $v = \operatorname{Im} h$ in $\mathbb{D}$. Taking $h - \operatorname{Re} h(0)$ instead of h, one can assume that $\operatorname{Re} h(0) = 0$. By Theorem 8.2.3 applied to ih, we see that the function h extends continuously on $\overline{\mathbb{D}}$ and h is uniformly 1/2-Hölder continuous on $\overline{\mathbb{D}}$. Hence the function

$u := \operatorname{Re} h : \overline{\mathbb{D}} \to \mathbb{R}$ is uniformly 1/2-Hölder continuous on $\overline{\mathbb{D}}$ with a uniform constant C_2. Furthermore, u is uniformly bounded on $\overline{\mathbb{D}}$, since

$$|u(\zeta)| = |u(\zeta) - u(0)| \le C_2\sqrt{|\zeta|}, \qquad \zeta \in \overline{\mathbb{D}}.$$

Let $g(\zeta) := \widetilde{f_1}(\zeta)e^{-h(\zeta)}$ and $G(\zeta) := g(\zeta)/\zeta$. Then $g \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ and $G \in \mathcal{O}(\mathbb{D}_*) \cap \mathcal{C}((\overline{\mathbb{D}})_*)$. Note that, for $\zeta \in \mathbb{T}$,

$$|g(\zeta)| = |\zeta\rho(\zeta)\overline{\nu_{D,1}(f(\zeta))}e^{-h(\zeta)}| \le \rho(\zeta)e^{-u(\zeta)},$$

which, combined with Proposition 4.6, the uniform boundedness of u and the maximum principle, gives uniform boundedness of g in $\overline{\mathbb{D}}$. The function G is uniformly bounded in $\overline{\mathbb{D}} \cap B_n(\zeta_1, 2C_1)$. Moreover, for $\zeta \in \mathbb{T} \cap B_n(\zeta_1, 2C_1)$,

$$\begin{aligned} G(\zeta) &= \rho(\zeta)\overline{\nu_{D,1}(f(\zeta))}e^{-u(\zeta) - i\operatorname{Im}\log\varphi(\zeta)} \\ &= \rho(\zeta)\overline{\nu_{D,1}(f(\zeta))}e^{-u(\zeta)+\operatorname{Re}\log\varphi(\zeta)}e^{-\log\varphi(\zeta)} = \rho(\zeta)e^{-u(\zeta)+\operatorname{Re}\log\varphi(\zeta)} \in \mathbb{R}. \end{aligned}$$

By the Reflection Principle one can extend G holomorphically past $\mathbb{T} \cap B_n(\zeta_1, 2C_1)$ to a function (denoted by the same letter) uniformly bounded in $B_n(\zeta_1, 2C_2)$, where the constant C_2 is uniform. Hence, from the Cauchy formula, G is uniformly Lipschitz continuous in $B_n(\zeta_1, C_2)$, and consequently uniformly 1/2-Hölder continuous in $B_n(\zeta_1, C_2)$.

Finally, the functions G, h, $\nu_{D,1} \circ f$ are uniformly 1/2-Hölder continuous on $\mathbb{T} \cap B_n(\zeta_1, C_2)$ and $|\nu_{D,1} \circ f| > 1/2$ on $\mathbb{T} \cap B_n(\zeta_1, C_2)$, so the function $\rho = Ge^h/\overline{\nu_{D,1} \circ f}$ is uniformly 1/2-Hölder continuous on $\mathbb{T} \cap B_n(\zeta_1, C_2)$. ∎

Proposition 4.9. *Let $f : (\mathbb{D}, 0) \to (D, z)$ be an E-mapping. Then*

$$|\widetilde{f}(\zeta_1) - \widetilde{f}(\zeta_2)| \le C\sqrt{|\zeta_1 - \zeta_2|}, \qquad \zeta_1, \zeta_2 \in \overline{\mathbb{D}},$$

where C is uniform if $(D, z) \in \mathcal{D}(c)$.

Proof. By Propositions 4.7 and 4.8 we obtain the desired inequality for $\zeta_1, \zeta_2 \in \mathbb{T}$. Theorem 8.2.2 finishes the proof. ∎

5. Openness of the set of E-mappings. We shall show that perturbing a little a domain D equipped with an E-mapping, we obtain a domain which also has an E-mapping close to a given one.

5.1. Preliminary results

Proposition 5.1.1. *Let $f : \mathbb{D} \to D$ be an E-mapping. Then there exist domains $G, \widetilde{D}, \widetilde{G} \subset \mathbb{C}^n$ and a biholomorphism $\Phi : \widetilde{D} \to \widetilde{G}$ such that*

(1) *$\widetilde{D}, \widetilde{G}$ are neighborhoods of $\overline{D}, \overline{G}$ respectively;*
(2) *$\Phi(D) = G$;*
(3) *$g(\zeta) := \Phi(f(\zeta)) = (\zeta, 0, \ldots, 0)$, $\zeta \in \overline{\mathbb{D}}$;*
(4) *$\nu_G(g(\zeta)) = (\zeta, 0, \ldots, 0)$, $\zeta \in \mathbb{T}$;*
(5) *for any $\zeta \in \mathbb{T}$, a point $g(\zeta)$ is a point of strong linear convexity of G.*

Proof. Let U, V be the sets from Proposition 3.2.4. We claim that after a linear change of coordinates one can assume that $\widetilde{f}_1, \widetilde{f}_2$ do not have common zeroes in V.

Since $f' \bullet \widetilde{f} = 1$, at least one of the functions $\widetilde{f}_1, \ldots, \widetilde{f}_n$, say $\widetilde{f}_1$, is not identically equal to 0. Let $\lambda_1, \ldots, \lambda_m$ be all zeroes of $\widetilde{f}_1$ in V. We may find $\alpha \in \mathbb{C}^n$ such that

$$(\alpha_1 \widetilde{f}_1 + \cdots + \alpha_n \widetilde{f}_n)(\lambda_j) \neq 0, \qquad j = 1, \ldots, m.$$

Otherwise, for any $\alpha \in \mathbb{C}^n$ there would exist $j \in \{1, \ldots, m\}$ such that $\alpha \bullet \widetilde{f}(\lambda_j) = 0$, hence

$$\mathbb{C}^n = \bigcup_{j=1}^{m} \{\alpha \in \mathbb{C}^n : \alpha \bullet \widetilde{f}(\lambda_j) = 0\}.$$

The sets $\{\alpha \in \mathbb{C}^n : \alpha \bullet \widetilde{f}(\lambda_j) = 0\}$, $j = 1, \ldots, m$, are $(n-1)$-dimensional complex hyperplanes, so their finite union cannot be the space $\mathbb{C}^n$.

Of course, at least one of the numbers $\alpha_2, \ldots, \alpha_n$, say α_2, is non-zero. Let

$$A := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad B := (A^T)^{-1}.$$

We claim that B is a change of coordinates we are looking for. If r is a defining function of D then $r \circ B^{-1}$ is a defining function of $B(D)$, so $B(D)$ is a bounded strongly linearly convex domain with real-analytic boundary. Let us check that Bf is an E-mapping of $B(D)$ with associated mappings

$$A\widetilde{f} \in \mathcal{O}(\overline{\mathbb{D}}) \quad \text{and} \quad \rho \frac{|A\overline{\nabla r \circ f}|}{|\nabla r \circ f|} \in \mathcal{C}^\omega(\mathbb{T}). \tag{15}$$

The conditions (1) and (2) of Definition 1.11 are clear. For $\zeta \in \mathbb{T}$ we have

$$\begin{aligned} \overline{\nu_{B(D)}(Bf(\zeta))} &= \frac{\overline{\nabla(r \circ B^{-1})(Bf(\zeta))}}{|\nabla(r \circ B^{-1})(Bf(\zeta))|} \\ &= \frac{(B^{-1})^T \overline{\nabla r(f(\zeta))}}{|(B^{-1})^T \overline{\nabla r(f(\zeta))}|} = \frac{A\overline{\nabla r(f(\zeta))}}{|A\overline{\nabla r(f(\zeta))}|}, \end{aligned} \tag{16}$$

so

$$\zeta\rho(\zeta) \frac{|A\overline{\nabla r(f(\zeta))}|}{|\nabla r(f(\zeta))|} \overline{\nu_{B(D)}(Bf(\zeta))} = \zeta\rho(\zeta) A\overline{\nu_D(f(\zeta))} = A\widetilde{f}(\zeta). \tag{17}$$

Moreover, for $\zeta \in \mathbb{T}$, $z \in D$,

$$\begin{aligned}\langle Bz - Bf(\zeta), \nu_{B(D)}(Bf(\zeta))\rangle &= \overline{\nu_{B(D)}(Bf(\zeta))}^T (Bz - Bf(\zeta)) \\ &= \frac{\overline{\nabla r(f(\zeta))}^T B^{-1} B(z - f(\zeta))}{|(B^{-1})^T \overline{\nabla r(f(\zeta))}|} = \frac{|\nabla r(f(\zeta))|}{|(B^{-1})^T \overline{\nabla r(f(\zeta))}|} \overline{\nu_D(f(\zeta))}^T (z - f(\zeta)) \\ &= \frac{|\nabla r(f(\zeta))|}{|(B^{-1})^T \overline{\nabla r(f(\zeta))}|} \langle z - f(\zeta), \nu_D(f(\zeta))\rangle.\end{aligned}$$

Therefore, B is the desired linear change of coordinates, as claimed.

If necessary, we shrink the sets U, V associated with f to sets associated with Bf. There exist holomorphic functions $h_1, h_2 : V \to \mathbb{C}$ such that

$$h_1 \widetilde{f}_1 + h_2 \widetilde{f}_2 \equiv 1 \quad \text{in } V.$$

Generally, this is a well-known fact for functions on pseudoconvex domains, however in the present case it may be shown elementarily. Indeed, if $\widetilde{f}_1 \equiv 0$ or $\widetilde{f}_2 \equiv 0$ then it is obvious. In the opposite case, let $\widetilde{f}_j = F_j P_j$, $j = 1, 2$, where F_j are holomorphic, non-zero in V, and P_j are polynomials with all (finitely many) zeroes in V. Then P_j are relatively prime, so there are polynomials Q_j, $j = 1, 2$, such that

$$Q_1 P_1 + Q_2 P_2 \equiv 1.$$

Hence

$$\frac{Q_1}{F_1} \widetilde{f}_1 + \frac{Q_2}{F_2} \widetilde{f}_2 \equiv 1 \quad \text{in } V.$$

Consider the mapping $\Psi : V \times \mathbb{C}^{n-1} \to \mathbb{C}^n$ given by

$$\Psi_1(Z) := f_1(Z_1) - Z_2 \widetilde{f}_2(Z_1) - h_1(Z_1) \sum_{j=3}^{n} Z_j \widetilde{f}_j(Z_1), \tag{18}$$

$$\Psi_2(Z) := f_2(Z_1) + Z_2 \widetilde{f}_1(Z_1) - h_2(Z_1) \sum_{j=3}^{n} Z_j \widetilde{f}_j(Z_1), \tag{19}$$

$$\Psi_j(Z) := f_j(Z_1) + Z_j, \quad j = 3, \ldots, n. \tag{20}$$

We claim that Ψ is biholomorphic in $\Psi^{-1}(U)$. First of all, observe that $\Psi^{-1}(\{z\}) \neq \emptyset$ for any $z \in U$. Indeed, by Proposition 3.2.4 there exists (exactly one) $Z_1 \in V$ such that

$$(z - f(Z_1)) \bullet \widetilde{f}(Z_1) = 0.$$

The numbers $Z_j \in \mathbb{C}$, $j = 3, \ldots, n$, are uniquely determined by the equations

$$Z_j = z_j - f_j(Z_1).$$

At least one of the numbers $\widetilde{f}_1(Z_1), \widetilde{f}_2(Z_1)$, say $\widetilde{f}_1(Z_1)$, is non-zero. Let

$$Z_2 := \frac{z_2 - f_2(Z_1) + h_2(Z_1)\sum_{j=3}^n Z_j \widetilde{f}_j(Z_1)}{\widetilde{f}_1(Z_1)}.$$

Then we easily check that the equality

$$z_1 = f_1(Z_1) - Z_2\widetilde{f}_2(Z_1) - h_1(Z_1)\sum_{j=3}^n Z_j\widetilde{f}_j(Z_1)$$

is equivalent to $(z - f(Z_1)) \bullet \widetilde{f}(Z_1) = 0$, which is true.

To finish the proof of biholomorphicity of Ψ in $\Psi^{-1}(U)$ it suffices to check that Ψ is injective in $\Psi^{-1}(U)$. Let us take Z, W such that $\Psi(Z) = \Psi(W) = z \in U$. By a direct computation both $\zeta = Z_1 \in V$ and $\zeta = W_1 \in V$ solve the equation

$$(z - f(\zeta)) \bullet \widetilde{f}(\zeta) = 0.$$

From Proposition 3.2.4 we infer that it has exactly one solution. Hence $Z_1 = W_1$. By (20) we have $Z_j = W_j$ for $j = 3, \ldots, n$. Finally $Z_2 = W_2$ follows from one of the equations (18), (19).

