Multiplicity of solutions for a singular
p-laplacian elliptic equation

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Abstract. The existence of two continuous solutions for a nonlinear singular elliptic equation with natural growth in the gradient is proved for the Dirichlet problem in the unit ball centered at the origin. The first continuous solution is positive and maximal; it is obtained via the regularization method. The second continuous solution is zero at the origin, and follows by considering the corresponding radial ODE and by sub-sup solutions method.

1. Introduction. Consider the nonlinear problem

$$(1.1) \qquad \begin{cases} -\mathbf{div}(|\nabla u|^{p-2}\nabla u) = f(x) - \dfrac{\lambda}{u^m}|\nabla u|^p, \quad u > 0, \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\partial\Omega$, $1 < p < \infty$, $\lambda \neq 0$, $m \in \mathbb{R}$, and $f(x)$ is measurable in Ω. Such problems arise in the theory of non-Newtonian fluids (see [AM], [MP], [MRS]).

For $m > 0$, the equation in (1.1) is singular at points where $u = 0$.

When the lower order term $B(x, u, \eta)$ has no singularity at $u = 0$, the following nonlinear elliptic problems have been studied extensively (see [A], [AAA], [AB], [ADP], [BBM], [BMP1], [BMP2], [BO], [BST], [CC], [DGP], [DB], [DN], [FPR], [G], [GT], [L], [LU], [OP], [P], [PS], [T], [Tr], [YC] and references therein): Find $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, such that

$$\begin{cases} -\mathbf{div}A(x, u, \nabla u) + B(x, u, \nabla u) = 0 \quad \text{in } \mathcal{D}'(\Omega), \\ u - h \in W_0^{1,p}(\Omega), \end{cases}$$

where $h \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $-\mathbf{div}A$ is a Leray–Lions operator from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ ($p > 1$, $p' > 1$, $1/p + 1/p' = 1$), which includes the p-laplacian, and B is a nonlinear lower order term with natural growth in

the gradient. Under some additional assumptions on A and B, the existence and regularity of solutions were studied in a large number of papers: see for instance Abdellaoui, Dall'Aglio and Peral [ADP], Amann [A], Bensoussan, Boccardo and Murat [BBM], Boccardo, Murat and Puel [BMP1, BMP2], Cho and Choe [CC], Dall'Aglio, Giachetti and Puel [DGP], Drábek and Nicolosi [DN], Ferone, Posteraro and Rakotoson [FPR], Grenon and Trombetti [GT], Orsina and Puel [OP] and Porretta and Segura de León [PS] for the existence and DiBenedetto [DB], Ladyzhenskaya and Ural'tseva [LU], Lieberman [L] and Tolksdorf [T] for the regularity.

When the lower order term B may be singular at $u = 0$, to the best of our knowledge, the problem has received little attention. In [MRS], Michaux, Rakotoson and Shen studied a class of quasilinear mixed equations of Leray–Lions type; however, they only considered the case where the boundary function has a positive lower bound and established the existence and local regularity of solutions with the same lower bound, whereas the positive lower bound means disappearance of singularity of the lower order term and plays a dominant role in their proof. It is of interest to consider the singularity of the lower order term and to study the existence of solutions in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

In the present paper, we investigate the existence and multiplicity of solutions of problem (1.1) with $m > 1$ and $\lambda > 0$; we extend the existence results of [AMA, PV, Z1] and the multiplicity result of [Z2].

In [Z1], Zhou considered a class of nonlinear singular elliptic problems including (1.1) with $m = 1$, and established the existence of a positive solution under the assumptions $\lambda \geq (p-1)/p$ and $f \in L^\infty(\Omega)$ with ess $\inf_\Omega f > 0$. In the case of $p = 2$, problem (1.1) was studied in [AMA], [PV] and [Z2]. For $0 < m \leq 1$, Arcoya and Martínez-Aparicio [AMA] proved the existence of a positive distributional solution under the assumptions $\lambda > 0$ and $f \in L^\infty(\Omega)$ with $\inf_{\Omega'} f > 0$ for any $\Omega' \subset\subset \Omega$. For $m = 1$, Porru and Vitolo [PV] established the existence of a positive classical solution via a substitution if $\lambda > 0$ and $f \equiv \mathrm{const} > 0$. For $4/3 > m > 1$, Zhou [Z2] obtained two different continuous solutions: one that is maximal and positive and another that is zero at the origin, if $\Omega = B_1 := \{x \in \mathbb{R}^N;\ |x| < 1\}$, $f(x) = \overline{f}(|x|)$, $\overline{f} \in C[0,1]$ with $\overline{f} > 0$ on $[0,1]$ and $\lambda > \inf_{r \geq 1} \mathcal{H}(r)$, where

$$\mathcal{H}(r) = \frac{2(m-1) + N(2-m)}{2}\, r^{m-1} + \frac{(2-m)^2 \max_{[0,1]} \overline{f}}{4}\, r^{m-2}.$$

The second solution can be obtained by considering the corresponding ODE with mixed boundary conditions and by the sub-sup solutions method.

We point out that in the case of $m = 1$, by a certain transformation (see [Z1]), problem (1.1) can be transformed into either a Dirichlet problem or a boundary blowup problem without a gradient term; such problems have

been studied extensively in the past years (see the references cited in [Z1]). It is worth noting that for $m > 1$, it seems to be impossible to transform the equation in (1.1) into an equation without a gradient term.

As seen in Section 2, the case $m \geq 1$ is essentially different from the case $0 < m < 1$. Indeed, we show that in the latter case, problem (1.1) admits at most one solution (see Corollary 2.5), while in the former case it has at least two solutions. The main idea of dealing with the multiplicity is similar to that in [Z2]. Roughly speaking, the proof consists of two steps. In the first step, we obtain a maximal, positive, continuous solution for a general domain Ω (see Theorem 2.6), for which our idea is based on the regularization method, monotonicity technique and Lemma 3.7. In the second step, we will assume $\Omega = B_1$ and $f(x) = \bar{f}(|x|)$ and consider the radial ODE corresponding to the equation in (1.1):

$$(1.2) \quad (|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' - \frac{\lambda}{u^m}|u'|^p + \bar{f}(r) = 0, \quad 0 < r < 1.$$

To get the second solution, we consider (1.2) with the boundary conditions

$$(1.3) \qquad\qquad u(1) = u(0) = u'(0) = 0.$$

A function $u \in C^1[0,1]$ is called a solution to problem (1.2) and (1.3) if $u > 0$ in $(0, 1)$, $|u'|^{p-2}u' \in C^1(0,1)$, and it satisfies (1.2) and (1.3). By the sub-sup solutions method, we prove that under some additional assumptions on m, λ and $\bar{f}$, problem (1.2) and (1.3) has a positive solution u (see the proof of Theorem 2.7). Let $w(x) = u(r)$ with $r = |x|$. Then $w(x)$ is a solution with $w(0) = 0$ if $\Omega = B_1$ and some assumptions on m, λ and $\bar{f}$ are satisfied (see Theorem 2.7). Thus problem (1.1) may have at least two solutions (see Theorem 2.8).

This paper is organized as follows. In Section 2, we state the main results. The other sections are devoted to the proofs of these results.

2. Main results. Denote

$$\mathcal{F} = W^{1,p}(\Omega) \cap L^\infty(\Omega), \quad \mathcal{F}_0 = W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$
$$\mathcal{F}_0^+ = \{w \in \mathcal{F}_0; \ w \geq 0 \text{ a.e. in } \Omega\}.$$

DEFINITION 2.1. A function $u \in \mathcal{F}$ is called a *sup-solution* of the equation in (1.1) if $u > 0$ a.e. in Ω, $|\nabla u|^p/u^m \in L^1(\Omega)$, and

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla\varphi + \frac{\lambda}{u^m}|\nabla u|^p\varphi - f(x)\varphi \right) dx \geq 0, \quad \forall \varphi \in \mathcal{F}_0^+.$$

Similarly, a function $u \in \mathcal{F}$ is called a *sub-solution* of the equation in (1.1) if $u > 0$ a.e. in Ω, $|\nabla u|^p/u^m \in L^1(\Omega)$, and it satisfies the converse inequality.

DEFINITION 2.2. A function $u \in \mathcal{F}_0$ is called a *solution* of problem (1.1) if $u > 0$ a.e. in Ω, $|\nabla u|^p/u^m \in L^1(\Omega)$, and

$$\int_\Omega \left(|\nabla u|^{p-2} \nabla u \nabla \varphi + \frac{\lambda}{u^m} |\nabla u|^p \varphi - f(x)\varphi \right) dx = 0, \quad \forall \varphi \in \mathcal{F}_0.$$

PROPOSITION 2.3. *Let* $\lambda > 0$ *and* $f \in L^1(\Omega)$ *with* $f \le 0$ *a.e. in* Ω. *Then problem* (1.1) *has no solution.*

Proof. Assume that u is a solution of problem (1.1). Then $u \in \mathcal{F}_0^+$. Substituting $\varphi = u$ into the integral equality in Definition 2.2 yields $\int_\Omega |\nabla u|^p\, dx = 0$, which implies that $u = 0$ a.e. in Ω, a contradiction. ∎

PROPOSITION 2.4. *Let* $\lambda \ge 0$ *and* $m > 0$ *be constants, and let* $f \in L^q(\Omega)$ *with* $f(x) \ge 0$ *a.e. in* Ω, *where* $1/p + 1/q = 1$. *Assume that* u_2 *and* u_1 *are a sup-solution and a sub-solution of the equation in* (1.1), *respectively, with* $u_2 \ge u_1$ *on* $\partial\Omega$.

