On para-Nordenian structures

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Abstract. The aim of this paper is to investigate para-Nordenian properties of the Sasakian metrics in the cotangent bundle.

1. Introduction. Let \((M^n, g)\) be an \(n\)-dimensional Riemannian manifold, \(T^*M^n\) its cotangent bundle and \(\pi\) the natural projection \(T^*M^n \to M^n\). A system of local coordinates \((U, x^i)\), \(i = 1, \ldots, n\) on \(M^n\) induces on \(T^*M^n\) a system of local coordinates \((\pi^{-1}(U), x^i, x^\bar{i} = p_i)\), \(\bar{i} := n + i\ (i = 1, \ldots, 2n)\), where \(x^i = p_i\) are the components of the covector \(p\) in each cotangent space \(T^*_xM^n\), \(x \in U\), with respect to the natural coframe \(\{dx^i\}\), \(i = 1, \ldots, n\).

We denote by \(\mathcal{I}_r^s(M^n)\) (resp. \(\mathcal{I}_r^s(T^*M^n)\)) the module over \(F(M^n)\) (resp. \(F(T^*M^n)\)) of \(C^\infty\) tensor fields of type \((r, s)\), where \(F(M^n)\) (resp. \(F(T^*M^n)\)) is the ring of real-valued \(C^\infty\) functions on \(M^n\) (resp. \(T^*M^n\)).

Let \(X = X^i \frac{\partial}{\partial x^i}\) and \(\omega = \omega_i dx^i\) be the local expressions in \(U \subset M^n\) of a vector and a covector (1-form) field \(X \in \mathcal{I}_1^0(M^n)\) and \(\omega \in \mathcal{I}_0^1(M^n)\), respectively. Then the complete and horizontal lifts \(C_X, H_X \in \mathcal{I}_1^0(T^*M^n)\) of \(X \in \mathcal{I}_1^0(M^n)\) and the vertical lift \(V \omega \in \mathcal{I}_1^0(T^*M^n)\) of \(\omega \in \mathcal{I}_0^1(M^n)\) are given, respectively, by

\[
C_X = X^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^i},
\]

\[
H_X = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^i},
\]

\[
V \omega = \sum_i \omega_i \frac{\partial}{\partial x^i}
\]

with respect to the natural frame \(\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\bar{i}}\}\), where \(\Gamma^h_{ij}\) are the components of the Levi-Civita connection \(\nabla_g\) on \(M^n\) (see [11] for more details).

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For each \( x \in M^n \) the scalar product \( g^{-1} = (g^{ij}) \) is defined on the cotangent space \( \pi^{-1}(x) = T^*_x(M^n) \) by
\[
g^{-1}(\omega, \theta) = g^{ij} \omega_i \theta_j
\]
for all \( \omega, \theta \in \mathfrak{S}_1^0(M^n) \).

A Sasakian metric \( Sg \) is defined on \( T^*M^n \) by the following three equations
\[
\begin{align*}
(1.4) & \quad Sg(V\omega, V\theta) = V(g^{-1}(\omega, \theta)) = g^{-1}(\omega, \theta) \circ \pi, \\
(1.5) & \quad Sg(V\omega, HY) = 0, \\
(1.6) & \quad Sg(HX, HY) = V(g(X,Y)) = g(X,Y) \circ \pi
\end{align*}
\]
for any \( X, Y \in \mathfrak{S}_1^1(M^n) \) and \( \omega, \theta \in \mathfrak{S}_0^1(M^n) \). Since any tensor field of type \((0,2)\) on \( T^*M^n \) is completely determined by its action on vector fields of type \( HX \) and \( V\omega \) (see [11, p. 280]), it follows that \( Sg \) is completely determined by (1.4)–(1.6).

From (1.1) and (1.2) we notice that the complete lift \( C^X \) of \( X \in \mathfrak{S}_1^1(M^n) \) is expressed by
\[
\begin{equation}
(1.7)
C^X = HX - V(p(\nabla X)),
\end{equation}
\]
where \( p(\nabla X) = p_i(\nabla_i X^i)dx^h \).

Using (1.4)–(1.7), we have
\[
\begin{equation}
(1.8)
Sg(C^X, C^Y) = V(g(X,Y)) + V(g^{-1}(p(\nabla X), p(\nabla Y))),
\end{equation}
\]
where \( g^{-1}(p(\nabla X), p(\nabla Y)) = g^{ij}(p_i \nabla_i X^l)(p_k \nabla_j Y^k) \).