Let $G := \Psi^{-1}(D)$, $\widetilde{D} := U$, $\widetilde{G} := \Psi^{-1}(U)$, $\Phi := \Psi^{-1}$. We have

$$\Psi_j(\zeta, 0, \ldots, 0) = f_j(\zeta), \quad j = 1, \ldots, n,$$

so $\Phi(f(\zeta)) = (\zeta, 0, \ldots, 0)$, $\zeta \in \overline{\mathbb{D}}$. Put $g(\zeta) := \Phi(f(\zeta))$, $\zeta \in \overline{\mathbb{D}}$. Note that the entries of the matrix $\Psi'(g(\zeta))$ are

$$\frac{\partial\Psi_1}{\partial Z_1}(g(\zeta)) = f_1'(\zeta),\ \frac{\partial\Psi_1}{\partial Z_2}(g(\zeta)) = -\widetilde{f}_2(\zeta),\ \frac{\partial\Psi_1}{\partial Z_j}(g(\zeta)) = -h_1(\zeta)\widetilde{f}_j(\zeta),\ j \geq 3,$$

$$\frac{\partial\Psi_2}{\partial Z_1}(g(\zeta)) = f_2'(\zeta),\ \frac{\partial\Psi_2}{\partial Z_2}(g(\zeta)) = \widetilde{f}_1(\zeta),\quad \frac{\partial\Psi_2}{\partial Z_j}(g(\zeta)) = -h_2(\zeta)\widetilde{f}_j(\zeta),\ j \geq 3,$$

$$\frac{\partial\Psi_k}{\partial Z_1}(g(\zeta)) = f_k'(\zeta),\ \frac{\partial\Psi_k}{\partial Z_2}(g(\zeta)) = 0,\qquad \frac{\partial\Psi_k}{\partial Z_j}(g(\zeta)) = \delta_j^k,\ j, k \geq 3.$$

Thus $\Psi'(g(\zeta))^T\widetilde{f}(\zeta) = (1, 0, \ldots, 0)$, $\zeta \in \overline{\mathbb{D}}$ (as $f' \bullet \widetilde{f} = 1$). Let us take a defining function r of D. Then $r \circ \Psi$ is a defining function of G. Therefore,

$$\nu_G(g(\zeta)) = \frac{\nabla(r\circ\Psi)(g(\zeta))}{|\nabla(r\circ\Psi)(g(\zeta))|} = \frac{\overline{\Psi'(g(\zeta))}^T\nabla r(f(\zeta))}{|\overline{\Psi'(g(\zeta))}^T\nabla r(f(\zeta))|}$$

$$= \frac{\overline{\Psi'(g(\zeta))}^T\overline{\frac{\widetilde{f}(\zeta)}{\zeta\rho(\zeta)}}|\nabla r(f(\zeta))|}{\left|\overline{\Psi'(g(\zeta))}^T\overline{\frac{\widetilde{f}(\zeta)}{\zeta\rho(\zeta)}}|\nabla r(f(\zeta))|\right|} = g(\zeta), \quad \zeta \in \mathbb{T}.$$

It remains to prove the fifth condition. By (5) (see Definition 1.1) we have

to show that

$$\sum_{j,k=1}^{n} \frac{\partial^2 (r\circ\Psi)}{\partial z_j \partial \overline{z}_k}(g(\zeta))X_j\overline{X}_k > \Big|\sum_{j,k=1}^{n} \frac{\partial^2 (r\circ\Psi)}{\partial z_j \partial z_k}(g(\zeta))X_jX_k\Big| \tag{21}$$

for $\zeta\in\mathbb{T}$ and $X\in(\mathbb{C}^n)_*$ with

$$\sum_{j=1}^{n} \frac{\partial (r\circ\Psi)}{\partial z_j}(g(\zeta))X_j = 0,$$

i.e. $X_1=0$. We have

$$\begin{aligned}\sum_{j,k=1}^{n} \frac{\partial^2 (r\circ\Psi)}{\partial z_j \partial \overline{z}_k}(g(\zeta))X_j\overline{X}_k &= \sum_{j,k,s,t=1}^{n} \frac{\partial^2 r}{\partial z_s \partial \overline{z}_t}(f(\zeta))\frac{\partial \Psi_s}{\partial z_j}(g(\zeta))\overline{\frac{\partial \Psi_t}{\partial z_k}(g(\zeta))}X_j\overline{X}_k\\ &= \sum_{s,t=1}^{n} \frac{\partial^2 r}{\partial z_s \partial \overline{z}_t}(f(\zeta))Y_s\overline{Y}_t,\end{aligned}$$

where

$$Y := \Psi'(g(\zeta))X.$$

Note that $Y\neq 0$. Additionally

$$\sum_{s=1}^{n} \frac{\partial r}{\partial z_s}(f(\zeta))Y_s = \sum_{j,s=1}^{n} \frac{\partial r}{\partial z_s}(f(\zeta))\frac{\partial \Psi_s}{\partial z_j}(g(\zeta))X_j = \sum_{j=1}^{n} \frac{\partial (r\circ\Psi)}{\partial z_j}(g(\zeta))X_j = 0.$$

Therefore, by the strong linear convexity of D at $f(\zeta)$,

$$\sum_{s,t=1}^{n} \frac{\partial^2 r}{\partial z_s \partial \overline{z}_t}(f(\zeta))Y_s\overline{Y}_t > \Big|\sum_{s,t=1}^{n} \frac{\partial^2 r}{\partial z_s \partial z_t}(f(\zeta))Y_sY_t\Big|.$$

To finish the proof observe that

$$\begin{aligned}&\Big|\sum_{j,k=1}^{n} \frac{\partial^2 (r\circ\Psi)}{\partial z_j \partial z_k}(g(\zeta))X_jX_k\Big|\\ &\quad= \Big|\sum_{j,k,s,t=1}^{n} \frac{\partial^2 r}{\partial z_s \partial z_t}(f(\zeta))\frac{\partial \Psi_s}{\partial z_j}(g(\zeta))\frac{\partial \Psi_t}{\partial z_k}(g(\zeta))X_jX_k\\ &\qquad+ \sum_{j,k,s=1}^{n} \frac{\partial r}{\partial z_s}(f(\zeta))\frac{\partial^2 \Psi_s}{\partial z_j \partial z_k}(g(\zeta))X_jX_k\Big|\\ &\quad= \Big|\sum_{s,t=1}^{n} \frac{\partial^2 r}{\partial z_s \partial z_t}(f(\zeta))Y_sY_t + \sum_{j,k=2}^{n}\sum_{s=1}^{n} \frac{\partial r}{\partial z_s}(f(\zeta))\frac{\partial^2 \Psi_s}{\partial z_j \partial z_k}(g(\zeta))X_jX_k\Big|\end{aligned}$$

and

$$\frac{\partial^2 \Psi_s}{\partial z_j \partial z_k}(g(\zeta)) = 0, \qquad j,k\geq 2,\ s\geq 1,$$

which gives (21). ∎

REMARK 5.1.2. Let D be a bounded domain in $\mathbb{C}^n$ and let $f : \mathbb{D} \to D$ be a (weak) stationary mapping such that ∂D is real-analytic in a neighborhood of $f(\mathbb{T})$. Assume moreover that U is a neighborhood of $f(\overline{\mathbb{D}})$, a mapping $\Theta : U \to \mathbb{C}^n$ is biholomorphic onto its image and the set $D \cap U$ is connected. Then $\Theta \circ f$ is a (weak) stationary mapping of $G := \Theta(D \cap U)$.

In particular, if U_1, U_2 are neighborhoods of the closures of domains D_1, D_2 with real-analytic boundaries and $\Theta : U_1 \to U_2$ is a biholomorphism such that $\Theta(D_1) = D_2$, then Θ maps (weak) stationary mappings of D_1 onto (weak) stationary mappings of D_2.

Proof. Actually, it is clear that the first two conditions of the definition of (weak) stationary mappings are preserved by Θ. To show the third one we proceed as in (15)–(17). Let $f : \mathbb{D} \to D$ be a (weak) stationary mapping. The candidates for the mappings in the condition (3) (resp. (3′)) of Definition 1.11 for $\Theta \circ f$ in the domain G are

$$((\Theta' \circ f)^{-1})^T \widetilde{f} \quad \text{and} \quad \rho \frac{|((\Theta' \circ f)^{-1})^T \overline{\nabla r \circ f}|}{|\nabla r \circ f|}.$$

Indeed, for $\zeta \in \mathbb{T}$,

$$\begin{aligned}\overline{\nu_G(\Theta(f(\zeta)))} &= \frac{\overline{\nabla(r \circ \Theta^{-1})(\Theta(f(\zeta)))}}{|\nabla(r \circ \Theta^{-1})(\Theta(f(\zeta)))|} = \frac{[(\Theta^{-1})'(\Theta(f(\zeta)))]^T \overline{\nabla r(f(\zeta))}}{|[(\Theta^{-1})'(\Theta(f(\zeta)))]^T \overline{\nabla r(f(\zeta))}|} \\ &= \frac{(\Theta'(f(\zeta))^{-1})^T \overline{\nabla r(f(\zeta))}}{|(\Theta'(f(\zeta))^{-1})^T \overline{\nabla r(f(\zeta))}|},\end{aligned}$$

hence

$$\begin{aligned}\zeta\rho(\zeta) \frac{|(\Theta'(f(\zeta))^{-1})^T \overline{\nabla r(f(\zeta))}|}{|\nabla r(f(\zeta))|} \overline{\nu_G(\Theta(f(\zeta)))} &= \zeta\rho(\zeta)(\Theta'(f(\zeta))^{-1})^T \overline{\nu_D(f(\zeta))} \\ &= (\Theta'(f(\zeta))^{-1})^T \widetilde{f}(\zeta). \ \blacksquare\end{aligned}$$

5.2. Situation (†). Consider the following situation, denoted by (†) (with data D_0 and U_0):

- D_0 is a bounded domain in $\mathbb{C}^n$, $n \geq 2$;
- $f_0 : \overline{\mathbb{D}} \ni \zeta \mapsto (\zeta, 0, \ldots, 0) \in \overline{D}_0$, $\zeta \in \overline{\mathbb{D}}$;
- $f_0(\mathbb{D}) \subset D_0$;
- $f_0(\mathbb{T}) \subset \partial D_0$;
- $\nu_{D_0}(f_0(\zeta)) = (\zeta, 0, \ldots, 0)$, $\zeta \in \mathbb{T}$;
- for any $\zeta \in \mathbb{T}$, a point $f_0(\zeta)$ is a point of strong linear convexity of D_0;
- ∂D_0 is real-analytic in a neighborhood U_0 of $f_0(\mathbb{T})$ with a function r_0;
- $|\nabla r_0| = 1$ on $f_0(\mathbb{T})$; in particular, $r_{0z}(f_0(\zeta)) = (\overline{\zeta}/2, 0, \ldots, 0)$, $\zeta \in \mathbb{T}$.

Since r_0 is real-analytic on $U_0 \subset \mathbb{R}^{2n}$, it extends in a natural way to a holomorphic function in a neighborhood $U_0^{\mathbb{C}} \subset \mathbb{C}^{2n}$ of U_0. Without loss of

generality we may assume that r_0 is bounded on $U_0^{\mathbb{C}}$. Set

$$X_0 = X_0(U_0, U_0^{\mathbb{C}}) := \{r \in \mathcal{O}(U_0^{\mathbb{C}}) : r(U_0) \subset \mathbb{R} \text{ and } r \text{ is bounded}\},$$

which equipped with the sup-norm is a (real) Banach space.

Remark 5.2.1. Lempert considered the case when U_0 is a neighborhood of the boundary of a bounded domain D_0 with real-analytic boundary. We shall need more general results to prove the 'localization property'.

5.3. General lemmas. We keep the notation from Subsection 5.2 and assume situation (†).

Let us introduce some additional objects we shall be dealing with and let us prove more general lemmas (their generality will be useful in the next section).

Consider the Sobolev space $W^{2,2}(\mathbb{T}, \mathbb{C}^m)$ of functions $f : \mathbb{T} \to \mathbb{C}^m$ whose first two derivatives (in the sense of distributions) are in $L^2(\mathbb{T})$. The $W^{2,2}$-norm is denoted by $\|\cdot\|_W$. For the basic properties of $W^{2,2}(\mathbb{T}, \mathbb{C}^m)$ see Appendix.

Put

$$B := \{f \in W^{2,2}(\mathbb{T}, \mathbb{C}^n) : f \text{ extends holomorphically on } \mathbb{D} \text{ and } f(0) = 0\},$$
$$B_0 := \{f \in B : f(\mathbb{T}) \subset U_0\}, \qquad B^* := \{\overline{f} : f \in B\},$$
$$Q := \{q \in W^{2,2}(\mathbb{T}, \mathbb{C}) : q(\mathbb{T}) \subset \mathbb{R}\}, \qquad Q_0 := \{q \in Q : q(1) = 0\}.$$

It is clear that B, B^*, Q and Q_0 equipped with the norm $\|\cdot\|_W$ are (real) Banach spaces. Note that B_0 is an open neighborhood of f_0. In what follows, we identify $f \in B$ with its unique holomorphic extension on $\mathbb{D}$.

Let us define the projection

$$\pi : W^{2,2}(\mathbb{T}, \mathbb{C}^n) \ni f = \sum_{k=-\infty}^{\infty} a_k \zeta^k \mapsto \sum_{k=-\infty}^{-1} a_k \zeta^k \in B^*.$$

Note that $f \in W^{2,2}(\mathbb{T}, \mathbb{C}^n)$ extends holomorphically on $\mathbb{D}$ if and only if $\pi(f) = 0$ (and the extension is $\mathcal{C}^{1/2}$ on $\mathbb{T}$). Actually, it suffices to observe that $g(\zeta) := \sum_{k=-\infty}^{-1} a_k \zeta^k$, $\zeta \in \mathbb{T}$, extends holomorphically on $\mathbb{D}$ if and only if $a_k = 0$ for $k < 0$. This follows immediately from the fact that the mapping $\mathbb{T} \ni \zeta \mapsto g(\overline{\zeta}) \in \mathbb{C}^n$ extends holomorphically on $\mathbb{D}$.