(i) *If* $0 < m < 1$, *then* $u_2 \ge u_1$ *a.e. in* Ω.
(ii) *If* $m \ge 1$, *and if there exist positive constants* c_2, c_1 *such that* $u_i \ge c_i$ $(i = 1, 2)$, *then* $u_2 \ge u_1$ *a.e. in* Ω.

Here the precise meaning of "$u_2 \ge u_1$ *on* $\partial\Omega$" *is that* "$(u_1 - u_2)_+ \in W_0^{1,p}(\Omega)$", *where* $s_+ = \max\{0, s\}$.

Proof. Let $\xi = \lambda/(p-1)$, $\mu_m = 0$ if $0 < m < 1$ and $\mu_m = \min\{c_2, c_1\}$ if $m \ge 1$, and define $g : [\mu_m, \infty) \to \mathbb{R}$ by

$$g(s) = \int_{\mu_m}^s e^{\frac{\xi}{m-1} y^{1-m}} dy \quad \text{if } m \ne 1,$$

$$g(s) = \begin{cases} \dfrac{s^{1-\xi}}{1-\xi} & (\xi \ne 1) \\ \ln(s) & (\xi = 1) \end{cases} \quad \text{if } m = 1.$$

Clearly, $g'(s) > 0$ and $g''(s) \le 0$, for all $s > \mu_m$. Then it follows from Definition 2.1 that

$$\int_\Omega (|\nabla g(u_2)|^{p-2} \nabla g(u_2) \nabla \varphi - (g'(u_2))^{p-1} f(x)\varphi)\, dx \ge 0,$$

$$\int_\Omega (|\nabla g(u_1)|^{p-2} \nabla g(u_1) \nabla \varphi - (g'(u_1))^{p-1} f(x)\varphi)\, dx \le 0$$

for any $\varphi \in \mathcal{F}_0^+$, so

$$\int_\Omega (|\nabla g(u_1)|^{p-2} \nabla g(u_1) - |\nabla g(u_2)|^{p-2} \nabla g(u_2)) \nabla \varphi\, dx$$

$$+ \int_\Omega f(x)((g'(u_2))^{p-1} - (g'(u_1))^{p-1})\varphi\, dx \le 0.$$

Note that $\varphi = (g(u_1) - g(u_2))_+ \in \mathcal{F}_0^+$. Substituting it into the above integral inequality yields

$$\int_{\Omega} (|\nabla g(u_1)|^{p-2} \nabla g(u_1) - |\nabla g(u_2)|^{p-2} \nabla g(u_2)) \nabla (g(u_1) - g(u_2))_+ \, dx$$

$$+ \int_{\Omega} f(x)((g'(u_2))^{p-1} - (g'(u_1))^{p-1})(g(u_1) - g(u_2))_+ \, dt \le 0.$$

Since $g' > 0$ and $g'' \le 0$ in (μ_m, ∞), we obtain

$$f(x)((g'(u_2))^{p-1} - (g'(u_1))^{p-1})(g(u_1) - g(u_2))_+ \ge 0 \quad \text{a.e. in } \Omega,$$

and hence

$$\int_{\Omega} (|\nabla g(u_1)|^{p-2} \nabla g(u_1) - |\nabla g(u_2)|^{p-2} \nabla g(u_2)) \nabla (g(u_1) - g(u_2))_+ dx \le 0.$$

By the inequality (cf. [D])

$$(2.1) \qquad (|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta') \cdot (\eta - \eta') \ge C(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2$$

for any $\eta, \eta' \in \mathbb{R}^N$, where C is a positive constant depending only on p, one arrives at

$$\int_{\Omega} (|\nabla g(u_1)| + |\nabla g(u_2)|)^{p-2} |\nabla (g(u_1) - g(u_2))_+|^2 \, dx = 0.$$

This shows that $(g(u_1) - g(u_2))_+ = 0$ a.e. in Ω, which implies that $u_2 \ge u_1$ a.e. in Ω. ∎

As an immediate consequence of Proposition 2.4, we obtain

COROLLARY 2.5. *Let* $\lambda > 0$ *and* $0 < m < 1$ *be constants, and let* $f \in L^q(\Omega)$ *with* $f \ge 0$ *a.e. in* Ω, *where* $1/p + 1/q = 1$. *Then problem* (1.1) *admits at most one solution in* $\mathcal{F}_0^+$.

The present paper will focus on the case of $m > 1$. We first obtain the existence of a solution for the general domain Ω.

THEOREM 2.6. *Let* $p > 1$, $\lambda > 0$, $2p/(p+1) > m > 1$, *and assume* $f \in L^\infty(\Omega)$ *with* ess $\inf_\Omega f(x) > 0$. *Then problem* (1.1) *admits one maximal, positive solution* u *in* $C^\alpha_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ *for some* $\alpha \in (0, 1)$.

For $\Omega = B_1 \equiv \{x \in \mathbb{R}^N; |x| < 1\}$, we establish the existence of the second solution of problem (1.1), which can be stated as follows.

THEOREM 2.7. *Let* $\Omega = B_1$ *and* $f(x) \equiv \bar{f}(|x|)$, *and let* $p > m > 1$. *Assume that* $\bar{f} \in C[0,1]$ *with* $\bar{f} > 0$ *on* $[0,1]$. *If* $\lambda > \inf_{r \ge 1} \mathcal{X}(r)$, *then problem* (1.1) *admits one solution* w *in* $C^1(\overline{\Omega})$ *with* $w > 0$ *in* $\Omega \setminus \{0\}$ *and* $w(0) = 0$. *Moreover, there exist two positive constants* C_2 *and* C_1 *with* $C_2 \ge C_1$ *such that*

$$(2.2) \qquad C_1 \le |x|^{(m-p)/p} w(x) \le C_2, \quad 0 \le |x| \le 1/4.$$

Here $\mathcal{X} : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$\mathcal{X}(r) = \frac{p(m-1) + N(p-m)}{p} r^{m-1} + \frac{(p-m)^p \max_{[0,1]} \bar{f}}{p^p} r^{m-p}.$$

By Theorems 2.6–2.7, we immediately obtain the following multiplicity result.

THEOREM 2.8. *Let $\Omega = B_1$ and $f(x) \equiv \bar{f}(|x|)$, and let $p > 1$, $2p/(p+1)$ $> m > 1$. Assume that $\bar{f} \in C[0,1]$ with $\bar{f} > 0$ on $[0,1]$. If $\lambda > \inf_{r \geq 1} \mathcal{X}(r)$, then problem (1.1) has at least two solutions.*

REMARK 2.9. *Let $p > m > 1$, and denote*

$$S^* = \begin{cases} S_0, & S_0 \geq 1, \\ 1, & S_0 < 1, \end{cases} \quad S_0 = \left(\frac{(p-m)^{p+1} \max_{[0,1]} \bar{f}}{p^{p-1}(m-1)[p(m-1) + N(p-m)]} \right)^{1/(p-1)}.$$

Then $\inf_{s \geq 1} \mathcal{X}(s) = \mathcal{X}(S^*)$. Indeed, since $\lim_{s \to 0^+} \mathcal{X}(s) = \lim_{s \to \infty} \mathcal{X}(s) = \infty$, $\mathcal{X}(s)$ must reach a minimum at some $s \in (0, \infty)$ satisfying $\mathcal{X}'(s) = 0$, which gives $s = S_0$, so $\inf_{s > 0} \mathcal{X}(s) = \mathcal{X}(S_0)$. Since $\mathcal{X}'(s) \geq 0$ for all $s \geq S_0$, we see that $\inf_{s \geq 1} \mathcal{X}(s) = \mathcal{X}(S_0)$ if $S_0 \geq 1$, and $\inf_{s \geq 1} \mathcal{X}(s) = \mathcal{X}(1)$ if $S_0 < 1$.