Since the tensor field \( Sg \in \mathfrak{S}_2^0(T^*M^n) \) is also completely determined by its action on vector fields of type \( C^X \) and \( C^Y \) (see [11, p. 237]), we have an alternative characterization of \( Sg \): a Sasakian metric \( Sg \) on \( T^*M^n \) is completely determined by the condition (1.8).

Sasakian metrics on the tangent bundle were introduced in [9] by the Japanese geometer S. Sasaki. Sasakian metrics (diagonal lifts of metrics) on tangent bundles were also studied in [3], [11]. In the more general case of tensor bundles of type \((1,q)\), \((0,q)\) and \((p,q)\), Sasakian metrics and their geodesics were considered in [1], [6], [7]. Sasakian metrics on the frame bundle were first considered by K. P. Mok [5] (see [2] for more details). This paper is concerned with para-Nordenian properties of the Sasakian metric on the cotangent bundle.

2. Levi-Civita connection of \( Sg \). On \( U \subset M^n \), we put
\[
X(i) = \frac{\partial}{\partial x^i}, \quad \theta(i) = dx^i, \quad i = 1, \ldots, n.
\]
Then from (1.2) and (1.3) we see that \( HX_{(i)} \) and \( V\theta^{(i)} \) have local expressions

\[
(i) \quad \tilde{e}_{(i)} = HX_{(i)} = \frac{\partial}{\partial x^i} + \sum_h p_a \Gamma^a_{ih} \frac{\partial}{\partial x^h},
\]

(2.1)

\[
(ii) \quad \tilde{e}_{(i)} = V\theta^{(i)} = \frac{\partial}{\partial x^i}.
\]

(2.2)

We call the set \( \{\tilde{e}_{(\alpha)}\} = \{\tilde{e}_{(i)}, \tilde{e}_{(i)}\} = \{HX_{(i)}, V\theta^{(i)}\} \) the frame adapted to the Levi-Civita connection \( \nabla_g \). The indices \( \alpha, \beta, \ldots = 1, \ldots, 2n \) indicate the indices with respect to the adapted frame.

From equations (1.2), (1.3), (2.1) and (2.2), we see that \( HX \) and \( V\omega \) have the components

\[
(iii) \quad HX = X^i \tilde{e}_{(i)}, \quad HX = (HX^\alpha) = \begin{pmatrix} X^i \\ 0 \end{pmatrix},
\]

(2.3)

\[
(iv) \quad V\omega = \sum_i \omega_i \tilde{e}_{(i)}, \quad V\omega = (V\omega^\alpha) = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix}
\]

(2.4)

with respect to the adapted frame \( \{\tilde{e}_{(\alpha)}\} \), where \( X^i \) and \( \omega_i \) are the local components of \( X \in \mathfrak{S}_0^1(M^n) \) and \( \omega \in \mathfrak{S}_0^1(M^n) \), respectively.

Let \( S\nabla \) be the Levi-Civita connection determined by the Sasakian metric \( Sg \). The components of \( S\nabla \) are given by

\[
S\Gamma^h_{ji} = \Gamma^h_{ji}, \quad S\Gamma^h_{ji} = S\Gamma^h_{j i} = S\Gamma^h_{j i} = 0,
\]

(2.5)

\[
S\Gamma^h_{j i} = \frac{1}{2} p_m R^h_{j m}, \quad S\Gamma^h_{j i} = \frac{1}{2} p_m R^h_{j m},
\]

\[
S\Gamma^h_{j i} = \frac{1}{2} p_m R_{jih m}, \quad S\Gamma^h_{j i} = S\Gamma^h_{j i} = -\Gamma^i_{jh}
\]

with respect to the adapted frame \( \{\tilde{e}_{(\alpha)}\} \), where \( R_{ijk}^h \) are the local components of the curvature tensor \( R \) of \( \nabla_g \).

Let now \( \tilde{X}, \tilde{Y} \in \mathfrak{S}_0^1(T^*M^n) \) and \( \tilde{X} = \tilde{X}^\alpha \tilde{e}_{(\alpha)} \), \( \tilde{Y} = \tilde{Y}^\beta \tilde{e}_{(\beta)} \). The covariant derivative \( S\nabla_{\tilde{Y}} \tilde{X} \) along \( \tilde{Y} \) has components

\[
S\nabla_{\tilde{Y}} \tilde{X}^\alpha = \tilde{Y}^\beta \tilde{e}_{(\beta)} \tilde{X}^\alpha + S\Gamma^\alpha_{\gamma \beta} \tilde{X}^\beta \tilde{Y}^\gamma
\]

(2.6)

with respect to the adapted frame \( \{\tilde{e}_{(\alpha)}\} \).