Consider the mapping $\Xi : X_0 \times \mathbb{C}^n \times B_0 \times Q_0 \times \mathbb{R} \to Q \times B^* \times \mathbb{C}^n$ defined by

$$\Xi(r, v, f, q, \lambda) := (r \circ f, \pi(\zeta(1 + q)(r_z \circ f)), f'(0) - \lambda v),$$

where ζ is treated as the identity function on $\mathbb{T}$. We have the following

Lemma 5.3.1. *There exist a neighborhood V_0 of $(r_0, f_0'(0))$ in $X_0 \times \mathbb{C}^n$ and a real-analytic mapping $\Upsilon : V_0 \to B_0 \times Q_0 \times \mathbb{R}$ such that for any $(r, v) \in V_0$ we have $\Xi(r, v, \Upsilon(r, v)) = 0$.*

Let $\widetilde{\Xi} : X_0 \times \mathbb{C}^n \times B_0 \times Q_0 \times (0,1) \to Q \times B^* \times \mathbb{C}^n$ be defined as

$$\widetilde{\Xi}(r, w, f, q, \xi) := (r \circ f, \pi(\zeta(1+q)(r_z \circ f)), f(\xi) - w).$$

Analogously we have

LEMMA 5.3.2. *Let $\xi_0 \in (0,1)$. Then there exist a neighborhood W_0 of $(r_0, f_0(\xi_0))$ in $X_0 \times D_0$ and a real-analytic mapping $\widetilde{\Upsilon} : W_0 \to B_0 \times Q_0 \times (0,1)$ such that for any $(r, w) \in W_0$ we have $\widetilde{\Xi}(r, w, \widetilde{\Upsilon}(r, w)) = 0$.*

Proof of Lemmas 5.3.1 and 5.3.2. We will prove the first lemma. Then we will see that the proof of the second one reduces to that proof.

We claim that Ξ is real-analytic. The only problem is to show that the mapping

$$T : X_0 \times B_0 \ni (r, f) \mapsto r \circ f \in Q$$

is real-analytic (the real-analyticity of the mapping $X_0 \times B_0 \ni (r, f) \mapsto r_z \circ f \in W^{2,2}(\mathbb{T}, \mathbb{C}^n)$ follows from this claim).

Fix $r \in X_0$, $f \in B_0$ and take $\varepsilon > 0$ such that a $2n$-dimensional polydisc $P_{2n}(f(\zeta), \varepsilon)$ is contained in $U_0^{\mathbb{C}}$ for any $\zeta \in \mathbb{T}$. Then any function $\widetilde{r} \in X_0$ is holomorphic in $U_0^{\mathbb{C}}$, so it expands in a holomorphic power series convergent in $P_{2n}(f(\zeta), \varepsilon)$. Losing no generality we may assume that n-dimensional polydiscs $P_n(f(\zeta), \varepsilon)$, $\zeta \in \mathbb{T}$, satisfy $P_n(f(\zeta), \varepsilon) \subset U_0$. This gives an expansion of the function $\widetilde{r}$ at any point $f(\zeta)$, $\zeta \in \mathbb{T}$, in a series

$$\sum_{\alpha \in \mathbb{N}_0^{2n}} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \widetilde{r}}{\partial x^\alpha}(f(\zeta)) x^\alpha$$

convergent to $\widetilde{r}(f(\zeta) + x)$, provided that $x = (x_1, \ldots, x_{2n}) \in P_n(0, \varepsilon)$ (where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_{2n}$). Hence

$$T(r + \varrho, f + h) = \sum_{\alpha \in \mathbb{N}_0^{2n}} \frac{1}{\alpha!} \left(\frac{\partial^{|\alpha|} r}{\partial x^\alpha} \circ f \right) h^\alpha + \sum_{\alpha \in \mathbb{N}_0^{2n}} \frac{1}{\alpha!} \left(\frac{\partial^{|\alpha|} \varrho}{\partial x^\alpha} \circ f \right) h^\alpha \tag{22}$$

pointwise for $\varrho \in X_0$ and $h \in W^{2,2}(\mathbb{T}, \mathbb{C}^n)$ with $\|h\|_{\sup} < \varepsilon$.

Put $P := \bigcup_{\zeta \in \mathbb{T}} P_{2n}(f(\zeta), \varepsilon)$ and for $\widetilde{r} \in X_0$ put $\|\widetilde{r}\|_P := \sup_P |\widetilde{r}|$. Let $\widetilde{r}$ be equal to r or to ϱ, where ϱ lies in a neighborhood of 0 in X_0. The Cauchy inequalities give

$$\left| \frac{\partial^{|\alpha|} \widetilde{r}}{\partial x^\alpha}(f(\zeta)) \right| \le \frac{\alpha! \|\widetilde{r}\|_P}{\varepsilon^{|\alpha|}}, \qquad \zeta \in \mathbb{T}. \tag{23}$$

Therefore,

$$\left\| \frac{\partial^{|\alpha|} \widetilde{r}}{\partial x^\alpha} \circ f \right\|_W \le C_1 \frac{\alpha! \|\widetilde{r}\|_P}{\varepsilon^{|\alpha|}}$$

for some $C_1 > 0$.

There is $C_2 > 0$ such that

$$\|gh^\alpha\|_W \le C_2^{|\alpha|+1}\|g\|_W\|h_1\|_W^{\alpha_1}\cdot\ldots\cdot\|h_{2n}\|_W^{\alpha_{2n}}$$

for $g \in W^{2,2}(\mathbb{T},\mathbb{C})$, $h \in W^{2,2}(\mathbb{T},\mathbb{C}^n)$, $\alpha \in \mathbb{N}_0^{2n}$ (see Appendix for a proof of this fact). Using the above inequalities we infer that

$$\sum_{\alpha\in\mathbb{N}_0^{2n}} \left\| \frac{1}{\alpha!}\left(\frac{\partial^{|\alpha|}\widetilde{r}}{\partial x^\alpha}\circ f\right)h^\alpha\right\|_W$$

is convergent if $\|h\|_W$ is small enough. Therefore, the series in (22) are absolutely convergent in the norm $\|\cdot\|_W$, whence T is real-analytic.

To show the existence of V_0 and Υ we will make use of the Implicit Function Theorem. More precisely, we shall show that the partial derivative

$$\Xi_{(f,q,\lambda)}(r_0, f_0'(0), f_0, 0, 1) : B \times Q_0 \times \mathbb{R} \to Q \times B^* \times \mathbb{C}^n$$

is an isomorphism. Observe that for any $(\widetilde{f},\widetilde{q},\widetilde{\lambda}) \in B \times Q_0 \times \mathbb{R}$ the following equality holds:

$$\begin{aligned}\Xi_{(f,q,\lambda)}(r_0, f_0'(0), f_0, 0, 1)(\widetilde{f},\widetilde{q},\widetilde{\lambda}) &= \frac{d}{dt}\Xi(r_0, f_0'(0), f_0 + t\widetilde{f}, t\widetilde{q}, 1 + t\widetilde{\lambda})\Big|_{t=0}\\ &= ((r_{0z}\circ f_0)\widetilde{f} + (r_{0\overline{z}}\circ f_0)\overline{\widetilde{f}}, \pi(\zeta\widetilde{q}r_{0z}\circ f_0 + \zeta(r_{0zz}\circ f_0)\widetilde{f} + \zeta(r_{0z\overline{z}}\circ f_0)\overline{\widetilde{f}}),\\ &\qquad \widetilde{f}'(0) - \widetilde{\lambda}f_0'(0)),\end{aligned}$$

where we treat $r_{0z}, r_{0\overline{z}}$ as row vectors, $\widetilde{f}, \overline{\widetilde{f}}$ as column vectors and $r_{0zz} = \left[\frac{\partial^2 r_0}{\partial z_j \partial z_k}\right]_{j,k=1}^n$, $r_{0z\overline{z}} = \left[\frac{\partial^2 r_0}{\partial z_j \partial \overline{z}_k}\right]_{j,k=1}^n$ as $n \times n$ matrices.

By the Bounded Inverse Theorem, it suffices to show that the derivative $\Xi_{(f,q,\lambda)}(r_0, f_0'(0), f_0, 0, 1)$ is bijective, i.e. for any $(\eta,\varphi,v) \in Q \times B^* \times \mathbb{C}^n$ there exists exactly one $(\widetilde{f},\widetilde{q},\widetilde{\lambda}) \in B \times Q_0 \times \mathbb{R}$ satisfying

$$(r_{0z}\circ f_0)\widetilde{f} + (r_{0\overline{z}}\circ f_0)\overline{\widetilde{f}} = \eta, \tag{24}$$

$$\pi(\zeta\widetilde{q}r_{0z}\circ f_0 + \zeta(r_{0zz}\circ f_0)\widetilde{f} + \zeta(r_{0z\overline{z}}\circ f_0)\overline{\widetilde{f}}) = \varphi, \tag{25}$$

$$\widetilde{f}'(0) - \widetilde{\lambda}f_0'(0) = v. \tag{26}$$

First, we show that $\widetilde{\lambda}$ and $\widetilde{f}_1$ are uniquely determined. Observe that, in view of the assumptions, (24) is just

$$\frac{1}{2}\overline{\zeta}\widetilde{f}_1 + \frac{1}{2}\zeta\overline{\widetilde{f}}_1 = \eta$$

or equivalently

$$\operatorname{Re}(\widetilde{f}_1/\zeta) = \eta \quad (\text{on } \mathbb{T}). \tag{27}$$

Note that (27) uniquely determines $\widetilde{f}_1/\zeta \in W^{2,2}(\mathbb{T},\mathbb{C}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ up to an imaginary additive constant, which may be computed using (26). Actually, $\eta = \operatorname{Re} G$ on $\mathbb{T}$ for some function $G \in W^{2,2}(\mathbb{T},\mathbb{C}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$. To see this,

let us expand $\eta(\zeta) = \sum_{k=-\infty}^{\infty} a_k\zeta^k$, $\zeta \in \mathbb{T}$. From the equality $\eta(\zeta) = \overline{\eta(\zeta)}$, $\zeta \in \mathbb{T}$, we get

$$\sum_{k=-\infty}^{\infty} a_k\zeta^k = \sum_{k=-\infty}^{\infty} \overline{a}_k\zeta^{-k} = \sum_{k=-\infty}^{\infty} \overline{a}_{-k}\zeta^k, \qquad \zeta \in \mathbb{T}, \tag{28}$$

so $a_{-k} = \overline{a}_k$, $k \in \mathbb{Z}$. Hence

$$\eta(\zeta) = a_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}(a_k\zeta^k) = \operatorname{Re}\Big(a_0 + 2\sum_{k=1}^{\infty} a_k\zeta^k\Big), \qquad \zeta \in \mathbb{T}.$$

Set

$$G(\zeta) := a_0 + 2\sum_{k=1}^{\infty} a_k\zeta^k, \qquad \zeta \in \mathbb{D}.$$

This series is convergent for $\zeta \in \mathbb{D}$, so $G \in \mathcal{O}(\mathbb{D})$. Further, the function G extends continuously on $\overline{\mathbb{D}}$ (to the function denoted by the same letter) and the extension lies in $W^{2,2}(\mathbb{T}, \mathbb{C})$. Clearly, $\eta = \operatorname{Re} G$ on $\mathbb{T}$.

We are searching for $C \in \mathbb{R}$ such that the functions $\widetilde{f}_1 := \zeta(G + iC)$ and $\theta := \operatorname{Im}(\widetilde{f}_1/\zeta)$ satisfy

$$\eta(0) + i\theta(0) = \widetilde{f}_1'(0)$$

and

$$\eta(0) + i\theta(0) - \widetilde{\lambda}\operatorname{Re} f_{01}'(0) - i\widetilde{\lambda}\operatorname{Im} f_{01}'(0) = \operatorname{Re} v_1 + i\operatorname{Im} v_1.$$

But

$$\eta(0) - \widetilde{\lambda}\operatorname{Re} f_{01}'(0) = \operatorname{Re} v_1,$$

which yields $\widetilde{\lambda}$ and then $\theta(0)$, and consequently the number C. Having $\widetilde{\lambda}$ and once again using (26), we find uniquely determined $\widetilde{f}_2'(0), \ldots, \widetilde{f}_n'(0)$.

Therefore, the equations (24) and (26) are satisfied by uniquely determined $\widetilde{f}_1$, $\widetilde{\lambda}$ and $\widetilde{f}_2'(0), \ldots, \widetilde{f}_n'(0)$.