3. Proof of Theorem 2.6. Let $\epsilon \in (0, 1/2)$, and define $H_\epsilon : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by

$$H_\epsilon(x, s, \xi) = \lambda \frac{\text{sgn}(s)}{[I(s) + \epsilon]^m} |\xi|^p - f(x),$$

where $I(s) = s$ if $s \geq 0$, $I(s) = 0$ if $s < 0$. Clearly, we have

$$|H_\epsilon(x, s, \xi)| \leq \frac{\lambda}{\epsilon^m} |\xi|^p + |f|_\infty$$

for almost every $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. Hence it follows from Theorem 1 in [BMP1] that for any fixed $\epsilon \in (0, 1/2)$, there exists a function $u_\epsilon \in \mathcal{F}_0$ such that

$$(3.1) \qquad \int_\Omega (|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \nabla \varphi + H_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi) \, dx = 0, \quad \forall \varphi \in \mathcal{F}_0.$$

LEMMA 3.1. *$u_\epsilon \geq 0$ a.e. in Ω for all $\epsilon \in (0, 1/2)$.*

Proof. Since $u_\epsilon \in \mathcal{F}_0$, we have $(u_\epsilon)_- \in \mathcal{F}_0$, where $s_- = \max\{0, -s\}$. Substituting $\varphi = (u_\epsilon)_-$ into (3.1) yields

$$\int_\Omega \left(|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \nabla (u_\epsilon)_- + \lambda \frac{\text{sgn}(u_\epsilon)(u_\epsilon)_-}{[I(u_\epsilon) + \epsilon]^m} |\nabla u_\epsilon|^p - f(x)(u_\epsilon)_- \right) dx = 0.$$

This leads to

$$\int_\Omega \left(|\nabla (u_\epsilon)_-|^p + \lambda \frac{|(u_\epsilon)_-|}{[I(u_\epsilon) + \epsilon]^m} |\nabla u_\epsilon|^p \right) dx = - \int_\Omega f(x)(u_\epsilon)_- \, dx \leq 0,$$

which implies that

$$\int_\Omega |\nabla(u_\epsilon)_-|^p \, dx = 0,$$

therefore, $(u_\epsilon)_- = 0$ a.e. in Ω, i.e. $u_\epsilon \geq 0$ a.e. in Ω. $\blacksquare$

By Lemma 3.1, one derives from (3.1) that

$$(3.2) \quad \int_\Omega \left(|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \nabla \varphi + \frac{\lambda}{(u_\epsilon + \epsilon)^m}|\nabla u_\epsilon|^p \varphi - f(x)\varphi \right) dx = 0, \quad \forall \varphi \in \mathcal{F}_0.$$

Denote $U_\epsilon = u_\epsilon + \epsilon$. Then $U_\epsilon \geq \epsilon$, and

$$(3.3) \quad \int_\Omega \left(|\nabla U_\epsilon|^{p-2}\nabla U_\epsilon \nabla \varphi + \frac{\lambda}{U_\epsilon^m}|\nabla U_\epsilon|^p \varphi - f(x)\varphi \right) dx = 0, \quad \forall \varphi \in \mathcal{F}_0.$$

Next our aim is to show that the limit $\lim_{\epsilon \to 0^+} u_\epsilon(x) = u(x)$ exists for almost every $x \in \Omega$, and the limit function u is a solution of problem (1.1). Before giving the proof, we emphasize that the main difficulty is twofold: singularity and natural growth of the nonlinear lower order term. To overcome the difficulties, we will establish a locally uniform positive lower bound of u_ϵ and prove the strong compactness:

$$(3.4) \qquad \nabla U_\epsilon^{(p-m)/p} \to \nabla u^{(p-m)/p} \quad \text{strongly in } L^1(\Omega) \ (\epsilon \to 0^+).$$

Due to the singularity, it seems difficult to deal with (3.4) as in the papers mentioned above. Our method relies on the monotonicity of u_ϵ in ϵ, Lemma 3.7 and a compactness argument.

It is worth noting that, to establish (3.4), the condition $\lambda \geq (p-1)/p$ is needed in [Z1] for $m = 1$, while for $m > 1$ the present paper does not impose any condition on λ except $\lambda > 0$ (see the proof of Lemma 3.6). Moreover, we have to require $m < 2p/(p+1)$ in order to apply Lebesgue's dominated convergence theorem to the proof of (3.4). To show (3.4), we need to establish some nontrivial, uniform estimates on u_ϵ. The following lemma gives uniform upper and lower bounds for u_ϵ.

LEMMA 3.2. *For all $\epsilon \in (0, 1/2)$,*

$$(3.5) \qquad\qquad \Phi \geq u_\epsilon \geq C\Phi^{p/(p-m)} \qquad \text{a.e. in } \Omega,$$

where $C > 0$ is independent of ϵ, and $\Phi \in C_0^1(\overline{\Omega})$ denotes the unique positive solution of the problem

$$-\mathbf{div}(|\nabla v|^{p-2}\nabla v) = |f|_\infty \quad \text{in } \mathcal{D}'(\Omega), \quad v \in C_0^1(\overline{\Omega}),$$

(see [BMP1] for the existence of Φ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, [L] for the global $C^{1,\alpha}$ regularity and [D] for the positivity).

Proof. As $\lambda > 0$, it follows from (3.2) that

$$\int_{\Omega} (|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \nabla \varphi - |f|_\infty \varphi) \, dx \leq 0, \quad \forall \varphi \in \mathcal{F}_0^+,$$

thus, by Proposition 2.4, one obtains $u_\epsilon \leq \Phi$ a.e. in Ω.

Below we show the second estimate in (3.5). Let $W_\epsilon = V + \epsilon$ with $V = C\Phi^{p/(p-m)}$, where $C \in (0, 1)$ satisfies

$$\left(\frac{Cp}{p-m}\right)^{p-1} |f|_\infty \max_{\overline{\Omega}} \Phi^{m(p-1)/(p-m)} + \frac{\lambda C^{p-m} p^p}{(p-m)^p} \max_{\overline{\Omega}} |\nabla \Phi|^p \leq \operatorname*{ess\,inf}_{\Omega} f(x).$$

Clearly, $W_\epsilon \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. By simple calculations and using the properties of Φ, we deduce that

$$-\mathbf{div}(|\nabla W_\epsilon|^{p-2} \nabla W_\epsilon) + \lambda \frac{|\nabla W_\epsilon|^p}{W_\epsilon^m} - f(x)$$

$$\leq -\mathbf{div}(|\nabla V|^{p-2} \nabla V) + \lambda \frac{|\nabla V|^p}{V^m} - \operatorname*{ess\,inf}_{\Omega} f(x)$$

$$= -\left(\frac{Cp}{p-m}\right)^{p-1} \Phi^{m(p-1)/(p-m)} \operatorname{div}(|\nabla \Phi|^{p-2} \nabla \Phi)$$

$$\quad - \frac{m(p-1)(Cp)^{p-1}}{(p-m)^p} \Phi^{p(m-1)/(p-m)} |\nabla \Phi|^p + \frac{\lambda C^{p-m} p^p}{(p-m)^p} |\nabla \Phi|^p - \operatorname*{ess\,inf}_{\Omega} f(x)$$

$$= \left(\frac{Cp}{p-m}\right)^{p-1} |f|_\infty \Phi^{m(p-1)/(p-m)} - \frac{m(p-1)(Cp)^{p-1}}{(p-m)^p} \Phi^{p(m-1)/(p-m)} |\nabla \Phi|^p$$

$$\quad + \frac{\lambda C^{p-m} p^p}{(p-m)^p} |\nabla \Phi|^p - \operatorname*{ess\,inf}_{\Omega} f(x)$$

$$\leq \left(\frac{Cp}{p-m}\right)^{p-1} |f|_\infty \max_{\overline{\Omega}} \Phi^{m(p-1)/(p-m)} + \frac{\lambda C^{p-m} p^p}{(p-m)^p} \max_{\overline{\Omega}} |\nabla \Phi|^p - \operatorname*{ess\,inf}_{\Omega} f(x)$$

$$\leq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

By Proposition 2.4 and (3.3), we derive that $U_\epsilon \geq W_\epsilon$ a.e. in Ω, i.e. $u_\epsilon \geq V$ a.e. in Ω. $\blacksquare$

By (3.5), it is easy to derive that for any compact subset Ω' of Ω, there exists a positive constant C_0, independent of ϵ, such that

$$u_\epsilon \geq C_0 \quad \text{a.e. in } \Omega',$$

therefore

$$|H_\epsilon(x, u_\epsilon(x), \xi)| \leq \frac{\lambda}{C_0^m} |\xi|^p + |f|_\infty$$

for almost every $x \in \Omega'$ and for all $\xi \in \mathbb{R}^N$. Consequently, all assumptions of Theorem 6.1 in [MRS] (see also [R], [RT]) for local Hölder continuity are satisfied. We conclude that u_ϵ is uniformly bounded in $C^\alpha(\Omega')$ for some

$\alpha \in (0,1)$. Using the Arzelà–Ascoli theorem, there exist a subsequence of $\{u_\epsilon\}$, still denoted by $\{u_\epsilon\}$, and a function $u \in C_{\text{loc}}^\alpha(\Omega)$, such that, as $\epsilon \to 0^+$,

$$(3.6) \qquad u_\epsilon \to u \quad \text{uniformly in } \Omega'.$$

Hence, u is continuous in Ω and satisfies, by (3.5),

$$(3.7) \qquad \Phi \geq u \geq C\Phi^{p/(p-m)} \quad \text{in } \Omega.$$

Then $\lim_{x \to \partial\Omega} u(x) = 0$. Define $u = 0$ on $\partial\Omega$. Then $u \in C(\overline{\Omega})$.

The following lemma shows the monotonicity of u_ϵ in ϵ.