Using (2.3)–(2.6), we have

**Theorem 2.1.** Let \( M^n \) be a Riemannian manifold with metric \( g \) and \( S\nabla \) be the Levi-Civita connection of the cotangent bundle \( T^*M^n \) equipped with the Sasakian metric \( Sg \). Then \( S\nabla \) satisfies

(i) \( S\nabla_{V\omega} V\theta = 0 \),

(ii) \( S\nabla_{V\omega} HX = \frac{1}{2} H(p(g^{-1} \circ R(\ , X)\tilde{\omega})) \),

(iii) \( S\nabla_{HX} V\theta = V(\nabla_X\theta) + \frac{1}{2} H(p(g^{-1} \circ R(\ , X)\tilde{\theta})) \),

(iv) \( S\nabla_{HX} HY = H(\nabla_X Y) + \frac{1}{2} V(pR(X, Y)) \)
for all $X, Y \in \mathcal{I}_1^1(M^n)$ and $\omega, \theta \in \mathcal{I}_0^0(M^n)$, where $\tilde{\omega} = g^{-1} \circ \omega \in \mathcal{I}_1^1(M^n)$, $R( , X)\tilde{\omega} \in \mathcal{I}_1^1(M^n)$, $g^{-1} \circ R( , X)\tilde{\omega} \in \mathcal{I}_0^2(M^n)$.

3. Para-Nordenian structures on $(T^*M^n, S^g)$. An almost paracomplex manifold is an almost product manifold $(M^n, \varphi)$, $\varphi^2 = I$, such that the two eigenbundles $T^+M^n$ and $T^-M^n$ associated to the two eigenvalues $+1$ and $-1$ of $\varphi$, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. Considering the paracomplex structure $\varphi$, we obtain the set $\{I, \varphi\}$ on $M^n$, which is an isomorphic representation of the algebra of order 2, called the algebra of paracomplex (or double) numbers and denoted by $R(j)$, $j^2 = 1$.

A tensor field $\omega \in \mathcal{I}_q^0(M^{2n})$ is said to be pure with respect to the paracomplex structure $\varphi$ if

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \cdots = \omega(X_1, X_2, \ldots, \varphi X_q)$$

for any $X_1, X_2, \ldots, X_q \in \mathcal{I}_q^1(M^n)$.

We define the operator $\phi_\varphi$ associated with $\varphi$ and applied to the pure tensor field $\omega$ by (see [10])

$$(\phi_\varphi \omega)(Y, X_1, \ldots, X_q) = (\varphi Y)(\omega(X_1, \ldots, X_q)) - Y(\omega(\varphi X_1, X_2, \ldots, X_q)) + \omega((L_X\varphi)Y, X_2, \ldots, X_q) + \cdots + \omega(X_1, X_2, \ldots, (L_X\varphi)Y),$$

where $L_X$ denotes the Lie derivative with respect to $X$. We note that $\phi_\varphi \omega \in \mathcal{I}_{q+1}^0(M^{2n})$.

If $\phi_\varphi \omega = 0$, then $\omega$ is said to be almost paraholomorphic with respect to the paracomplex algebra $R(j)$ (see [4], [8]).

A Riemannian manifold $(M^{2n}, g)$ with an almost paracomplex structure $\varphi$ is said to be almost para-Nordenian if the Riemannian metric $g$ is pure with respect to $\varphi$. It is well known that the almost para-Nordenian manifold is para-Kähler ($\nabla_g \varphi = 0$) if and only if $g$ is paraholomorphic ($\phi_\varphi g = 0$) (see [8]).

Let $(T^*M^n, S^g)$ be the cotangent bundle with the Sasakian metric $S^g$. We define a tensor field $F$ of type $(1, 1)$ on $T^*M^n$ by

$$F^{H}X = V\tilde{X},$$

$$F^{V}\omega = H\tilde{\omega}$$

for any $X \in \mathcal{I}_0^1(M^n)$ and $\omega \in \mathcal{I}_1^0(M^n)$, where $\tilde{X} = g \circ X \in \mathcal{I}_1^1(M^n)$, $\tilde{\omega} = g^{-1} \circ \omega \in \mathcal{I}_0^1(M^n)$. Then we obtain

$$F^2 = I.$$
Indeed, by virtue of (3.1) we have

$$F^2(HX) = F(FHX) = F(V\tilde{X}) = H\tilde{X} = HX,$$

$$F^2(V\omega) = F(FV\omega) = F(H\tilde{\omega}) = V\tilde{\omega} = V\omega$$

for any $X \in \mathcal{O}_0(M^n)$ and $\omega \in \mathcal{O}_0(M^n)$, which implies $F^2 = I$.