Consider (25), which is a system of n equations with unknowns $\widetilde{q}, \widetilde{f}_2, \ldots, \widetilde{f}_n$. Observe that $\widetilde{q}$ appears only in the first of the equations and the remaining $n-1$ equations mean exactly that the mapping

$$\zeta(r_{0\widehat{z}\widehat{z}} \circ f_0)\widehat{\widetilde{f}} + \zeta(r_{0\widehat{z}\widehat{z}} \circ f_0)\widehat{\overline{\widetilde{f}}} - \psi \tag{29}$$

extends holomorphically on $\mathbb{D}$; here $\widehat{a} := (a_2, \ldots, a_n)$ and $\psi \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1})$ may be obtained from φ and $\widetilde{f}_1$. To see this, write (25) in the form

$$\pi(F_1 + \zeta F_2 + \zeta F_3) = (\varphi_1, \ldots, \varphi_n),$$

where

$$F_1 := (\widetilde{q}, 0, \ldots, 0),$$
$$F_2 := (A_j)_{j=1}^n, \quad A_j := \sum_{k=1}^n (r_{0z_jz_k} \circ f_0)\widetilde{f}_k,$$
$$F_3 := (B_j)_{j=1}^n, \quad B_j := \sum_{k=1}^n (r_{0z_j\overline{z}_k} \circ f_0)\overline{\widetilde{f}}_k.$$

It follows that

$$\widetilde{q} + \zeta A_1 + \zeta B_1 - \varphi_1 \quad \text{and} \quad \zeta A_j + \zeta B_j - \varphi_j, \quad j = 2, \ldots, n,$$

should extend holomorphically on $\mathbb{D}$ and

$$\psi := \big(\varphi_j - \zeta(r_{0z_jz_1} \circ f_0)\widetilde{f}_1 - \zeta(r_{0z_j\overline{z}_1} \circ f_0)\overline{\widetilde{f}}_1\big)_{j=2}^n.$$

Put

$$g(\zeta) := \widehat{\widetilde{f}}(\zeta)/\zeta, \quad \alpha(\zeta) := \zeta^2 r_{0\widehat{z}\widehat{z}}(f_0(\zeta)), \quad \beta(\zeta) := r_{0\widehat{z}\widehat{\overline{z}}}(f_0(\zeta)).$$

Observe that $\alpha(\zeta)$, $\beta(\zeta)$ are $(n-1) \times (n-1)$ matrices depending real-analytically on ζ, and $g(\zeta)$ is a column vector in $\mathbb{C}^{n-1}$. This allows us to reduce (29) to the following problem: find a unique $g \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ such that

$$\alpha g + \beta\overline{g} - \psi \text{ extends holomorphically on } \mathbb{D} \text{ and } g(0) = \widehat{\widetilde{f}}'(0). \tag{30}$$

The fact that every $f_0(\zeta)$ is a point of strong linear convexity of D_0 may be written as

$$|X^T\alpha(\zeta)X| < X^T\beta(\zeta)\overline{X}, \quad \zeta \in \mathbb{T},\ X \in (\mathbb{C}^{n-1})_*. \tag{31}$$

Note that $\beta(\zeta)$ is self-adjoint and strictly positive, hence using Proposition 8.4.1 we get a mapping $H \in \mathcal{O}(\overline{\mathbb{D}}, \mathbb{C}^{(n-1)\times(n-1)})$ such that $\det H \neq 0$ on $\overline{\mathbb{D}}$ and $HH^* = \beta$ on $\mathbb{T}$. With this notation, (30) is equivalent to

$$H^{-1}\alpha g + H^*\overline{g} - H^{-1}\psi \text{ extends holomorphically on } \mathbb{D} \tag{32}$$

or, if we denote $h := H^T g$, $\gamma := H^{-1}\alpha(H^T)^{-1}$, to

$$\gamma h + \overline{h} - H^{-1}\psi \text{ extends holomorphically on } \mathbb{D}. \tag{33}$$

For any $\zeta \in \mathbb{T}$ the operator norm of the symmetric matrix $\gamma(\zeta)$ is uniformly less than 1. In fact, from (31), for any $X \in \mathbb{C}^{n-1}$ with $|X| = 1$,

$$\begin{aligned}|X^T\gamma(\zeta)X| &= |X^TH(\zeta)^{-1}\alpha(\zeta)(H(\zeta)^T)^{-1}X| \\ &< X^TH(\zeta)^{-1}\beta(\zeta)\overline{(H(\zeta)^T)^{-1}X} \\ &= X^TH(\zeta)^{-1}H(\zeta)H(\zeta)^*\overline{(H(\zeta)^T)^{-1}}\,\overline{X} = |X|^2 = 1,\end{aligned}$$

so, by a compactness argument, $|X^T\gamma(\zeta)X| \le 1 - \widetilde{\varepsilon}$ for some $\widetilde{\varepsilon} > 0$ independent of ζ and X. Thus $\|\gamma(\zeta)\| \le 1 - \widetilde{\varepsilon}$ by Proposition 8.4.2.

We have to prove that there is a unique solution $h \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ of (33) such that $h(0) = a$ with a given $a \in \mathbb{C}^{n-1}$.

Define the operator

$$P : W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}) \ni \sum_{k=-\infty}^{\infty} a_k \zeta^k \mapsto \overline{\sum_{k=-\infty}^{-1} a_k \zeta^k} \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}),$$

where $a_k \in \mathbb{C}^{n-1}$, $k \in \mathbb{Z}$. We will show that a mapping $h \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ satisfies (33) and $h(0) = a$ if and only if it is a fixed point of the mapping

$$K : W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}) \ni h \mapsto P(H^{-1}\psi - \gamma h) + a \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}).$$

Indeed, take $h \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ and suppose that $h(0) = a$ and $\gamma h + \overline{h} - H^{-1}\psi$ extends holomorphically on $\mathbb{D}$. Then

$$h = a + \sum_{k=1}^{\infty} a_k \zeta^k, \quad \overline{h} = \overline{a} + \sum_{k=1}^{\infty} \overline{a}_k \zeta^{-k} = \sum_{k=-\infty}^{-1} \overline{a}_{-k} \zeta^k + \overline{a},$$

$$P(h) = 0, \qquad P(\overline{h}) = \sum_{k=1}^{\infty} a_k \zeta^k = h - a$$

and

$$P(\gamma h + \overline{h} - H^{-1}\psi) = 0,$$

which implies

$$P(H^{-1}\psi - \gamma h) = h - a$$

and finally $K(h) = h$. Conversely, suppose that $K(h) = h$. Then

$$P(H^{-1}\psi - \gamma h) = h - a = \sum_{k=1}^{\infty} a_k \zeta^k + a_1 - a, \qquad P(h) = 0$$

and

$$P(\overline{h}) = \sum_{k=1}^{\infty} a_k \zeta^k = h - a_1,$$

from which follows that

$$P(\gamma h + \overline{h} - H^{-1}\psi) = P(\overline{h}) - P(H^{-1}\psi - \gamma h) = a - a_1$$

and

$$P(\gamma h + \overline{h} - H^{-1}\psi) = 0 \quad \text{iff} \quad a = a_1.$$

Observe that $h(0) = K(h)(0) = P(H^{-1}\psi - \gamma h)(0) + a = a$.

We shall make use of the Banach Fixed Point Theorem. To do this, consider $W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1})$ equipped with the following norm:

$$\|h\|_\varepsilon := \|h\|_L + \varepsilon \|h'\|_L + \varepsilon^2 \|h''\|_L,$$

where $\varepsilon > 0$ and $\|\cdot\|_L$ is the L^2-norm (it is a Banach space). We will prove that K is a contraction with respect to the norm $\|\cdot\|_\varepsilon$ for sufficiently small ε. Indeed, there is $\widetilde{\varepsilon} > 0$ such that, for any $h_1, h_2 \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1})$,

$$\begin{aligned}(34)\qquad \|K(h_1) - K(h_2)\|_L &= \|P(\gamma(h_2 - h_1))\|_L \\ &\le \|\gamma(h_2 - h_1)\|_L \le (1 - \widetilde{\varepsilon})\|h_2 - h_1\|_L.\end{aligned}$$

Moreover,

$$\begin{aligned}(35)\qquad \|K(h_1)' - K(h_2)'\|_L &= \|P(\gamma h_2)' - P(\gamma h_1)'\|_L \\ &\le \|(\gamma h_2)' - (\gamma h_1)'\|_L = \|\gamma'(h_2 - h_1) + \gamma(h_2' - h_1')\|_L.\end{aligned}$$

Furthermore,

$$\begin{aligned}(36)\qquad &\|K(h_1)'' - K(h_2)''\|_L \\ &\le \|\gamma''(h_2 - h_1)\|_L + 2\|\gamma'(h_2' - h_1')\|_L + \|\gamma(h_1'' - h_2'')\|_L.\end{aligned}$$

Using the finiteness of $\|\gamma'\|$, $\|\gamma''\|$ and putting (34), (35), (36) together, we see that there exists $\varepsilon > 0$ such that K is a contraction with respect to $\|\cdot\|_\varepsilon$.

We have found $\widetilde{f}$ and $\widetilde{\lambda}$ satisfying (24), (26), and the last $n-1$ equations from (25) are satisfied as well.

It remains to show that there exists a unique $\widetilde{q} \in Q_0$ such that $\widetilde{q} + \zeta A_1 + \zeta B_1 - \varphi_1$ extends holomorphically on $\mathbb{D}$.

Comparing the coefficients as in (28), we see that if

$$\pi(\zeta A_1 + \zeta B_1 - \varphi_1) = \sum_{k=-\infty}^{-1} a_k \zeta^k$$

then $\widetilde{q}$ has to be taken as

$$-\sum_{k=-\infty}^{-1} a_k \zeta^k - \sum_{k=0}^{\infty} b_k \zeta^k$$

with $b_k := \overline{a}_{-k}$ for $k \ge 1$ and $b_0 \in \mathbb{R}$ uniquely determined by $\widetilde{q}(1) = 0$.

Let us show that Lemma 5.3.2 follows from the proof of 5.3.1. Since $\widetilde{\Xi}$ is real-analytic, it suffices to prove that the derivative

$$\widetilde{\Xi}_{(f,q,\xi)}(r_0, f_0(\xi_0), f_0, 0, \xi_0) : B \times Q_0 \times \mathbb{R} \to Q \times B^* \times \mathbb{C}^n$$

is invertible. For $(\widetilde{f}, \widetilde{q}, \widetilde{\xi}) \in B \times Q_0 \times \mathbb{R}$ we get

$$\begin{aligned}\widetilde{\Xi}_{(f,q,\xi)}(r_0, f_0(\xi_0), f_0, 0, \xi_0)(\widetilde{f}, \widetilde{q}, \widetilde{\xi}) &= \frac{d}{dt}\widetilde{\Xi}(r_0, f_0(\xi_0), f_0 + t\widetilde{f}, t\widetilde{q}, \xi_0 + t\widetilde{\xi})\Big|_{t=0} \\ = ((r_{0z} \circ f_0)\widetilde{f} + (r_{0\overline{z}} \circ f_0)\overline{\widetilde{f}}, &\pi(\zeta\widetilde{q} r_{0z} \circ f_0 + \zeta(r_{0zz} \circ f_0)\widetilde{f} + \zeta(r_{0z\overline{z}} \circ f_0)\overline{\widetilde{f}}), \\ &\widetilde{f}(\xi_0) + \widetilde{\xi} f_0'(\xi_0)).\end{aligned}$$

We have to show that for any $(\eta, \varphi, w) \in Q \times B^* \times \mathbb{C}^n$ there exists exactly one $(\widetilde{f}, \widetilde{q}, \widetilde{\xi}) \in B \times Q_0 \times \mathbb{R}$ satisfying

$$(r_{0z} \circ f_0)\widetilde{f} + (r_{0\overline{z}} \circ f_0)\overline{\widetilde{f}} = \eta, \tag{37}$$

$$\pi(\zeta \widetilde{q} r_{0z} \circ f_0 + \zeta(r_{0zz} \circ f_0)\widetilde{f} + \zeta(r_{0z\overline{z}} \circ f_0)\overline{\widetilde{f}}) = \varphi, \tag{38}$$

$$\widetilde{f}(\xi_0) + \widetilde{\xi} f_0'(\xi_0) = w. \tag{39}$$

The equation (37) turns out to be

$$\mathrm{Re}(\widetilde{f}_1/\zeta) = \eta \quad (\text{on } \mathbb{T}). \tag{40}$$

The equation above uniquely determines $\widetilde{f}_1/\zeta \in W^{2,2}(\mathbb{T}, \mathbb{C}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ up to an imaginary additive constant, which may be computed using (39). Indeed, there exists $G \in W^{2,2}(\mathbb{T}, \mathbb{C}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ such that $\eta = \mathrm{Re}\, G$ on $\mathbb{T}$. We are searching for $C \in \mathbb{R}$ such that the functions $\widetilde{f}_1 := \zeta(G + iC)$ and $\theta := \mathrm{Im}(\widetilde{f}_1/\zeta)$ satisfy

$$\xi_0 \eta(\xi_0) + i\xi_0 \theta(\xi_0) = \widetilde{f}_1(\xi_0)$$

and

$$\xi_0(\eta(\xi_0) + i\theta(\xi_0)) + \widetilde{\xi}\, \mathrm{Re}\, f_{01}'(\xi_0) + i\widetilde{\xi}\, \mathrm{Im}\, f_{01}'(\xi_0) = \mathrm{Re}\, w_1 + i\, \mathrm{Im}\, w_1.$$

But

$$\xi_0 \eta(\xi_0) + \widetilde{\xi}\, \mathrm{Re}\, f_{01}'(\xi_0) = \mathrm{Re}\, w_1,$$

which yields $\widetilde{\xi}$ and then $\theta(\xi_0)$, and consequently the number C. Having $\widetilde{\xi}$ and once again using (39), we find uniquely determined $\widetilde{f}_2(\xi_0), \ldots, \widetilde{f}_n(\xi_0)$.

Therefore, the equations (37) and (39) are satisfied by uniquely determined $\widetilde{f}_1$, $\widetilde{\xi}$ and $\widetilde{f}_2(\xi_0), \ldots, \widetilde{f}_n(\xi_0)$.

In the remaining part of the proof we change the second condition of (30) to

$$g(\xi_0) = \widehat{\widetilde{f}}(\xi_0)/\xi_0$$

and we have to prove that there is a unique solution $h \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ of (33) such that $h(\xi_0) = a$ with a given $a \in \mathbb{C}^{n-1}$. Let τ be an automorphism of $\mathbb{D}$ (so it extends holomorphically near $\overline{\mathbb{D}}$), which maps 0 to ξ_0, i.e.

$$\tau(\xi) := \frac{\xi_0 - \xi}{1 - \overline{\xi}_0 \xi}, \qquad \xi \in \mathbb{D}.$$

Let the mappings P, K be as before. Then $h \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ satisfies (33) and $h(\xi_0) = a$ if and only if $h \circ \tau \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ satisfies (33) and $(h \circ \tau)(0) = a$. We already know that there is exactly one $\widetilde{h} \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1}) \cap \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ satisfying (33) and $\widetilde{h}(0) = a$. Setting $h := \widetilde{h} \circ \tau^{-1}$, we get the claim. ∎

5.4. Topology in a class of domains with real-analytic boundaries. We introduce a concept of a domain being close to another domain. Let $D_0 \subset \mathbb{C}^n$ be a bounded domain with real-analytic boundary. Then there exist a neighborhood U_0 of ∂D_0 and a real-analytic defining function $r_0 : U_0 \to \mathbb{R}$ such that ∇r_0 does not vanish in U_0 and

$$D_0 \cap U_0 = \{z \in U_0 : r_0(z) < 0\}.$$

Definition 5.4.1. We say that domains D *tend to* D_0 if one can choose their defining functions $r \in X_0$ so that they tend to r_0 in X_0.