LEMMA 3.3. $u_{\epsilon_2} \geq u_{\epsilon_1}$ *in* Ω *for* $1/2 > \epsilon_2 > \epsilon_1 > 0$.

Proof. Let $V_{\epsilon_2} = u_{\epsilon_2} + \epsilon_1$. Then $U_{\epsilon_1} - V_{\epsilon_2} \in \mathcal{F}_0$. It follows from the equality (3.3) with $\epsilon = \epsilon_2$ that

$$\int_\Omega \left(|\nabla V_{\epsilon_2}|^{p-2}\nabla V_{\epsilon_2}\nabla\varphi + \frac{\lambda}{V_{\epsilon_2}^m}|\nabla V_{\epsilon_2}|^p\varphi - f(x)\varphi \right) dx \geq 0, \quad \forall \varphi \in \mathcal{F}_0^+.$$

By Proposition 2.4, we see that $V_{\epsilon_2} \geq U_{\epsilon_1}$ in Ω, that is, $u_{\epsilon_2} \geq u_{\epsilon_1}$ in Ω. ∎

From Lemma 3.3 and (3.6), we derive that

$$(3.8) \qquad u_\epsilon \geq u \quad \text{in } \Omega,$$

$$(3.9) \qquad u_\epsilon \to u \quad \text{in } \Omega \ (\epsilon \to 0^+).$$

LEMMA 3.4. $\nabla u_\epsilon \to \nabla u$ *strongly in* $L^p(\Omega)$ $(\epsilon \to 0^+)$.

Proof. Substituting $\varphi = u_\epsilon$ into (3.2) yields

$$\int_\Omega |\nabla u_\epsilon|^p \, dx + \lambda \int_\Omega \frac{u_\epsilon}{(u_\epsilon + \epsilon)^m} |\nabla u_\epsilon|^p \, dx = \int_\Omega f(x)u_\epsilon \, dx,$$

therefore, using the first estimate of (3.5) yields

$$(3.10) \qquad \int_\Omega |\nabla u_\epsilon|^p \, dx \leq C,$$

from which and (3.9) we deduce that $\nabla u \in L^p(\Omega)$, and up to a subsequence

$$\nabla u_\epsilon \to \nabla u \quad \text{weakly in } L^p(\Omega) \ (\epsilon \to 0^+).$$

Then substituting $\varphi = u_\epsilon - u$ into (3.2) yields

$$\int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon(\nabla u_\epsilon - \nabla u) \, dx$$

$$+ \lambda \int_\Omega \frac{u_\epsilon - u}{(u_\epsilon + \epsilon)^m} |\nabla u_\epsilon|^p \, dx = \int_\Omega f(x)(u_\epsilon - u) \, dx.$$

By (3.8), we have

$$\int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon(\nabla u_\epsilon - \nabla u) \, dx \leq \int_\Omega f(x)(u_\epsilon - u) \, dx,$$

therefore

$$\varlimsup_{\epsilon\to 0^+}\int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon(\nabla u_\epsilon - \nabla u)\,dx \le 0.$$

Noticing

$$\lim_{\epsilon\to 0^+}\int_\Omega |\nabla u|^{p-2}\nabla u(\nabla u_\epsilon - \nabla u)\,dx = 0,$$

we obtain

$$\varlimsup_{\epsilon\to 0^+}\int_\Omega (|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon - |\nabla u|^{p-2}\nabla u)(\nabla u_\epsilon - \nabla u)\,dx \le 0.$$

Recalling the inequality (2.1), we find that

$$(3.11)\qquad \lim_{\epsilon\to 0^+}\int_\Omega (|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon - |\nabla u|^{p-2}\nabla u)(\nabla u_\epsilon - \nabla u)\,dx = 0,$$

so

$$\int_\Omega (|\nabla u_\epsilon| + |\nabla u|)^{p-2}|\nabla u_\epsilon - \nabla u|^2\,dx \to 0 \quad (\epsilon\to 0^+).$$

From this and using Hölder's inequality we get

$$\int_\Omega |\nabla u_\epsilon - \nabla u|^p\,dx$$

$$\le \int_\Omega (|\nabla u_\epsilon| + |\nabla u|)^{p-1}|\nabla u_\epsilon - \nabla u|\,dx$$

$$= \int_\Omega (|\nabla u_\epsilon| + |\nabla u|)^{p/2}\frac{|\nabla u_\epsilon - \nabla u|}{(|\nabla u_\epsilon| + |\nabla u|)^{1-p/2}}\,dx$$

$$\le \left(\int_\Omega (|\nabla u_\epsilon| + |\nabla u|)^p\,dx\right)^{1/2}\cdot\left(\int_\Omega \frac{|\nabla u_\epsilon - \nabla u|^2}{(|\nabla u_\epsilon| + |\nabla u|)^{2-p}}\,dx\right)^{1/2}$$

$$\le C\left(\int_\Omega \frac{|\nabla u_\epsilon - \nabla u|^2}{(|\nabla u_\epsilon| + |\nabla u|)^{2-p}}\,dx\right)^{1/2} \to 0 \quad (\epsilon\to 0^+).\ \blacksquare$$

Lemma 3.4 immediately implies that

$$(3.12)\qquad \nabla u_\epsilon \to \nabla u \quad \text{a.e. in } \Omega\ (\epsilon\to 0^+).$$

LEMMA 3.5. *For all $\epsilon\in(0,1/2)$, we have*

$$(3.13)\qquad \int_\Omega \frac{|\nabla u_\epsilon|^p}{(u_\epsilon + \epsilon)^m}\,dx \le \frac{1}{\lambda}\int_\Omega f(x)\,dx,$$

i.e.

$$\int_\Omega |\nabla U_\epsilon^{(p-m)/p}|^p\,dx \le \frac{(p-m)^p}{\lambda p^p}\int_\Omega f(x)\,dx.$$

Proof. Clearly, $\frac{u_\epsilon}{u_\epsilon+\sigma} \in \mathcal{F}_0$ for any $\sigma > 0$. Substituting it into (3.2) yields

$$\sigma \int_\Omega \frac{|\nabla u_\epsilon|^p}{(u_\epsilon + \sigma)^2} \, dx + \lambda \int_\Omega \frac{|\nabla u_\epsilon|^p}{(u_\epsilon + \epsilon)^m} \frac{u_\epsilon}{u_\epsilon + \sigma} \, dx = \int_\Omega f(x) \frac{u_\epsilon}{u_\epsilon + \sigma} \, dx,$$

therefore

$$\lambda \int_\Omega \frac{|\nabla u_\epsilon|^p}{(u_\epsilon + \epsilon)^m} \frac{u_\epsilon}{u_\epsilon + \sigma} \, dx \le \int_\Omega f(x) \frac{u_\epsilon}{u_\epsilon + \sigma} \, dx.$$

Note that $u_\epsilon > 0$ in Ω. Then for any fixed $\epsilon \in (0, 1/2)$ we have

$$\frac{u_\epsilon}{u_\epsilon + \sigma} \to 1 \quad \text{in } \Omega \ (\sigma \to 0^+).$$

Letting $\sigma \to 0^+$ and using Lebesgue's dominated convergence theorem, we obtain

$$\lambda \int_\Omega \frac{|\nabla u_\epsilon|^p}{(u_\epsilon + \epsilon)^m} \, dx \le \int_\Omega f(x) \, dx. \quad \blacksquare$$

Now passing to the limit in (3.13) as $\epsilon \to 0^+$ and using Fatou's lemma and noticing (3.9), (3.7) and (3.12) yield

$$\int_\Omega \frac{|\nabla u|^p}{u^m} \, dx \le \frac{1}{\lambda} \int_\Omega f(x) \, dx.$$

This shows that

(3.14) $\qquad |\nabla u|^p / u^m \in L^1(\Omega) \quad$ (i.e. $|\nabla u^{(p-m)/p}| \in L^p(\Omega)$).

The following lemma is the key to the whole proof.

LEMMA 3.6. $\nabla U_\epsilon^{(p-m)/p} \to \nabla u^{(p-m)/p}$ *strongly in* $L^p(\Omega)$ $(\epsilon \to 0^+)$.

To show Lemma 3.6, we need the following

LEMMA 3.7. *Let Φ be as in Lemma 3.2. Then*

$$\int_\Omega [\Phi(x)]^{-s} \, dx < \infty \quad \text{if and only if} \quad s < 1.$$

Proof. We first claim that

(3.15) $\qquad \int_\Omega [d(x)]^{-s} \, dx < \infty \quad$ if and only if $\quad s < 1,$

where $d(x) = \text{dist}(x, \partial\Omega)$. Denote by Ψ_1 the eigenfunction corresponding to the first eigenvalue λ_1 of $-\Delta$ in Ω with homogeneous Dirichlet boundary condition. Theorem 2 from [V] yields $\partial\Psi_1/\partial\nu > 0$ on $\partial\Omega$, where ν denotes the interior unit normal to $\partial\Omega$, so $\Psi_1(x) \ge Cd(x)$ for all $x \in \overline{\Omega}$. On the other hand, since $\Psi_1 \in C^1(\overline{\Omega})$, we have $\Psi_1(x) \le C|x - x_0|$ for any $x_0 \in \partial\Omega$ and for all $x \in \overline{\Omega}$. In addition, there exists some $\overline{x} \in \partial\Omega$ such that $d(x) = |x - \overline{x}|$,

and hence $\Psi_1(x) \leq Cd(x)$ for all $x \in \overline{\Omega}$. Since (see [LM])

$$\int_{\Omega} [\Psi_1(x)]^{-s}\,dx < \infty \quad \text{if and only if} \quad s < 1,$$

we find that (3.15) holds.

Theorem 5 from [V] yields $\partial\Phi/\partial\nu > 0$ on $\partial\Omega$, hence $\Phi(x) \geq Cd(x)$ for all $x \in \overline{\Omega}$. By the same reasoning as for Ψ_1, one can derive that $\Phi(x) \leq Cd(x)$ for all $x \in \overline{\Omega}$. Therefore, there exist two positive constants C_2, C_1 with $C_2 \geq C_1$ such that

$$C_1 d(x) \leq \Phi(x) \leq C_2 d(x), \quad \forall x \in \overline{\Omega}.$$

This and (3.15) imply the desired result. ∎

Proof of Lemma 3.6. Clearly, we have

$$\nabla U_\epsilon^{(p-m)/p} \to \nabla u^{(p-m)/p} \quad \text{weakly in } L^p(\Omega) \ (\epsilon \to 0^+).$$

Let $V_\epsilon = u + \epsilon$. Note that for fixed $\epsilon \in (0, 1/2)$,

$$\varphi = \left(\frac{p-m}{p}\right)^{p-1} U_\epsilon^{-(p-1)m/p}(U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \in \mathcal{F}_0.$$

Substituting it into (3.3) yields

$$(3.16) \quad \int_{\Omega} |\nabla U_\epsilon^{(p-m)/p}|^{p-2} \nabla U_\epsilon^{(p-m)/p} \nabla (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p})\,dx$$

$$= \left(\frac{p-m}{p}\right)^{p-1} \frac{(p-1)m}{p} \int_{\Omega} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(p+(p-1)m)/p}}(U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p})\,dx$$

$$- \lambda\left(\frac{p-m}{p}\right)^{p-1} \int_{\Omega} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(2p-1)m/p}}(U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p})\,dx$$

$$+ \left(\frac{p-m}{p}\right)^{p-1} \int_{\Omega} f(x) U_\epsilon^{-(p-1)m/p}(U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p})\,dx$$

$$:= I_{1\epsilon} + I_{2\epsilon} + I_{3\epsilon}.$$

Below we shall show that

$$(3.17) \quad \overline{\lim_{\epsilon \to 0^+}} \int_{\Omega} |\nabla U_\epsilon^{(p-m)/p}|^{p-2} \nabla U_\epsilon^{(p-m)/p} \nabla (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p})\,dx \leq 0.$$

By the first estimate in (3.5), there exists a constant $\delta \in (0, 1)$ such that for all $\epsilon \in (0, \tau)$, where $\tau = \min\left\{\frac{1}{2}, \frac{1}{2}\left(\frac{p\lambda}{(p-1)m}\right)^{1/(m-1)}\right\}$, we have

$$u_\epsilon < \tau \quad \text{in } \Omega_\delta := \{x \in \Omega;\ \text{dist}(x, \partial\Omega) < \delta\},$$

thus, for all $\epsilon \in (0, \tau)$,

$$U_\epsilon^{m-1} = (u_\epsilon + \epsilon)^{m-1} < (2\tau)^{m-1} \leq \frac{p\lambda}{(p-1)m} \quad \text{in } \Omega_\delta.$$

This and $U_\epsilon \geq V_\epsilon$ imply that

$$\int_{\Omega_\delta} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(p+(p-1)m)/p}} (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx$$

$$= \int_{\Omega_\delta} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(2p-1)m/p}} U_\epsilon^{m-1} (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx$$

$$\leq \frac{p\lambda}{(p-1)m} \int_{\Omega_\delta} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(2p-1)m/p}} (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx$$

for all $\epsilon \in (0, \tau)$, therefore

$$I_{1\epsilon} = \left(\frac{p-m}{p}\right)^{p-1} \frac{(p-1)m}{p} \int_{\Omega_\delta} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(p+(p-1)m)/p}} (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx$$

$$+ \left(\frac{p-m}{p}\right)^{p-1} \frac{(p-1)m}{p} \int_{\Omega-\Omega_\delta} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(p+(p-1)m)/p}} (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx$$

$$\leq \lambda \left(\frac{p-m}{p}\right)^{p-1} \int_{\Omega_\delta} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(2p-1)m/p}} (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx$$

$$+ \left(\frac{p-m}{p}\right)^{p-1} \frac{(p-1)m}{p} \int_{\Omega-\Omega_\delta} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(p+(p-1)m)/p}} (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx$$

$$\leq -I_{2\epsilon} + \left(\frac{p-m}{p}\right)^{p-1} \frac{(p-1)m}{p} \int_{\Omega-\Omega_\delta} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(p+(p-1)m)/p}} (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx,$$

i.e.

$$(3.18) \quad I_{1\epsilon} + I_{2\epsilon}$$

$$\leq \left(\frac{p-m}{p}\right)^{p-1} \frac{(p-1)m}{p} \int_{\Omega-\Omega_\delta} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(p+(p-1)m)/p}} (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx.$$

On the other hand, by (3.13) and (3.6), we have

$$\int_{\Omega-\Omega_\delta} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^{(p+(p-1)m)/p}} (U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx$$

$$= \int_{\Omega-\Omega_\delta} \frac{|\nabla U_\epsilon|^p}{U_\epsilon^m} \left[1 - \left(\frac{V_\epsilon}{U_\epsilon}\right)^{(p-m)/p}\right] dx$$

$$\leq C \max_{\Omega-\Omega_\delta} \left[1 - \left(\frac{V_\epsilon}{U_\epsilon}\right)^{(p-m)/p}\right] \to 0 \quad (\epsilon \to 0^+).$$

This together with (3.18) implies that

(3.19) $$\overline{\lim_{\epsilon \to 0^+}} (I_{1\epsilon} + I_{2\epsilon}) \le 0.$$

Next let us estimate $I_{3\epsilon}$. Since $1 < m < 2p/(p+1)$, it follows that $p(m-1)/(p-m) < 1$. By Lemma 3.7, one obtains

$$\int_{\Omega} \Phi^{p(1-m)/(p-m)} \, dx < \infty \quad (\text{i.e. } \Phi^{p(1-m)/(p-m)} \in L^1(\Omega)).$$

Since $U_\epsilon \ge V_\epsilon \ge \epsilon$, by noticing $m > 1$ and by using the second estimate in (3.5), we have

$$0 \le U_\epsilon^{-(p-1)m/p}(U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \le U_\epsilon^{1-m} \le C\Phi^{p(1-m)/(p-m)} \text{ in } \Omega.$$

Note that

$$U_\epsilon^{-(p-1)m/p}(U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \to 0 \quad \text{in } \Omega \ (\epsilon \to 0^+).$$

Using Lebesgue's dominated convergence theorem, we get

$$I_{3\epsilon} = \int_{\Omega} f(x) U_\epsilon^{-(p-1)m/p}(U_\epsilon^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx \to 0 \quad (\epsilon \to 0^+).$$

This together with (3.16) and (3.19) implies (3.17).

Noticing

$$0 \le 1 - \left(\frac{u}{V_\epsilon}\right)^{m/p} \le 1 \quad \text{in } \Omega, \quad 1 - \left(\frac{u}{V_\epsilon}\right)^{m/p} \to 0 \quad \text{in } \Omega \quad (\epsilon \to 0^+),$$

and using Lebesgue's dominated convergence theorem and (3.14), we obtain, as $\epsilon \to 0^+$,

$$\int_{\Omega} |\nabla(u^{(p-m)/p} - V_\epsilon^{(p-m)/p})|^p \, dx = \left(\frac{p-m}{p}\right)^p \int_{\Omega} \frac{|\nabla u|^p}{u^m} \left|1 - \left(\frac{u}{V_\epsilon}\right)^{m/p}\right|^p dx \to 0,$$

therefore, by Hölder's inequality and by using (3.13), we have

$$\left| \int_{\Omega} |\nabla U_\epsilon^{(p-m)/p}|^{p-2} \nabla U_\epsilon^{(p-m)/p} \nabla(u^{(p-m)/p} - V_\epsilon^{(p-m)/p}) \, dx \right|$$

$$\le \left(\int_{\Omega} |\nabla U_\epsilon^{(p-m)/p}|^p \, dx \right)^{(p-1)/p} \cdot \left(\int_{\Omega} |\nabla(u^{(p-m)/p} - V_\epsilon^{(p-m)/p})|^p \, dx \right)^{1/p}$$

$$\le C \left(\int_{\Omega} |\nabla(u^{(p-m)/p} - V_\epsilon^{(p-m)/p})|^p \, dx \right)^{1/p} \to 0 \quad (\epsilon \to 0^+).$$