Remark 5.4.2. If $r \in X_0$ is near r_0 with respect to the topology on X_0, then $\{z \in U_0 : r(z) = 0\}$ is a compact real-analytic hypersurface which bounds a bounded domain. We denote it by D^r.

Moreover, if D^{r_0} is strongly linearly convex then so is D^r provided that r is near r_0.

5.5. Statement of the main result of this section

Remark 5.5.1. Assume that D^r is a strongly linearly convex domain bounded by a real-analytic hypersurface $\{z \in U_0 : r(z) = 0\}$. Let $\xi \in (0,1)$ and $w \in (\mathbb{C}^n)_*$. Then a function $f \in B_0$ satisfies the conditions

$$f \text{ is a weak stationary mapping of } D^r, \qquad f(0) = 0, \qquad f(\xi) = w$$

if and only if there exists $q \in Q_0$ such that $q > -1$ and $\widetilde{\Xi}(r, w, f, q, \xi) = 0$.

Actually, from $\widetilde{\Xi}(r, w, f, q, \xi) = 0$ we deduce immediately that $r \circ f = 0$ on $\mathbb{T}$, $f(\xi) = w$ and $\pi(\zeta(1+q)(r_z \circ f)) = 0$. From the first equality we get $f(\mathbb{T}) \subset \partial D^r$. From the last one we infer that the condition $(3')$ of Definition 1.11 is satisfied (with $\rho := (1+q)|r_z \circ f|$). Since D^r is strongly linearly convex, $\overline{D^r}$ is polynomially convex (use the fact that projections of $\mathbb{C}$-convex domains are $\mathbb{C}$-convex as well, and that D^r is smooth). In particular,

$$f(\overline{\mathbb{D}}) = f(\widehat{\mathbb{T}}) \subset \widehat{f(\mathbb{T})} \subset \widehat{\overline{D^r}} = \overline{D^r},$$

where $\widehat{S} := \{z \in \mathbb{C}^m : |P(z)| \le \sup_S |P| \text{ for any } P \in \mathbb{C}[z_1, \ldots, z_m]\}$ is the polynomial hull of a set $S \subset \mathbb{C}^m$.

Note that this implies $f(\mathbb{D}) \subset D^r$, by virtue of the fact that ∂D^r does not contain non-constant analytic discs (as D^r is strongly pseudoconvex).

The opposite implication is clear.

In a similar way we show that for any $v \in (\mathbb{C}^n)_*$ and $\lambda > 0$, a function $f \in B_0$ satisfies the conditions

$$f \text{ is a weak stationary mapping of } D^r, \qquad f(0) = 0, \qquad f'(0) = \lambda v$$

if and only if there exists $q \in Q_0$ such that $q > -1$ and $\Xi(r, v, f, q, \lambda) = 0$.

PROPOSITION 5.5.2. *Let $D_0 \subset \mathbb{C}^n$, $n \geq 2$, be a strongly linearly convex domain with real-analytic boundary and let $f_0 : \mathbb{D} \to D_0$ be an E-mapping.*

(1) *Let $\xi_0 \in (0,1)$. Then there exist a neighborhood W_0 of $(r_0, f_0(\xi_0))$ in $X_0 \times D_0$ and real-analytic mappings*

$$\Lambda : W_0 \to \mathcal{C}^{1/2}(\overline{\mathbb{D}}), \qquad \Omega : W_0 \to (0,1)$$

such that

$$\Lambda(r_0, f_0(\xi_0)) = f_0, \qquad \Omega(r_0, f_0(\xi_0)) = \xi_0$$

and for any $(r,w) \in W_0$ the mapping $f := \Lambda(r,w)$ is an E-mapping of D^r satisfying

$$f(0) = f_0(0) \quad \text{and} \quad f(\Omega(r,w)) = w.$$

(2) *There exist a neighborhood V_0 of $(r_0, f_0'(0))$ in $X_0 \times \mathbb{C}^n$ and a real-analytic mapping*

$$\Gamma : V_0 \to \mathcal{C}^{1/2}(\overline{\mathbb{D}})$$

such that

$$\Gamma(r_0, f_0'(0)) = f_0$$

and for any $(r,v) \in V_0$ the mapping $f := \Gamma(r,v)$ is an E-mapping of D^r satisfying

$$f(0) = f_0(0) \quad \text{and} \quad f'(0) = \lambda v \quad \text{for some } \lambda > 0.$$

Proof. Observe that Proposition 5.1.1 provides us with a mapping $g_0 = \Phi \circ f_0$ and a domain $G_0 := \Phi(D_0)$ giving a data for situation (†) (here ∂D_0 is contained in U_0). Clearly, $\rho_0 := r_0 \circ \Phi^{-1}$ is a defining function of G_0.

Using Lemmas 5.3.1 and 5.3.2, we get neighborhoods V_0 of $(\rho_0, g_0'(0))$ and W_0 of $(\rho_0, g_0(\xi_0))$ and real-analytic mappings Υ, $\widetilde{\Upsilon}$ such that $\Xi(\rho, v, \Upsilon(\rho, v)) = 0$ on V_0 and $\widetilde{\Xi}(\rho, w, \widetilde{\Upsilon}(\rho, w)) = 0$ on W_0. Define

$$\widehat{\Lambda} := \pi_B \circ \widetilde{\Upsilon}, \qquad \Omega := \pi_{\mathbb{R}} \circ \widetilde{\Upsilon}, \qquad \widehat{\Gamma} := \pi_B \circ \Upsilon,$$

where

$$\pi_B : B \times Q_0 \times \mathbb{R} \to B, \qquad \pi_{\mathbb{R}} : B \times Q_0 \times \mathbb{R} \to \mathbb{R}$$

are the projections.

If ρ is sufficiently close to ρ_0 then the hypersurface $\{\rho = 0\}$ bounds a strongly linearly convex domain. Moreover, $\widehat{\Lambda}(\rho, w)$ and $\widehat{\Gamma}(\rho, v)$ are weak stationary mappings of G^ρ (see Remark 5.5.1).

Composing $\widehat{\Lambda}(\rho, w)$ and $\widehat{\Gamma}(\rho, v)$ with Φ^{-1} and making use of Remark 5.1.2, we get weak stationary mappings of D^r, where $r := \rho \circ \Phi$. To show that they are E-mappings we proceed as follows. If D^r is sufficiently close to D_0 (this depends on the distance between ρ and ρ_0) then the domain D^r is strongly linearly convex, so by the results of Section 2,

$$\Lambda(r,w) := \Phi^{-1} \circ \widehat{\Lambda}(\rho, w) \quad \text{and} \quad \Gamma(r,v) := \Phi^{-1} \circ \widehat{\Gamma}(\rho, v)$$

are stationary mappings. Further, they are close to f_0 provided that r is sufficiently close to r_0. Therefore, their winding numbers are equal. Thus $\Lambda(r,w)$ and $\Gamma(r,v)$ satisfy the condition (4) of Definition 1.12, i.e. they are E-mappings. ∎

6. Localization property

PROPOSITION 6.1. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a domain such that ∂D is real-analytic and strongly convex in a neighborhood of $a \in \partial D$. Then for any sufficiently small neighborhood V_0 of a there is a weak stationary mapping f of $D \cap V_0$ such that $f(\mathbb{T}) \subset \partial D$. In particular, f is a weak stationary mapping of D.*

Proof. Let r be a real-analytic defining function in a neighborhood of a. The problem we are dealing with has a local character, so replacing r with $r \circ \Psi$, where Ψ is a local biholomorphism near a, we may assume that $a = (0,\ldots,0,1)$ and a defining function of D near a is $r(z) = -1+|z|^2+h(z-a)$, where h is real-analytic in a neighborhood of 0 and $h(z) = O(|z|^3)$ as $z \to 0$ (cf. [8, p. 321]).

Following [5], let us consider the mappings

$$A_t(z) := \left((1-t^2)^{1/2}\frac{z'}{1+tz_n}, \frac{z_n+t}{1+tz_n}\right), \qquad z = (z', z_n) \in \mathbb{C}^{n-1}\times\mathbb{D},\ t \in (0,1),$$

which restricted to $\mathbb{B}_n$ are automorphisms. Let

$$r_t(z) := \begin{cases} \dfrac{|1+tz_n|^2}{1-t^2} r(A_t(z)), & t \in (0,1), \\ -1+|z|^2, & t = 1. \end{cases}$$

It is clear that $f_{(1)}(\zeta) = (\zeta, 0, \ldots, 0)$, $\zeta \in \mathbb{D}$, is a stationary mapping of $\mathbb{B}_n$. We want to have situation (†), which will allow us to use Lemma 5.3.1 (or Lemma 5.3.2). Note that r_t does not converge to r_1 as $t \to 1$. However, $r_t \to r_1$ in $X_0(U_0, U_0^{\mathbb{C}})$, where U_0 is a neighborhood of $f_{(1)}(\mathbb{T})$ contained in $\{z \in \mathbb{C}^n : \operatorname{Re} z_n > -1/2\}$ and $U_0^{\mathbb{C}}$ is sufficiently small (remember that $h(z) = O(|z|^3)$).

Therefore, making use of Lemma 5.3.1 for t sufficiently close to 1, we obtain stationary mappings $f_{(t)}$ of $D_t := \{z \in \mathbb{C}^n : r_t(z) < 0,\ \operatorname{Re} z_n > -1/2\}$ such that $f_{(t)} \to f_{(1)}$ in the $W^{2,2}$-norm (so also in the sup-norm). Actually, it follows from Lemma 5.3.1 that one may take $f_{(t)} := \pi_B \circ \Upsilon(r_t, f'_{(1)}(0))$ (keeping the notation from that lemma). The argument used in Remark 5.5.1 shows that $f_{(t)}$ satisfies conditions (1′), (2′) and (3′) of Definition 1.11. Since the non-constant function $r \circ A_t \circ f_{(t)}$ is subharmonic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and $r \circ A_t \circ f_{(t)} = 0$ on $\mathbb{T}$, we see from the maximum principle that $f_{(t)}$ maps $\mathbb{D}$ in D_t. Therefore, $f_{(t)}$ are weak stationary mappings for t close to 1.

In particular,

$$f_{(t)}(\mathbb{D}) \subset 2\mathbb{B}_n \cap \{z \in \mathbb{C}^n : \operatorname{Re} z_n > -1/2\}$$

provided that t is close to 1. The mappings A_t have the following important property:

$$A_t(2\mathbb{B}_n \cap \{z \in \mathbb{C}^n : \operatorname{Re} z_n > -1/2\}) \to \{a\}, \qquad t \to 1,$$

in the sense of the Hausdorff distance.

Therefore, we find from Remark 5.1.2 that $g_{(t)} := A_t \circ f_{(t)}$ is a stationary mapping of D. Since $g_{(t)}$ maps $\mathbb{D}$ onto an arbitrarily small neighborhood of a provided that t is sufficiently close to 1, we immediately get the assertion. ∎

7. Proofs of Theorems 1.3 and 1.14. We start this section with the following

LEMMA 7.1. *For any different $z, w \in D$ (resp. for any $z \in D$, $v \in (\mathbb{C}^n)_*$) there exists an E-mapping $f : \mathbb{D} \to D$ such that $f(0) = z$, $f(\xi) = w$ for some $\xi \in (0,1)$ (resp. $f(0) = z$, $f'(0) = \lambda v$ for some $\lambda > 0$).*

Proof. Fix different $z, w \in D$ (resp. $z \in D$, $v \in (\mathbb{C}^n)_*$).

First, consider the case when D is bounded strongly convex with real-analytic boundary. Without loss of generality one may assume that $0 \in D \Subset \mathbb{B}_n$. We need some properties of the Minkowski functionals.

Let μ_G be the Minkowski functional of a domain $G \subset \mathbb{C}^n$ containing the origin, i.e.

$$\mu_G(x) := \inf\{s > 0 : x/s \in G\}, \qquad x \in \mathbb{C}^n.$$

Assume that G is bounded strongly convex with real-analytic boundary. We shall show that

- $\mu_G - 1$ is a defining function of G, real-analytic outside 0;
- $\mu_G^2 - 1$ is a defining function of G, real-analytic outside 0 and strongly convex outside 0.

Clearly, $G = \{x \in \mathbb{R}^{2n} : \mu_G(x) < 1\}$. Setting

$$q(x,s) := r(x/s), \qquad (x,s) \in U_0 \times U_1,$$

where r is a real-analytic defining function of G and $U_0 \subset \mathbb{R}^{2n}$, $U_1 \subset \mathbb{R}$ are neighborhoods of ∂G and 1 respectively, we have

$$\frac{\partial q}{\partial s}(x,s) = -\frac{1}{s^2}\left\langle \nabla r\left(\frac{x}{s}\right), x\right\rangle_{\mathbb{R}} \neq 0$$

for (x,s) such that $x \in \partial G$ and $s = \mu_G(x) = 1$ (since $0 \in G$, the vector $-x$ hooked at the point x points to the inside of G, so it is not orthogonal to the normal vector at x). By the Implicit Function Theorem for the equation

$q = 0$, the function μ_G is real-analytic in a neighborhood V_0 of ∂G. To see that μ_G is also real-analytic outside 0, fix $x_0 \in (\mathbb{R}^{2n})_*$. Then the set

$$W_0 := \{x \in \mathbb{R}^{2n} : x/\mu_G(x_0) \in V_0\}$$

is open and contains x_0. Since

$$\mu_G(x) = \mu_G(x_0)\mu_G\left(\frac{x}{\mu_G(x_0)}\right), \qquad x \in W_0,$$

the function μ_G is real-analytic in W_0. Therefore, we can apply d/ds to both sides of $\mu_G(sx) = s\mu_G(x),\ x \neq 0,\ s > 0$, to obtain

$$\langle \nabla\mu_G(x), x\rangle_{\mathbb{R}} = \mu_G(x), \qquad x \neq 0,$$

so $\nabla\mu_G \neq 0$ in $(\mathbb{R}^{2n})_*$.