From this and (3.17) it follows that

$$\overline{\lim_{\epsilon \to 0}} \int_\Omega |\nabla U_\epsilon^{(p-m)/p}|^{p-2} \nabla U_\epsilon^{(p-m)/p} \nabla (U_\epsilon^{(p-m)/p} - u^{(p-m)/p})\, dx \le 0.$$

Since

$$\lim_{\epsilon \to 0^+} \int_\Omega |\nabla u^{(p-m)/p}|^{p-2} \nabla u^{(p-m)/p} \nabla (U_\epsilon^{(p-m)/p} - u^{(p-m)/p})\, dx = 0,$$

we derive that

$$\overline{\lim_{\epsilon \to 0^+}} \int_\Omega (|\nabla U_\epsilon^{(p-m)/p}|^{p-2} \nabla U_\epsilon^{(p-m)/p} - |\nabla u^{(p-m)/p}|^{p-2} \nabla u^{(p-m)/p})$$

$$\times \nabla (U_\epsilon^{(p-m)/p} - u^{(p-m)/p})\, dx \le 0.$$

Recalling the inequality (2.1), we have

$$\lim_{\epsilon \to 0^+} \int_\Omega (|\nabla U_\epsilon^{(p-m)/p}|^{p-2} \nabla U_\epsilon^{(p-m)/p} - |\nabla u^{(p-m)/p}|^{p-2} \nabla u^{(p-m)/p})$$

$$\times \nabla (U_\epsilon^{(p-m)/p} - u^{(p-m)/p})\, dx = 0.$$

Then the same reasoning as in Lemma 3.4 completes the proof.

Lemma 3.6 immediately implies that

$$\int_\Omega \frac{|\nabla u_\epsilon|^p}{(u_\epsilon + \epsilon)^m}\, dx \to \int_\Omega \frac{|\nabla u|^p}{u^m}\, dx \qquad (\epsilon \to 0^+).$$

Then it is easy to check that u is a solution of problem (1.1).

It remains to show that u is maximal. Let w be a solution of problem (1.1). Then $w_\epsilon = w + \epsilon$ satisfies

$$\int_\Omega \left(|\nabla w_\epsilon|^{p-2} \nabla w_\epsilon \nabla \varphi + \frac{\lambda}{w_\epsilon^m} |\nabla w_\epsilon|^p \varphi - f(x)\varphi \right) dx \le 0, \qquad \forall \varphi \in \mathcal{F}_0^+.$$

Recalling Proposition 2.4 and (3.3), we see that

$$U_\epsilon = u_\epsilon + \epsilon \ge w_\epsilon = w + \epsilon \quad \text{a.e. in } \Omega,$$

i.e. $u_\epsilon \ge w$ a.e. in Ω and passing to the limit in ϵ gives $u \ge w$ a.e. in Ω. The proof of Theorem 2.6 is complete.

4. Proof of Theorem 2.7. Under the hypotheses of Theorem 2.7, we first show the existence of a positive solution for problem (1.2) and (1.3). Let $\epsilon \in (0,1)$, and define $\overline{H}_\epsilon : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{H}_\epsilon(r, v, \xi) = -\frac{N-1}{r + \epsilon^{1/\alpha}} |\xi|^{p-2}\xi + \lambda \frac{|\xi|^p}{[I_\epsilon(v)]^m} - \overline{f}(r),$$

where $\alpha = p/(p-m)$, and $I_\epsilon(v) = v + \epsilon^2$ if $v \geq 0$, $I_\epsilon(v) = \epsilon^2$ if $v < 0$. By the inequality $a^{p-1} \leq a^p + 1$ $(a \geq 0)$, we have

(4.1)
$$|\overline{H}_\epsilon(r,v,\xi)| \leq \frac{N-1}{\epsilon^{1/\alpha}}|\xi|^{p-1} + \frac{\lambda}{\epsilon^{2m}}|\xi|^p + \max_{[0,1]}\overline{f}$$

$$\leq \frac{N-1}{\epsilon^{1/\alpha}}(1 + |\xi|^p) + \frac{\lambda}{\epsilon^{2m}}|\xi|^p + \max_{[0,1]}\overline{f}$$

$$\leq \left(\frac{N-1}{\epsilon^{1/\alpha}} + \frac{\lambda}{\epsilon^{2m}} + \max_{[0,1]}\overline{f}\right)\mathcal{H}(|\xi|)$$

for all $(r,v,\xi) \in (0,1) \times \mathbb{R} \times \mathbb{R}$, where $\mathcal{H}(s) = 1 + s^p$ for $s \geq 0$. Denote $\mathcal{M} = \{u \in C^1(0,1); |u'|^{p-2}u' \in C^1(0,1)\}$ and define an operator $\mathcal{L}_\epsilon : \mathcal{M} \to C(0,1)$ by

$$(\mathcal{L}_\epsilon u)(r) = -(|u'|^{p-2}u')' + \overline{H}_\epsilon(r,u,u'), \quad 0 < r < 1.$$

Consider the problem

(4.2)
$$\begin{cases} (\mathcal{L}_\epsilon u)(r) = 0, & 0 < r < 1, \\ u(1) = u(0) = 0. \end{cases}$$

A function u is called a *sup-solution* [*sub-solution*] of problem (4.2) if $\mathcal{L}_\epsilon u \geq$ [$\leq$] 0 in $(0,1)$, and $u(r) \geq$ [$\leq$] 0 at $r = 0,1$.

We will apply the sub-sup solutions method (cf. Theorem 1 and Remark 2.4 in [JG]) to show the existence of solutions of problem (4.2). Since $\int_0^\infty (s^{p-1}/\mathcal{H}(r))\,dr = \infty$, the condition (2.3) in [JG] is satisfied. Then it suffices to find a sub-solution and a sup-solution to obtain a solution.

LEMMA 4.1. *Let* $W = C\Psi^\alpha$ *with* $\alpha = p/(p-m)$, *where*

(4.3)
$$\Psi(r) = \frac{p-1}{p}\left[\left(\frac{1}{2}\right)^{p/(p-1)} - \left|\frac{1}{2} - r\right|^{p/(p-1)}\right],$$

and the constant $C \in (0,1)$ *is such that*

$$(C\alpha)^{p-1} + (N-1)(C\alpha)^{p-1}2^{(\alpha-1)(p-1)-1} + \lambda C^{p-m}\alpha^p \leq \min_{[0,1]}\overline{f}.$$

Then W *is a sub-solution of problem* (4.2).

Proof. It is easy to check that Ψ has the following properties:

  (i) $\Psi > 0$ in $(0,1)$, $\Psi \in C^1[0,1]$,
  (ii) $(|\Psi'|^{p-2}\Psi')' = -1$ in $(0,1)$, $\Psi(1) = \Psi(0) = 0$,
  (iii) $|\Psi(r)| \leq r$ and $|\Psi'(r)| \leq 1$, for all $r \in [0,1]$.

Using the properties of Ψ and noticing $(\alpha - 1)(p - 1) > 1$, one arrives at

$$\mathcal{L}_\epsilon W = -(|W'|^{p-2}W')' - \frac{N-1}{r + \epsilon^{1/\alpha}}|W'|^{p-2}W' + \frac{\lambda}{(W + \epsilon^2)^m}|W'|^p - \bar{f}(r)$$

$$\leq -(|W'|^{p-2}W')' - \frac{N-1}{r + \epsilon^{1/\alpha}}|W'|^{p-2}W' + \frac{\lambda}{W^m}|W'|^p - \bar{f}(r)$$

$$= -(C\alpha)^{p-1}\Psi^{(\alpha-1)(p-1)}(|\Psi'|^{p-2}\Psi')'$$

$$\quad - (C\alpha)^{p-1}(\alpha-1)(p-1)\Psi^{(\alpha-1)(p-1)-1}|\Psi'|^p$$

$$\quad - (N-1)(C\alpha)^{p-1}\frac{\Psi^{(\alpha-1)(p-1)}}{r + \epsilon^{1/\alpha}}|\Psi'|^{p-2}\Psi' + \lambda C^{p-m}\alpha^p|\Psi'|^p - \bar{f}(r)$$

$$= (C\alpha)^{p-1}\Psi^{(\alpha-1)(p-1)} - (C\alpha)^{p-1}(\alpha-1)(p-1)\Psi^{(\alpha-1)(p-1)-1}|\Psi'|^p$$

$$\quad - (N-1)(C\alpha)^{p-1}\frac{\Psi^{(\alpha-1)(p-1)}}{r + \epsilon^{1/\alpha}}|\Psi'|^{p-2}\Psi' + \lambda C^{p-m}\alpha^p|\Psi'|^p - \bar{f}(r)$$

$$\leq (C\alpha)^{p-1}\Psi^{(\alpha-1)(p-1)} + (N-1)(C\alpha)^{p-1}(r + \epsilon^{1/\alpha})^{(\alpha-1)(p-1)-1}|\Psi'|^{p-1}$$

$$\quad + \lambda C^{p-m}\alpha^p|\Psi'|^p - \min_{[0,1]}\bar{f}(r) \quad (\text{note } \Psi(r) \leq r + \epsilon^{1/\alpha})$$

$$\leq (C\alpha)^{p-1} + (N-1)(C\alpha)^{p-1}2^{(\alpha-1)(p-1)-1} + \lambda C^{p-m}\alpha^p - \min_{[0,1]}\bar{f}(r)$$

$$\leq 0, \quad 0 < r < 1.$$

Thus the lemma follows. ■

Let $\inf_{r \geq 1}\mathcal{X}(r) \equiv \delta$. Then it follows from the definition of infimum and $\lambda > \delta$ that for $\delta_0 = (\lambda - \delta)/2 > 0$, there exists some $C_* \geq 1$, such that

$$\mathcal{X}(C_*) < \delta + \delta_0 < \lambda.$$

LEMMA 4.2. *There exists a constant $\epsilon_0 \in (0,1)$ such that for any $\epsilon \in (0, \epsilon_0)$, $V_\epsilon = C_*(r + \epsilon^{1/\alpha})^\alpha$ is a sup-solution of problem (4.2).*