Furthermore, $\nabla\mu_G^2 = 2\mu_G\nabla\mu_G$, so $\mu_G^2 - 1$ is also a defining function of G. To show that $u := \mu_G^2$ is strongly convex outside 0, let us prove that

$$X^T\mathcal{H}_a X > 0, \qquad a \in \partial G,\ X \in (\mathbb{R}^{2n})_*,$$

where $\mathcal{H}_x := \mathcal{H}u(x)$ for $x \in (\mathbb{R}^{2n})_*$. Applying $\partial/\partial x_j$ to both sides of

$$u(sx) = s^2 u(x), \qquad x, s \neq 0,$$

we get

$$\frac{\partial u}{\partial x_j}(sx) = s\frac{\partial u}{\partial x_j}(x), \tag{41}$$

and further applying d/ds we obtain

$$\sum_{k=1}^{2n} \frac{\partial^2 u}{\partial x_j \partial x_k}(sx)x_k = \frac{\partial u}{\partial x_j}(x).$$

In particular,

$$x^T\mathcal{H}_x y = \sum_{j,k=1}^{2n} \frac{\partial^2 u}{\partial x_k \partial x_j}(x)x_k y_j = \langle \nabla u(x), y\rangle_{\mathbb{R}}, \qquad x \in (\mathbb{R}^{2n})_*,\ y \in \mathbb{R}^{2n}.$$

Let $a \in \partial G$. Since $\langle \nabla\mu_G(a), a\rangle_{\mathbb{R}} = \mu_G(a) = 1$, we have $a \notin T_G^{\mathbb{R}}(a)$. Any $X \in (\mathbb{R}^{2n})_*$ can be represented as $\alpha a + \beta Y$, where $Y \in T_G^{\mathbb{R}}(a)$, $\alpha, \beta \in \mathbb{R}$, $(\alpha, \beta) \neq (0, 0)$. Then

$$\begin{aligned} X^T\mathcal{H}_a X &= \alpha^2 a^T\mathcal{H}_a a + 2\alpha\beta a^T\mathcal{H}_a Y + \beta^2 Y^T\mathcal{H}_a Y \\ &= \alpha^2\langle \nabla u(a), a\rangle_{\mathbb{R}} + 2\alpha\beta\langle \nabla u(a), Y\rangle_{\mathbb{R}} + \beta^2 Y^T\mathcal{H}_a Y \\ &= \alpha^2 2\mu_G(a)\langle \nabla\mu_G(a), a\rangle_{\mathbb{R}} + \beta^2 Y^T\mathcal{H}_a Y = 2\alpha^2 + \beta^2 Y^T\mathcal{H}_a Y. \end{aligned}$$

Since G is strongly convex, the Hessian of any defining function is strictly positive on the real tangent space, i.e. $Y^T\mathcal{H}_a Y > 0$ if $Y \in (T_G^{\mathbb{R}}(a))_*$. Hence $X^T\mathcal{H}_a X \geq 0$. Note that $X^T\mathcal{H}_a X = 0$ cannot happen, since then $\alpha = 0$,

consequently $\beta \neq 0$ and $Y^T \mathcal{H}_a Y = 0$. On the other hand $Y = X/\beta \neq 0$, a contradiction.

Applying $\partial/\partial x_k$ to both sides of (41), we obtain

$$\frac{\partial^2 u}{\partial x_j \partial x_k}(sx) = \frac{\partial^2 u}{\partial x_j \partial x_k}(x), \qquad x, s \neq 0,$$

and for $a, X \in (\mathbb{R}^{2n})_*$,

$$X^T \mathcal{H}_a X = X^T \mathcal{H}_{a/\mu_G(a)} X > 0.$$

Let us consider the sets

$$D_t := \{x \in \mathbb{C}^n : t\mu_D^2(x) + (1-t)\mu_{\mathbb{B}_n}^2(x) < 1\}, \qquad t \in [0,1].$$

Since the functions $t\mu_D^2 + (1-t)\mu_{\mathbb{B}_n}^2$ are real-analytic in $(\mathbb{C}^n)_*$ and strongly convex in $(\mathbb{C}^n)_*$, D_t are strongly convex domains with real-analytic boundaries satisfying

$$D = D_1 \Subset D_{t_2} \Subset D_{t_1} \Subset D_0 = \mathbb{B}_n \qquad \text{if } 0 < t_1 < t_2 < 1.$$

It is clear that $\mu_{D_t} = \sqrt{t\mu_D^2 + (1-t)\mu_{\mathbb{B}_n}^2}$. Further, if t_1 is close to t_2 then D_{t_1} is close to D_{t_2} with respect to the topology introduced in Section 5. We want to show that D_t are in some family $\mathcal{D}(c)$. Only the interior and exterior ball conditions need verification.

There exists $\delta > 0$ such that $\delta\mathbb{B}_n \Subset D$. Further, $\nabla\mu_{D_t}^2 \neq 0$ in $(\mathbb{R}^{2n})_*$. Set

$$M := \sup\left\{\frac{\mathcal{H}\mu_{D_t}^2(x;X)}{|\nabla\mu_{D_t}^2(y)|} : t \in [0,1],\ x, y \in 2\overline{\mathbb{B}}_n \setminus \delta\mathbb{B}_n,\ X \in \mathbb{R}^{2n},\ |X| = 1\right\}.$$

It is a positive number since the functions $\mu_{D_t}^2$ are strongly convex in $(\mathbb{R}^{2n})_*$ and the 'sup' of the continuous, positive function is taken over a compact set. Let

$$r := \min\left\{\frac{1}{2M}, \frac{\operatorname{dist}(\partial D, \delta\mathbb{B}_n)}{2}\right\}.$$

For fixed $t \in [0,1]$ and $a \in \partial D_t$ put $a' := a - r\nu_{D_t}(a)$. In particular, $\overline{B_n(a', r)} \subset 2\overline{\mathbb{B}}_n \setminus \delta\mathbb{B}_n$. Let us define

$$h(x) := \mu_{D_t}^2(x) - \frac{|\nabla\mu_{D_t}^2(a)|}{2|a-a'|}(|x-a'|^2 - r^2), \qquad x \in 2\overline{\mathbb{B}}_n \setminus \delta\mathbb{B}_n.$$

We have $h(a) = 1$ and

$$\nabla h(x) = \nabla\mu_{D_t}^2(x) - \frac{|\nabla\mu_{D_t}^2(a)|}{|a-a'|}(x-a').$$

For $x = a$, dividing the right side by $|\nabla\mu_{D_t}^2(a)|$, we get the difference of the same normal vectors $\nu_{D_t}(a)$, so $\nabla h(a) = 0$. Moreover, for $|X| = 1$,

$$\mathcal{H}h(x;X) = \mathcal{H}\mu_{D_t}^2(x;X) - \frac{|\nabla\mu_{D_t}^2(a)|}{r} \leq M|\nabla\mu_{D_t}^2(a)| - 2M|\nabla\mu_{D_t}^2(a)| < 0.$$

It follows that $h \le 1$ in any convex set S such that $a \in S \subset 2\overline{\mathbb{B}}_n \setminus \delta\mathbb{B}_n$. Indeed, assume the contrary. Then there is $y \in S$ such that $h(y) > 1$. Let us join a and y with a line segment and consider the function

$$g : [0,1] \ni t \mapsto h(ta + (1-t)y) \in \mathbb{R}.$$

Since a is a strong local maximum point of h, the function g has a local minimum at some point $t_0 \in (0,1)$. Hence

$$0 \le g''(t_0) = \mathcal{H}h(t_0 a + (1-t_0)y; a - y),$$

which is impossible.

Setting $S := \overline{B_n(a', r)}$, we get

$$\mu^2_{D_t}(x) \le 1 + \frac{|\nabla \mu^2_{D_t}(a)|}{2|a-a'|}(|x-a'|^2 - r^2) < 1$$

for $x \in B_n(a', r)$, i.e. $x \in D_t$.

The proof of the exterior ball condition is similar. Set

$$m := \inf\left\{\frac{\mathcal{H}\mu^2_{D_t}(x;X)}{|\nabla \mu^2_{D_t}(y)|} : t \in [0,1],\ x, y \in (\overline{\mathbb{B}}_n)_*,\ X \in \mathbb{R}^{2n},\ |X| = 1\right\}.$$

Note that $m > 0$. Actually, the homogeneity of μ_{D_t} implies $\mathcal{H}\mu^2_{D_t}(sx; X) = \mathcal{H}\mu^2_{D_t}(x; X)$ and $\nabla\mu^2_{D_t}(sx) = s\nabla\mu^2_{D_t}(x)$ for $x \ne 0$, $X \in \mathbb{R}^{2n}$, $s > 0$. Therefore, there are positive constants C_1, C_2 such that $C_1 \le \mathcal{H}\mu^2_{D_t}(x; X)$ for $x \ne 0$, $X \in \mathbb{R}^{2n}$, $|X| = 1$ and $|\nabla\mu^2_{D_t}(y)| \le C_2$ for $y \in \overline{\mathbb{B}}_n$. In particular, $m \ge C_1/C_2$.

Let $R := 2/m$. For fixed $t \in [0,1]$ and $a \in \partial D_t$ put $a'' := a - R\nu_{D_t}(a)$. Let us define

$$\widetilde{h}(x) := \mu^2_{D_t}(x) - \frac{|\nabla\mu^2_{D_t}(a)|}{2|a-a''|}(|x-a''|^2 - R^2), \qquad x \in \overline{\mathbb{B}}_n.$$

We have $\widetilde{h}(a) = 1$ and

$$\nabla\widetilde{h}(x) = \nabla\mu^2_{D_t}(x) - \frac{|\nabla\mu^2_{D_t}(a)|}{|a-a''|}(x - a''),$$

so $\nabla\widetilde{h}(a) = 0$. Moreover, for $x \in (\overline{\mathbb{B}}_n)_*$ and $|X| = 1$,

$$\mathcal{H}\widetilde{h}(x; X) = \mathcal{H}\mu^2_{D_t}(x; X) - \frac{|\nabla\mu^2_{D_t}(a)|}{R} \ge m|\nabla\mu^2_{D_t}(a)| - m|\nabla\mu^2_{D_t}(a)|/2 > 0.$$

Therefore, a is a strong local minimum point of $\widetilde{h}$.

Now using the properties listed above we may deduce that $\widetilde{h} \ge 1$ in $\overline{\mathbb{B}}_n$. We proceed as before: seeking a contradiction suppose that there is $y \in \overline{\mathbb{B}}_n$ such that $\widetilde{h}(y) < 1$. Moving y a little (if necessary) we may assume that 0 does not lie on the segment joining a and y. Then the function $\widetilde{g}(t) := \widetilde{h}(ta + (1-t)y)$ attains its local maximum at some point $t_0 \in (0,1)$. The second derivative of $\widetilde{g}$ at t_0 is non-positive, which gives a contradiction with the positivity of the Hessian of $\widetilde{h}$.

Hence, we get

$$\frac{|\nabla \mu^2_{D_t}(a)|}{2|a-a''|}(|x-a''|^2-R^2) \le \mu^2_{D_t}(x)-1<0$$

for $x \in D_t$, so $D_t \subset B_n(a'', R)$.

Let T be the set of all $t \in [0,1]$ such that there is an E-mapping $f_t : \mathbb{D} \to D_t$ with $f_t(0) = z$, $f_t(\xi_t) = w$ for some $\xi_t \in (0,1)$ (resp. $f_t(0) = z$, $f_t'(0) = \lambda_t v$ for some $\lambda_t > 0$). We claim that $T = [0,1]$. To prove this we will use the open-closed argument.

Clearly, $T \neq \emptyset$, as $0 \in T$. Moreover, T is open in $[0,1]$. Indeed, let $t_0 \in T$. It follows from Proposition 5.5.2 that there is a neighborhood T_0 of t_0 such that there exist E-mappings $f_t : \mathbb{D} \to D_t$ and $\xi_t \in (0,1)$ with $f_t(0) = z$, $f_t(\xi_t) = w$ for all $t \in T_0$ (resp. $\lambda_t > 0$ such that $f_t(0) = z$, $f_t'(0) = \lambda_t v$ for all $t \in T_0$).

To prove that T is closed, choose a sequence $\{t_m\} \subset T$ converging to some $t \in [0,1]$. We want to show that $t \in T$. Since f_{t_m} are E-mappings, they are complex geodesics. Therefore, making use of the inclusions $D \subset D_{t_m} \subset \mathbb{B}_n$, we find that there is a compact set $K \subset (0,1)$ (resp. a compact set $\widetilde{K} \subset (0,\infty)$) such that $\{\xi_{t_m}\} \subset K$ (resp. $\{\lambda_{t_m}\} \subset \widetilde{K}$). By Propositions 4.7 and 4.9 the functions f_{t_m} and $\widetilde{f}_{t_m}$ are equicontinuous in $\mathcal{C}^{1/2}(\overline{\mathbb{D}})$, and by Propositions 4.6 and 4.8 the functions ρ_{t_m} are uniformly bounded from above and below by positive numbers and equicontinuous in $\mathcal{C}^{1/2}(\mathbb{T})$. From the Arzelà–Ascoli Theorem there are a subsequence $\{s_m\} \subset \{t_m\}$ and mappings $f, \widetilde{f} \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}^{1/2}(\overline{\mathbb{D}})$, $\rho \in \mathcal{C}^{1/2}(\mathbb{T})$ such that $f_{s_m} \to f$, $\widetilde{f}_{s_m} \to \widetilde{f}$ uniformly on $\overline{\mathbb{D}}$, $\rho_{s_m} \to \rho$ uniformly on $\mathbb{T}$ and $\xi_{s_m} \to \xi \in (0,1)$ (resp. $\lambda_{s_m} \to \lambda > 0$).