Proof. Noticing $V_\epsilon \geq \epsilon$ and $(\alpha - 1)p = \alpha m$, one has

$$\mathcal{L}_\epsilon V_\epsilon = -(|V_\epsilon'|^{p-2}V_\epsilon')' - \frac{N-1}{r + \epsilon^{1/\alpha}}|V_\epsilon'|^{p-2}V_\epsilon' + \frac{\lambda}{(V_\epsilon + \epsilon^2)^m}|V_\epsilon'|^p - \bar{f}$$

$$= -(C_*\alpha)^{p-1}(\alpha-1)(p-1)(r + \epsilon^{1/\alpha})^{(\alpha-1)(p-1)-1}$$

$$\quad - (N-1)(C_*\alpha)^{p-1}(r + \epsilon^{1/\alpha})^{(\alpha-1)(p-1)-1}$$

$$\quad + \frac{\lambda C_*^{p-m}\alpha^p}{[1 + C_*^{-1}(r + \epsilon^{1/\alpha})^{-\alpha}\epsilon^2]^m} - \bar{f}.$$

Since $(\alpha - 1)(p - 1) > 1$, one obtains

$$
\begin{aligned}
\mathcal{L}_\epsilon V_\epsilon \geq \; & -(C_*\alpha)^{p-1}(\alpha-1)(p-1)(1+\epsilon^{1/\alpha})^{(\alpha-1)(p-1)-1} \\
& -(N-1)(C_*\alpha)^{p-1}(1+\epsilon^{1/\alpha})^{(\alpha-1)(p-1)-1} + \frac{\lambda C_*^{p-m}\alpha^p}{(1+C_*^{-1}\epsilon)^m} - \max_{[0,1]} \overline{f} \\
= \; & \lambda C_*^{p-m}\alpha^p - (C_*\alpha)^{p-1}[(\alpha-1)(p-1)+N-1] - \max_{[0,1]} \overline{f} + r(\epsilon) \\
= \; & C_*^{p-m}\alpha^p[\lambda - \mathcal{X}(C_*)] + r(\epsilon), \quad 0 < r < 1,
\end{aligned}
$$

where

$$
\begin{aligned}
r(\epsilon) = \; & (C_*\alpha)^{p-1}[(\alpha-1)(p-1)+N-1][1-(1+\epsilon^{1/\alpha})^{(\alpha-1)(p-1)-1}] \\
& + \lambda\alpha^p C_*^{p-m}[(1+C_*^{-1}\epsilon)^{-m}-1].
\end{aligned}
$$

Clearly, $r(\epsilon) \to 0$ as $\epsilon \to 0^+$. Since $\lambda > \mathcal{X}(C_*)$, there exists a constant $\epsilon_0 \in (0,1)$ such that for any $\epsilon \in (0,\epsilon_0)$,

$$
C_*^{p-m}\alpha^p[\lambda - \mathcal{X}(C_*)] + r(\epsilon) \geq 0, \quad 0 < r < 1.
$$

Therefore for any $\epsilon \in (0,\epsilon_0)$,

$$
\mathcal{L}_\epsilon V_\epsilon \geq 0, \quad 0 < r < 1.
$$

This shows that for any $\epsilon \in (0,\epsilon_0)$, V_ϵ is a sup-solution of (4.2). ∎

By Theorem 1 and Remark 2.4 in [JG] and Lemmas 4.1–4.2, we see that for any fixed $\epsilon \in (0,\epsilon_0)$, problem (4.2) has a solution $u_\epsilon \in C^1[0,1] \cap \mathcal{M}$ with

$$
(4.4) \qquad C_*(r+\epsilon^{1/\alpha})^\alpha \geq u_\epsilon \geq C\Psi^\alpha, \quad r \in [0,1].
$$

Hence u_ϵ satisfies

$$
(4.5) \quad -(|u'_\epsilon|^{p-2}u'_\epsilon)' - (N-1)\frac{|u'_\epsilon|^{p-2}u'_\epsilon}{r+\epsilon^{1/\alpha}} + \lambda\frac{|u'_\epsilon|^p}{(u_\epsilon+\epsilon^2)^m} = \overline{f}(r), r \in (0,1).
$$

Next we estimate u'_ϵ. We first obtain

LEMMA 4.3. *There exists a positive constant C, independent of ϵ, such that for all $\epsilon \in (0,\epsilon_0)$,*

$$
(4.6) \qquad |u'_\epsilon(r)| \leq C, \quad \forall r \in [0,1].
$$

Proof. Noticing $u_\epsilon(1) = u_\epsilon(0) = 0$ and $u_\epsilon \geq 0$ on $[0,1]$, we have

$$
(4.7) \qquad u'_\epsilon(0) \geq 0 \geq u'_\epsilon(1).
$$

Integrating (4.5) over $(0,1)$ yields

$$
-(|u'_\epsilon|^{p-2}u'_\epsilon)|_0^1 + \lambda\int_0^1 \frac{|u'_\epsilon|^p}{(u_\epsilon+\epsilon^2)^m}\,dr = (N-1)\int_0^1 \frac{|u'_\epsilon|^{p-2}u'_\epsilon}{r+\epsilon^{1/\alpha}}\,dr + \int_0^1 \overline{f}(r)\,dr,
$$

and by (4.7), we derive that

$$(4.8) \qquad \lambda \int_0^1 \frac{|u'_\epsilon|^p}{(u_\epsilon + \epsilon^2)^m} \, dr \le (N-1) \int_0^1 \frac{|u'_\epsilon|^{p-1}}{r + \epsilon^{1/\alpha}} \, dr + \int_0^1 \bar{f}(r) \, dr.$$

Using Young's inequality $ab \le \sigma a^l + \sigma^{-q/l} b^q$ $(a, b \ge 0,\ \sigma > 0,\ q, l > 1,$ $1/l + 1/q = 1)$ and taking $l = p/(p-1)$ and $q = p$, we deduce that

$$(4.9) \qquad \frac{|u'_\epsilon|^{p-1}}{r + \epsilon^{1/\alpha}} \le \sigma \frac{|u'_\epsilon|^p}{(r + \epsilon^{1/\alpha})^{p/(p-1)}} + \sigma^{1-p}.$$

On the other hand, by the first estimate in (4.4) we obtain

$$(4.10) \qquad u_\epsilon(r) + \epsilon^2 \le C_*(r + \epsilon^{1/\alpha})^\alpha + \epsilon^2 \le 2C_*(r + \epsilon^{1/\alpha})^\alpha, \qquad r \in [0, 1],$$

therefore, due to $(p - m)/(p - 1) < m$, we get

$$\frac{|u'_\epsilon|^p}{(r + \epsilon^{1/\alpha})^{p/(p-1)}} \le C \frac{|u'_\epsilon|^p}{(u_\epsilon + \epsilon^2)^{(p-m)/(p-1)}} \le C \frac{|u'_\epsilon|^p}{(u_\epsilon + \epsilon^2)^m}, \qquad r \in [0, 1],$$

where $C > 0$ are constants independent of ϵ. Combining this and (4.9) yields

$$(4.11) \qquad \frac{|u'_\epsilon|^{p-1}}{r + \epsilon^{1/\alpha}} \le \sigma C \frac{|u'_\epsilon|^p}{(u_\epsilon + \epsilon^2)^m} + \sigma^{1-p}, \qquad r \in [0, 1].$$

Now taking $\sigma = \frac{\lambda}{2(N-1)C}$ in (4.11), we deduce from (4.8) that

$$(4.12) \qquad \frac{\lambda}{2} \int_0^1 \frac{|u'_\epsilon|^p}{(u_\epsilon + \epsilon^2)^m} \, dr \le (N-1) \left(\frac{\lambda}{2(N-1)C} \right)^{1-p} + \int_0^1 \bar{f}(r) \, dr \le C.$$

This and (4.11) imply that

$$(4.13) \qquad \int_0^1 \frac{|u'_\epsilon|^{p-1}}{r + \epsilon^{1/\alpha}} \, dr \le C.$$