Clearly, $f(\overline{\mathbb{D}}) \subset \overline{D}_t$, $f(\mathbb{T}) \subset \partial D_t$ and $\rho > 0$. By the strong pseudoconvexity of D_t we get $f(\mathbb{D}) \subset D_t$.

The conditions (3′) and (4) of Definitions 1.11 and 1.12 follow from the uniform convergence of suitable functions. Therefore, f is a weak E-mapping of D_t, and hence an E-mapping of D_t, satisfying $f(0) = z$, $f(\xi) = w$ (resp. $f(0) = z$, $f'(0) = \lambda v$).

Let us go back to the general situation, that is, when a domain D is bounded strongly linearly convex with real-analytic boundary. Take $\eta \in \partial D$ such that $\max_{\zeta \in \partial D} |z-\zeta| = |z-\eta|$. Then η is a point of strong convexity of D. Indeed, by the Implicit Function Theorem one can assume that in a neighborhood of η the defining functions of D and $B := B_n(z, |z-\eta|)$ are of the form $r(x) := \widetilde{r}(\widetilde{x}) - x_{2n}$ and $q(x) := \widetilde{q}(\widetilde{x}) - x_{2n}$ respectively, where $x = (\widetilde{x}, x_{2n}) \in \mathbb{R}^{2n}$ is sufficiently close to η. From the inclusion $D \subset B$ it follows that $r - q \ge 0$ near η and $(r-q)(\eta) = 0$. Thus the Hessian $\mathcal{H}(r-q)(\eta)$ is weakly positive on $\mathbb{C}^n$. Since $\mathcal{H}q(\eta)$ is strictly positive on $T^{\mathbb{R}}_B(\eta)_* = T^{\mathbb{R}}_D(\eta)_*$, we find that $\mathcal{H}r(\eta)$ is strictly positive on $T^{\mathbb{R}}_D(\eta)_*$ as well.

By a continuity argument, there is a convex neighborhood V_0 of η such that all points from $\partial D \cap V_0$ are points of strong convexity of D. It follows from Proposition 6.1 (after shrinking V_0 if necessary) that there is a weak stationary mapping $g : \mathbb{D} \to D \cap V_0$ such that $g(\mathbb{T}) \subset \partial D$. In particular, g is a weak stationary mapping of D. Since $D \cap V_0$ is convex, the condition on the winding number is satisfied on $D \cap V_0$ (and hence on the whole D). Consequently, g is an E-mapping of D.

If $z = g(0)$, $w = g(\xi)$ for some $\xi \in \mathbb{D}$ (resp. $z = g(0)$, $v = g'(0)$) then there is nothing to prove. In the other case let us take curves $\alpha : [0,1] \to D$, $\beta : [0,1] \to D$ joining $g(0)$ and z, and $g(1/2)$ and w (resp. $\alpha : [0,1] \to D$, $\beta : [0,1] \to (\mathbb{C}^n)_*$ joining $g(0)$ and z, and $g'(0)$ and v). We may assume that the images of α and β are disjoint. Let T be the set of all $t \in [0,1]$ such that there is an E-mapping $g_t : \mathbb{D} \to D$ such that $g_t(0) = \alpha(t)$, $g_t(\xi_t) = \beta(t)$ for some $\xi_t \in (0,1)$ (resp. $g_t(0) = \alpha(t)$, $g_t'(0) = \lambda_t \beta(t)$ for some $\lambda_t > 0$). Again $T \neq \emptyset$, since $0 \in T$. Using the results of Section 4 just as before (but for one domain), we see that T is closed.

Since $\widetilde{k}_D$ is symmetric, it follows from Proposition 5.5.2(1) that the set T is open in $[0,1]$ (first we move along α, then by symmetry we move along β). Therefore, g_1 is the E-mapping for z, w.

In the case of κ_D we change a point and then we change a direction. To be more precise, consider the set S of all $s \in [0,1]$ such that there is an E-mapping $h_s : \mathbb{D} \to D$ with $h_s(0) = \alpha(s)$. Then $0 \in S$, by Proposition 5.5.2(1) the set S is open in $[0,1]$ and, by results of Section 4 again, it is closed. Hence $S = [0,1]$. Now we may join $h_1'(0)$ and v with a curve $\gamma : [0,1] \to \mathbb{C}^n$. Let us define R as the set of all $r \in [0,1]$ such that there is an E-mapping $\widetilde{h}_r : \mathbb{D} \to D$ with $\widetilde{h}_r(0) = h_1(0)$, $\widetilde{h}_r'(0) = \sigma_r \gamma(1-r)$ for some $\sigma_r > 0$. Then $1 \in R$, by Proposition 5.5.2(2) the set R is open in $[0,1]$, and by Section 4 it is closed. Hence $R = [0,1]$, so $\widetilde{h}_0$ is the E-mapping for z, v. ∎

Now we are in a position to prove the main results of Lempert's paper.

Proof of Theorem 1.3 (real-analytic case). It follows from Lemma 7.1 that for any different points $z, w \in D$ (resp. $z \in D$, $v \in (\mathbb{C}^n)_*$) one may find an E-mapping passing through them (resp. $f(0) = z$, $f'(0) = v$). On the other hand, it follows from Proposition 3.1.1 that E-mappings have left inverses, so they are complex geodesics. ∎

Proof of Theorem 1.14 (real-analytic case). This is a direct consequence of Lemma 7.1 and Corollary 3.1.5. ∎

$\mathcal{C}^2$-smooth case

LEMMA 7.2. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strongly pseudoconvex domain with $\mathcal{C}^2$-smooth boundary. Take $z \in D$ and let r be a defining function of D such that*

- $r \in \mathcal{C}^2(\mathbb{C}^n)$;
- $D = \{x \in \mathbb{C}^n : r(x) < 0\}$;
- $\mathbb{C}^n \setminus \overline{D} = \{x \in \mathbb{C}^n : r(x) > 0\}$;
- $|\nabla r| = 1$ *on* ∂D;
- $\sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(a) X_j \overline{X}_k \geq C|X|^2$ *for any* $a \in \partial D$ *and* $X \in \mathbb{C}^n$ *with some constant* $C > 0$.

Suppose that there is a sequence $\{r_m\}$ *of* $\mathcal{C}^2$*-smooth real-valued functions such that* $\partial^\alpha r_m / \partial x^\alpha \to \partial^\alpha r / \partial x^\alpha$ *locally uniformly for any* $\alpha \in \mathbb{N}_0^{2n}$ *with* $|\alpha| \leq 2$. *Let* D_m *be the connected component of the set* $\{x \in \mathbb{C}^n : r_m(x) < 0\}$ *such that* D_m *contains the point* z.

Then there is $c > 0$ *such that* (D_m, z) *and* (D, z) *belong to* $\mathcal{D}(c)$, $m \gg 1$.

Proof. Losing no generality assume that $D \Subset \mathbb{B}_n$. Note that the conditions (1), (5), (6) of Definition 4.1 are clearly satisfied. To find c satisfying (2), we take $s > 0$ such that $\mathcal{H}r(x; X) < s|X|^2$ for $x \in \overline{\mathbb{B}}_n$ and $X \in (\mathbb{R}^{2n})_*$. Then $\mathcal{H}r_m(x; X) < 2s|X|^2$ for $x \in \overline{\mathbb{B}}_n$, $X \in (\mathbb{R}^{2n})_*$ and $m \gg 1$. Let $U_0 \subset \mathbb{B}_n$ be an open neighborhood of ∂D such that $|\nabla r|$ is on U_0 between $3/4$ and $5/4$. Note that $\partial D_m \subset U_0$ and $|\nabla r_m| \in (1/2, 3/2)$ on U_0 for $m \gg 1$.

Fix m and $a \in \partial D_m$ and put $b := a - R\nu_{D_m}(a)$, where the small number $R > 0$ will be specified later. There is $t > 0$ such that $\nabla r_m(a) = 2t(a - b)$. Note that t may be arbitrarily large provided that R was small enough. We take $t := 2s$ and $R := |\nabla r_m(a)|/t$. Then we have $\mathcal{H}r_m(x; X) < 2t|X|^2$ for $x \in \overline{\mathbb{B}}_n$, $X \in (\mathbb{R}^{2n})_*$ and $m \gg 1$. Hence the function

$$h(x) := r_m(x) - t(|x - b|^2 - R^2), \qquad x \in \mathbb{C}^n,$$

attains at a its global maximum on $\overline{\mathbb{B}}_n$ (since a is a strong local maximum point and the Hessian of h is negative on the convex set $\overline{\mathbb{B}}_n$, cf. the proof of Lemma 7.1). Thus $h \leq 0$ on $\mathbb{B}_n$. From this we immediately get (2).

Note that it follows from (2) that $D_m = \{x \in \mathbb{C}^n : r_m(x) < 0\}$ for m big enough (i.e. the set $\{x \in \mathbb{C}^n : r_m(x) < 0\}$ is connected).

Moreover, the condition (2) implies the condition (3) as follows. We infer from Remark 4.4 that there is $c' > 0$ such that D satisfies (3) with c'. Let m_0 be such that the Hausdorff distance between ∂D and ∂D_m is smaller than $1/c'$ for $m \geq m_0$. There is $c'' > 0$ such that D_{m_0} satisfies (3) with c''. Losing no generality we may assume that $c'' < c'$. Take any $x, y \in D_m$. Since D_m satisfies (without loss of generality) the interior ball condition with radius $1/c'$, we infer that there are balls of radius $1/c'$ contained in D_m and containing x and y respectively. The centers of these balls lie in D_{m_0}. Using the fact that (D_{m_0}, z) lies in $\mathcal{D}(c'')$, we may join the chosen centers to balls of radius $1/(2c'')$ as in the condition (3), so we have found a chain consisting of balls of radii $1/c'$ and $1/c''$, joining x and y.

Thus we may join x and y with balls contained entirely in the constructed chain, of the same radii depending only on c' and c''.

Now we prove (4). We shall show that there is $c > c'$ such that every D_m satisfies (4) with c for m large enough. To do this let us cover ∂D with a finite number of balls B_j, $j = 1, \ldots, N$, from the condition (4) and let B'_j be a ball relatively contained in B_j such that $\{B'_j\}_{j=1}^N$ covers ∂D as well. Let Φ_j be mappings corresponding to B_j. Let ε be such that any ball of radius ε intersecting ∂D is relatively contained in B'_j for some j. Observe that any ball B of radius $\varepsilon/2$ intersecting ∂D_m is contained in a ball of radius ε intersecting ∂D; hence it is contained in B'_j for some j. Then the pair B, Φ_j satisfies the conditions (4)(b, c, d). Therefore, it suffices to check that there is $c > 2/\varepsilon$ such that each pair B'_j, Φ_j satisfies the condition (4) for D_m with c (and $m \gg 1$). This is possible since $\Phi_j(D_m) \subset \Phi_j(D)$, $D^\alpha \Phi_j(\partial D_m \cap B_j)$ converges to $D^\alpha \Phi_j(\partial D \cap B_j)$ for $|\alpha| \le 2$ and for any $w \in \Phi(\partial D \cap B_j)$ there is a ball of radius $2/\varepsilon$ containing $\Phi_j(D)$ and tangent to $\partial \Phi_j(D)$ at w. To be precise, we proceed as follows.

Let $a, b \in \mathbb{C}^n$ and $x \in \partial B_n(a, \widetilde{c})$, where $\widetilde{c} > c'$. Then a ball $B_n(2a - x, 2\widetilde{c})$ contains $B_n(a, \widetilde{c})$ and is tangent to $B_n(a, \widetilde{c})$ at x. There is a number $\eta = \eta(\delta, \widetilde{c}) > 0$, independent of a, b, x, such that the diameter of the set $B_n(b, \widetilde{c}) \setminus B_n(2a - x, 2\widetilde{c})$ is smaller than $\delta > 0$ whenever $|a - b| < \eta$ (this is a simple consequence of the triangle inequality).

Let $\widetilde{s} > 0$ be such that $\mathcal{H}(r \circ \Phi_j^{-1})(x; X) \ge 2\widetilde{s}|X|^2$ for $x \in U_j$, $j = 1, \ldots, N$, where U_j is an open neighborhood of $\Phi_j(\partial D \cap B_j)$. Then, for m large enough, $\mathcal{H}(r_m \circ \Phi_j^{-1})(x; X) \ge \widetilde{s}|X|^2$ for $x \in U_j$ and $\Phi_j(\partial D_m \cap B'_j) \subset U_j$, $j = 1, \ldots, N$. Repeating for the function

$$x \mapsto (r_m \circ \Phi_j^{-1})(x) - \widetilde{t}(|x - \widetilde{b}|^2 - \widetilde{R}^2)$$

the argument used in the interior ball condition with suitably chosen $\widetilde{t}$ and uniform $\widetilde{R} > c$, we find that there is a uniform $\widetilde{\varepsilon} > 0$ such that for any j, m and $w \in \Phi_j(\partial D_m \cap B'_j)$ there is a ball B of radius $\widetilde{R}$, tangent to $\Phi_j(\partial D_m \cap B'_j)$ at w, such that $\Phi_j(\partial D_m \cap B'_j) \cap B_n(w, \widetilde{\varepsilon}) \subset B$. Let $a_{j,m}(w)$ denote its center.

On the other hand, for any $w \in \Phi_j(\partial D_m \cap B'_j)$ there is $t > 0$ such that $w' = w + t\nu(w) \in \Phi_j(\partial D \cap B_j)$, where $\nu(w)$ is a normal vector to $\Phi_j(\partial D_m \cap B'_j)$ at w. Let $a_j(w')$ be the center of a ball of radius $\widetilde{R}$ tangent to $\Phi_j(\partial D \cap B_j)$ at w'. It follows that $|a_{j,m}(w) - a_j(w')| < \eta(\widetilde{\varepsilon}/2, \widetilde{R})$ provided that m is large enough.

Combining the facts presented above, we finish the proof of the exterior ball condition (with a radius dependent only on $\widetilde{\varepsilon}$ and $\widetilde{R}$). ∎

Proof of Theorems 1.3 and 1.14 ($\mathcal{C}^2$-smooth case). Losing no generality assume that $0 \in D \Subset \mathbb{B}_n$.

It follows from the Weierstrass Theorem that there is sequence $\{P_k\}$ of real polynomials on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ such that

$$\partial^\alpha P_k/\partial x^\alpha \to \partial^\alpha r/\partial x^\alpha \quad \text{uniformly on } \overline{\mathbb{B}}_n,$$

where $\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in \mathbb{N}_0^{2n}$ is such that $|\alpha| \le 2$. Consider the open set

$$\widetilde{D}_{k,\varepsilon} := \{x \in \mathbb{C}^n : P_k(x) + \varepsilon < 0\}.$$

Let $\varepsilon_m > 0$ be a sequence converging to 0 with $3\varepsilon_{m+1} < \varepsilon_m$. For any $m \in \mathbb{N}$ there is $k_m \in \mathbb{N}$ such that $\sup_{\overline{\mathbb{B}}_n} |P_{k_m} - r| < \varepsilon_m$. Putting $r_m := P_{k_m} + 2\varepsilon_m$, we get $r + \varepsilon_m < r_m < r + 3\varepsilon_m$ on $\overline{\mathbb{B}}_n$. In particular, $r_{m+1} < r_m$ on $\overline{\mathbb{B}}_n$.

Let D_m be the connected component of $D_{k_m,2\varepsilon_m}$ containing 0. It is a bounded strongly linearly convex domain with real-analytic boundary and r_m is its defining function provided that m is large enough. Moreover, $D_m \subset D_{m+1}$ and $\bigcup_m D_m = D$. Using properties of holomorphically invariant functions and metrics we get Theorem 1.3.

We are left with showing the claim that for any different $z, w \in D$ (resp. $z \in D$, $v \in (\mathbb{C}^n)_*$) there is a weak E-mapping for z, w (resp. for z, v). Fix $z \in D$ and $w \in D$ (resp. $v \in (\mathbb{C}^n)_*$). Then $z, w \in D_m$ (resp. $z \in D_m$), $m \gg 1$. Therefore, for any $m \gg 1$ one may find an E-mapping f_m of D_m for z, w (resp. for z, v). Since $(D_m, z) \in \mathcal{D}(c)$ for some uniform $c > 0$ ($m \gg 1$) (Lemma 7.2), we find that f_m, $\widetilde{f}_m$ and ρ_m satisfy the uniform estimates from Section 4. Thus, passing to a subsequence we may assume that $\{f_m\}$ converges uniformly on $\overline{\mathbb{D}}$ to a mapping $f \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}^{1/2}(\overline{\mathbb{D}})$ passing through z, w (resp. such that $f(0) = z$, $f'(0) = \lambda v$, $\lambda > 0$), that $\{\widetilde{f}_m\}$ converges uniformly on $\overline{\mathbb{D}}$ to a mapping $\widetilde{f} \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}^{1/2}(\overline{\mathbb{D}})$ and that $\{\rho_m\}$ is convergent uniformly on $\mathbb{T}$ to a positive function $\rho \in \mathcal{C}^{1/2}(\mathbb{T})$ (in particular, $f' \bullet \widetilde{f} = 1$ in $\mathbb{D}$, so $\widetilde{f}$ has no zeroes in $\overline{\mathbb{D}}$). We already know that this implies that f is a weak E-mapping of D.

To get $\mathcal{C}^{k-1-\varepsilon}$-smoothness of f and its associated mappings for $k = 3, 4, \ldots, \infty$, it suffices to repeat the proof of Proposition 5 in [5]. This is just the Webster Lemma (we have proved it in the real-analytic case—see Proposition 2.1). Namely, let

$$\psi : \partial D \ni z \mapsto (z, T_D^{\mathbb{C}}(z)) \in \mathbb{C}^n \times (\mathbb{P}^{n-1})_*,$$

where $\mathbb{P}^{n-1}$ is the $(n-1)$-dimensional complex projective space. Let $\pi : (\mathbb{C}^n)_* \to \mathbb{P}^{n-1}$ be the canonical projection.

By the Webster Lemma, $\psi(\partial D)$ is a totally real manifold of class $\mathcal{C}^{k-1}$. Observe that the mapping $(f, \pi \circ \widetilde{f}) : \overline{\mathbb{D}} \to \mathbb{C}^n \times \mathbb{P}^{n-1}$ is 1/2-Hölder continuous, is holomorphic in $\mathbb{D}$ and maps $\mathbb{T}$ into $\psi(\partial D)$. Therefore, it is $\mathcal{C}^{k-1-\varepsilon}$-smooth for any $\varepsilon > 0$, whence f is $\mathcal{C}^{k-1-\varepsilon}$-smooth. Since $\nu_D \circ f$ is of class $\mathcal{C}^{k-1-\varepsilon}$, it suffices to proceed as in the proof of Proposition 2.1. ∎

8. Appendix

8.1. Totally real submanifolds

LEMMA 8.1.1. *Let $M \subset \mathbb{C}^m$ be a totally real local $\mathcal{C}^\omega$ submanifold of real dimension m and let $z \in M$. Then there are neighborhoods $U, V \subset \mathbb{C}^m$ of 0 and z respectively and a biholomorphic mapping $\Phi : U \to V$ such that $\Phi(\mathbb{R}^m \cap U) = M \cap V$.*

LEMMA 8.1.2 (Reflection Principle). *Let $M \subset \mathbb{C}^m$ be a totally real local $\mathcal{C}^\omega$ submanifold of real dimension m. Let $V_0 \subset \mathbb{C}$ be a neighborhood of a point $\zeta_0 \in \mathbb{T}$ and let $g : \overline{\mathbb{D}} \cap V_0 \to \mathbb{C}^m$ be a continuous mapping. Suppose that $g \in \mathcal{O}(\mathbb{D} \cap V_0)$ and $g(\mathbb{T} \cap V_0) \subset M$. Then g extends holomorphically past $\mathbb{T} \cap V_0$.*

8.2. Some estimates of holomorphic functions of class $\mathcal{C}^\alpha$. Let us recall some theorems about functions holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. Concrete values of the constants M, K can be calculated by inspecting the proofs. In fact, it is only important that they do not depend on the functions involved.

THEOREM 8.2.1 (Hardy, Littlewood, cf. [3, Theorem 3, p. 411]). *Let $f \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$. Then for $\alpha \in (0, 1]$ the following conditions are equivalent:*

$$\exists M > 0 : \ |f(e^{i\theta}) - f(e^{i\theta'})| \le M|\theta - \theta'|^\alpha, \ \theta, \theta' \in \mathbb{R}; \tag{42}$$

$$\exists K > 0 : \ |f'(\zeta)| \le K(1 - |\zeta|)^{\alpha-1}, \ \zeta \in \mathbb{D}. \tag{43}$$

Moreover, if M satisfying (42) is given then K can be chosen as

$$2^{(1-3\alpha)/2}\pi^\alpha M \int_0^\infty \frac{t^\alpha}{1+t^2}\, dt$$

and if K satisfying (43) is given then M can be chosen as $(2/\alpha + 1)K$.

THEOREM 8.2.2 (Hardy, Littlewood, cf. [3, Theorem 4, p. 413]). *Let $f \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ be such that*

$$|f(e^{i\theta}) - f(e^{i\theta'})| \le M|\theta - \theta'|^\alpha, \qquad \theta, \theta' \in \mathbb{R},$$

for some $\alpha \in (0, 1]$ and $M > 0$. Then

$$|f(\zeta) - f(\zeta')| \le K|\zeta - \zeta'|^\alpha, \qquad \zeta, \zeta' \in \overline{\mathbb{D}},$$

where

$$K := \max\left\{2^{1-2\alpha}\pi^\alpha M, 2^{(3-5\alpha)/2}\pi^\alpha \alpha^{-1} M \int_0^\infty \frac{t^\alpha}{1+t^2}\, dt\right\}.$$

THEOREM 8.2.3 (Privalov, cf. [3, Theorem 5, p. 414]). *Let $f \in \mathcal{O}(\mathbb{D})$ be such that $\operatorname{Re} f$ extends continuously on $\overline{\mathbb{D}}$ and*

$$|\operatorname{Re} f(e^{i\theta}) - \operatorname{Re} f(e^{i\theta'})| \le M|\theta - \theta'|^\alpha, \qquad \theta, \theta' \in \mathbb{R},$$

for some $\alpha \in (0,1)$ *and* $M > 0$. *Then* f *extends continuously on* $\overline{\mathbb{D}}$ *and*

$$|f(\zeta) - f(\zeta')| \le K|\zeta - \zeta'|^\alpha, \qquad \zeta, \zeta' \in \overline{\mathbb{D}},$$

where

$$K := \max\Big\{2^{1-2\alpha}\pi^\alpha, 2^{(3-5\alpha)/2}\pi^\alpha\alpha^{-1}\int_0^\infty \frac{t^\alpha}{1+t^2}\,dt\Big\}$$
$$\times\left(\frac{2}{\alpha}+1\right)2^{(3-3\alpha)/2}\pi^\alpha M\int_0^\infty \frac{t^\alpha}{1+t^2}\,dt.$$

8.3. Sobolev space. The Sobolev space $W^{2,2}(\mathbb{T}) = W^{2,2}(\mathbb{T}, \mathbb{C}^m)$ is the space of functions $f : \mathbb{T} \to \mathbb{C}^m$, whose first two derivatives (in the sense of distributions) are in $L^2(\mathbb{T})$ (here we use the standard identification of functions on the unit circle and functions on $[0, 2\pi]$). Each such f is $\mathcal{C}^1$-smooth.

$W^{2,2}(\mathbb{T})$ is a complex Hilbert space with the scalar product

$$\langle f, g\rangle_W := \langle f, g\rangle_L + \langle f', g'\rangle_L + \langle f'', g''\rangle_L,$$

where

$$\langle \widetilde{f}, \widetilde{g}\rangle_L := \frac{1}{2\pi}\int_0^{2\pi} \langle \widetilde{f}(e^{it}), \widetilde{g}(e^{it})\rangle\,dt.$$

Let $\|\cdot\|_L$, $\|\cdot\|_W$ denote the norms induced by $\langle\cdot\,,-\rangle_L$ and $\langle\cdot\,,-\rangle_W$. The following characterization follows easily from Parseval's identity:

$$W^{2,2}(\mathbb{T}) = \Big\{f \in L^2(\mathbb{T}) : \sum_{k=-\infty}^{\infty}(1+k^2+k^4)|a_k|^2 < \infty\Big\},$$

where $a_k \in \mathbb{C}^m$ are the m-dimensional Fourier coefficients of f, i.e. $f(\zeta) = \sum_{k=-\infty}^{\infty} a_k\zeta^k$, $\zeta \in \mathbb{T}$. More precisely, Parseval's identity gives

$$\|f\|_W = \sqrt{\sum_{k=-\infty}^{\infty}(1+k^2+k^4)|a_k|^2}, \qquad f \in W^{2,2}(\mathbb{T}).$$

Note that $W^{2,2}(\mathbb{T}) \subset \mathcal{C}^{1/2}(\mathbb{T}) \subset \mathcal{C}(\mathbb{T})$ and both inclusions are continuous (in particular, both inclusions are real-analytic). Note also that

$$\|f\|_{\sup} \le \sum_{k=-\infty}^{\infty}|a_k| \le \sqrt{\sum_{k=-\infty}^{\infty}\frac{1}{1+k^2}\sum_{k=-\infty}^{\infty}(1+k^2)|a_k|^2} \le \frac{\pi}{\sqrt{3}}\|f\|_W.$$

Moreover, direct calculation shows that there exists $C > 0$ such that

$$\|h^\alpha\|_W \le C^{|\alpha|}\|h_1\|_W^{\alpha_1}\cdot\ldots\cdot\|h_{2n}\|_W^{\alpha_{2n}}, \qquad h \in W^{2,2}(\mathbb{T}, \mathbb{C}^n),\ \alpha \in \mathbb{N}_0^{2n}.$$

8.4. Matrices

PROPOSITION 8.4.1 (Lempert, [5, Théorème B]). *Let $A : \mathbb{T} \to \mathbb{C}^{n\times n}$ be a matrix-valued real-analytic mapping such that $A(\zeta)$ is self-adjoint and strictly positive for any $\zeta \in \mathbb{T}$. Then there exists $H \in \mathcal{O}(\overline{\mathbb{D}}, \mathbb{C}^{(n-1)\times(n-1)})$ such that* $\det H \neq 0$ *on $\overline{\mathbb{D}}$ and $HH^* = A$ on $\mathbb{T}$.*

PROPOSITION 8.4.2 ([9, Lemma 2.1]). *Let A be a complex symmetric $n \times n$ matrix. Then*

$$\|A\| = \sup\{|z^T Az| : z \in \mathbb{C}^n,\ |z| = 1\}.$$

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