Integrating (4.5) over (r_1, r_2) and integrating by parts, we have

$$(|u'_\epsilon|^{p-2} u'_\epsilon)|_{r_1}^{r_2} = \int_{r_1}^{r_2} \left(\frac{\lambda}{(u_\epsilon + \epsilon^2)^m} |u'_\epsilon|^p - \frac{N-1}{r + \epsilon^{1/\alpha}} |u'_\epsilon|^{p-2} u'_\epsilon - \bar{f}(r) \right).$$

Combining this with (4.12) and (4.13), we find that there exists a positive constant C, independent of ϵ, such that

$$(4.14) \qquad \left| |u'_\epsilon(r_2)|^{p-2} u'_\epsilon(r_2) - |u'_\epsilon(r_1)|^{p-2} u'_\epsilon(r_1) \right| \le C, \qquad \forall r_2, r_1 \in [0, 1].$$

Since $u_\epsilon(0) = u_\epsilon(1) = 0$ and $u_\epsilon \in C^1[0, 1]$, by the mean value theorem there exists $r_\epsilon \in (0, 1)$ such that $u'_\epsilon(r_\epsilon) = 0$. Then taking $r_1 = r_\epsilon$ in (4.14) gives

$$\left| |u'_\epsilon(r)|^{p-2} u'_\epsilon(r) \right| \le C, \qquad \forall r \in [0, 1].$$

This shows that $|u'_\epsilon(r)| \le C$ for all $r \in [0, 1]$. ∎

LEMMA 4.4. *For any $\delta \in (0, 1/2)$, there exists a positive constant C_δ, independent of ϵ, such that for all $\epsilon \in (0, \epsilon_0)$,*

$$(4.15) \qquad |u'_\epsilon(r_2) - u'_\epsilon(r_1)| \leq C_\delta |r_2 - r_1|^\beta, \quad \forall r_2, r_1 \in [\delta, 1-\delta],$$

where $\beta = 1/(p-1)$ if $p \geq 2$, $\beta = 1$ if $1 < p < 2$.

Proof. By (4.4) and (4.6), one derives from (4.5) that for any $\delta \in (0, 1/2)$, there exists a positive constant C_δ, independent of ϵ, such that for all $\epsilon \in (0, \epsilon_0)$,

$$(4.16) \qquad |(|u'_\epsilon|^{p-2} u'_\epsilon)'| \leq C_\delta, \quad \delta \leq t \leq 1-\delta.$$

Recalling the inequality (2.1) and using (4.16), one derives that if $p \geq 2$, then

$$\begin{aligned} |u'_\epsilon(r_2) - &u'_\epsilon(r_1)|^p \\ &\leq C^{-1}[u'_\epsilon(r_2) - u'_\epsilon(r_1)] \cdot [|u'_\epsilon(r_2)|^{p-2} u'_\epsilon(r_2) - |u'_\epsilon(r_1)|^{p-2} u'_\epsilon(r_1)] \\ &\leq C_\delta |u'_\epsilon(r_2) - u'_\epsilon(r_1)| \, |r_2 - r_1|, \quad \forall r_2, r_1 \in [\delta, 1-\delta], \end{aligned}$$

so

$$|u'_\epsilon(r_2) - u'_\epsilon(r_1)| \leq C_\delta |r_2 - r_1|^{1/(p-1)}, \quad \forall r_2, r_1 \in [\delta, 1-\delta],$$

and if $p \in (1, 2)$, then

$$\begin{aligned} |u'_\epsilon(r_2) - &u'_\epsilon(r_1)|^2 [|u'_\epsilon(r_2)| + |u'_\epsilon(r_1)|]^{p-2} \\ &\leq C^{-1}[u'_\epsilon(r_2) - u'_\epsilon(r_1)] \cdot [|u'_\epsilon(r_2)|^{p-2} u'_\epsilon(r_2) - |u'_\epsilon(r_1)|^{p-2} u'_\epsilon(r_1)] \\ &\leq C_\delta |u'_\epsilon(r_2) - u'_\epsilon(r_1)| \, |r_2 - r_1|, \quad \forall r_2, r_1 \in [\delta, 1-\delta], \end{aligned}$$

therefore, using (4.6) yields

$$|u'_\epsilon(r_2) - u'_\epsilon(r_1)| \leq C_\delta |r_2 - r_1| [|u'_\epsilon(r_2)| + |u'_\epsilon(r_1)|]^{2-p} \leq C_\delta |r_2 - r_1|$$

for all $r_2, r_1 \in [\delta, 1-\delta]$. $\blacksquare$

Using (4.6) and (4.15) and the Arzelà–Ascoli theorem, there exists a subsequence of $\{u_\epsilon\}$, still denoted by $\{u_\epsilon\}$, and a function $u \in C^1(0,1) \cap C[0,1]$ such that, as $\epsilon \to 0^+$,

$$(4.17) \qquad \begin{aligned} u_\epsilon &\to u \quad \text{uniformly in } C[0,1], \\ u_\epsilon &\to u \quad \text{uniformly in } C^1[\delta, 1-\delta], \end{aligned}$$

where $\delta \in (0, 1/2)$, and hence it follows from $u_\epsilon(1) = u_\epsilon(0) = 0$ and (4.4) that $u(1) = u(0) = 0$, and

$$(4.18) \qquad Cr^{p/(p-m)} \geq u(r) \geq C[\Psi(r)]^{p/(p-m)}, \quad r \in [0,1],$$

therefore $u > 0$ in $(0,1)$ and $u'(0) = \lim_{r \to 0} u(r)/r = 0$. Thus, (1.3) is satisfied.

We now show that u satisfies (1.2). Integrating (4.5) over (r_0, r) ($0 < r_0, r < 1$) and integrating by parts gives

$$|u_\epsilon'(r)|^{p-2}u_\epsilon'(r) = \int_{r_0}^{r} \left(\lambda \frac{|u_\epsilon'|^p}{(u_\epsilon + \epsilon^2)^m} - \frac{N-1}{r + \epsilon^{1/\alpha}} |u_\epsilon'|^{p-2}u_\epsilon' - \bar{f}(r) \right) dr$$
$$+ |u_\epsilon'(r_0)|^{p-2}u_\epsilon'(r_0).$$

Letting $\epsilon \to 0^+$ and using Lebesgue's dominated convergence theorem yield

$$(4.19) \qquad |u'(r)|^{p-2}u'(r) = \int_{r_0}^{r} \left(\lambda \frac{|u'|^p}{u^m} - \frac{N-1}{r} |u'|^{p-2}u' - \bar{f}(r) \right) dr$$
$$+ |u'(r_0)|^{p-2}u'(r_0).$$

This shows that $|u'(r)|^{p-2}u'(r) \in C^1(0,1)$, hence (1.2) is satisfied.

Next we prove that $u \in C^1[0,1]$. Letting $\epsilon \to 0^+$ in (4.12) and (4.13) and using Fatou's lemma yield

$$(4.20) \qquad \int_0^1 \frac{|u'|^p}{u^m} dr \le C, \qquad \int_0^1 \frac{|u'|^{p-1}}{r} dr \le C.$$

So, $|u'|^p/u^m, |u'|^{p-2}u'/r \in L^1[0,1]$. By (4.19), the function $\omega(r) = |u'(r)|^{p-2}u'(r) = \phi_p(u'(r))$ is absolutely continuous on $[0,1]$. Since $u'(r) = \phi_q(\omega(r))$, where $1/p + 1/q = 1$, we see that $u' \in C[0,1]$. Thus u is a positive solution of problem (1.2) and (1.3).

Proof of Theorem 2.7. Let $w(x) = u(r)$ with $r = |x|$. Clearly, $w(0) = 0$. Some calculations give

$$\nabla w = u'(r)\frac{x}{|x|}, \quad |\nabla w| = |u'(r)|, \quad \text{in } B_1 \setminus \{0\},$$

$$\mathbf{div}(|\nabla w|^{p-2}\nabla w) = (|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' \quad \text{in } B_1 \setminus \{0\}.$$

Thus w satisfies

$$-\mathbf{div}(|\nabla w|^{p-2}\nabla w) + \frac{\lambda}{w^m}|\nabla w|^p - \bar{f}(|x|) = 0 \quad \text{in } B_1 \setminus \{0\}.$$

Since $u \in C^1[0,1]$ and $u'(0) = 0$, $|\nabla w(0)| = 0$ and $w \in C^1(\overline{B}_1)$.

By the first estimate in (4.20), one obtains $|\nabla w|^p/w^m \in L^1(B_1)$.

In virtue of the above facts, it is not difficult to check that w is a solution of problem (1).

It remains to show (2.2). By a simple calculation, we obtain

$$\Psi'(r) = \left(\frac{1}{2} - r \right)^{1/(p-1)} \ge \left(\frac{1}{4} \right)^{1/(p-1)}, \quad \forall 0 \le r \le \frac{1}{4}.$$

Then

$$\Psi(r) \geq \left(\frac{1}{4}\right)^{1/(p-1)} r, \quad \forall 0 \leq r \leq \frac{1}{4}.$$

This together with (4.18) implies (2.2).

The proof of Theorem 2.7 is complete.

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