**Theorem 3.1.** The triple $(T^*M^n, Sg, F)$ is an almost para-Nordenian manifold.

**Proof.** We put

$$A(\tilde{X}, \tilde{Y}) = Sg(F\tilde{X}, \tilde{Y}) - Sg(\tilde{X}, F\tilde{Y})$$

for any $\tilde{X}, \tilde{Y} \in \mathcal{O}_0(T^*M^n)$. From (1.4)–(1.6) and (3.1), we have

$$A(HX, HY) = Sg(FHX, HY) - Sg(HX, FHY) = Sg(V\tilde{X}, HY) - Sg(HX, V\tilde{Y}) = 0,$$

$$A(HX, V\omega) = Sg(FHX, V\omega) - Sg(HX, FV\omega) = Sg(V\tilde{X}, V\omega) - Sg(HX, V\tilde{\omega}) = g^{-1}(g \circ X, \omega) - g(X, g^{-1} \circ \omega) = 0,$$

$$A(V\omega, HY) = -A(HY, V\omega) = 0,$$

$$A(V\omega, V\theta) = Sg(FV\omega, V\theta) - Sg(V\omega, FV\theta) = g^{-1}(H\tilde{\omega}, V\theta) - Sg(V\omega, H\tilde{\theta}) = 0,$$

i.e. $Sg$ is pure with respect to $F$. Thus Theorem 3.1 is proved. $\blacksquare$

We now consider the covariant derivative of $F$. Taking into account (i)–(iv) of Theorem 2.1 and (3.1), we obtain

$$\begin{align*}
(\mathcal{S}V_{HX}F)(HY) &= \mathcal{S}V_{HX}(FHY) - F(\mathcal{S}V_{HX}HY) \\
&= \mathcal{S}V_{HX}V\tilde{Y} - F(\mathcal{S}V_{HX}HY) \\
&= V(\nabla_X\tilde{Y}) + \frac{1}{2}H(p(g^{-1} \circ R(,X)Y)) - F(H(\nabla_XY) + \frac{1}{2}V(pR(X,Y))) \\
&= \frac{1}{2}H(pg^{-1} \circ (R(,X)Y - R(X,Y))),
\end{align*}$$

$$\begin{align*}
(\mathcal{S}V_{V\omega}F)(HY) &= \mathcal{S}V_{V\omega}(FHY) - F(\mathcal{S}V_{V\omega}HY) \\
&= \mathcal{S}V_{V\omega}V\tilde{Y} - \frac{1}{2}F(p(g^{-1} \circ R(,Y)\tilde{\omega})) \\
&= -\frac{1}{2}V(pR(,Y)\tilde{\omega}),
\end{align*}$$

$$\begin{align*}
(\mathcal{S}V_{HX}F)(V\theta) &= \mathcal{S}V_{HX}(FV\theta) - F(\mathcal{S}V_{HX}V\theta) \\
&= \mathcal{S}V_{HX}H\tilde{\theta} - F(V(\nabla_X\theta) + \frac{1}{2}H(p(g^{-1} \circ R(,X)\tilde{\theta}))) \\
&= H(\nabla_X\tilde{\theta}) + \frac{1}{2}V(pR(X,\tilde{\theta})) - H(g^{-1} \circ (\nabla_X\theta)) - \frac{1}{2}V(pg \circ (g^{-1} \circ R(,X)\tilde{\theta})) \\
&= \frac{1}{2}V(pR(X,\tilde{\theta}) - pR(,X)\tilde{\theta}),
\end{align*}$$

$$\begin{align*}
(\mathcal{S}V_{V\omega}F)(V\theta) &= \mathcal{S}V_{V\omega}(FV\theta) - F(\mathcal{S}V_{V\omega}V\theta) \\
&= \mathcal{S}V_{V\omega}H\tilde{\theta} - F(V(\nabla_X\theta) + \frac{1}{2}H(p(g^{-1} \circ R(,X)\tilde{\theta}))) \\
&= H(\nabla_X\tilde{\theta}) + \frac{1}{2}V(pR(X,\tilde{\theta})) - H(g^{-1} \circ (\nabla_X\theta)) - \frac{1}{2}V(pg \circ (g^{-1} \circ R(,X)\tilde{\theta})) \\
&= \frac{1}{2}V(pR(X,\tilde{\theta}) - pR(,X)\tilde{\theta}),
\end{align*}$$
Hence, we have

\[(\mathcal{L}_F V \theta) = \mathcal{L}_F V \theta - F(\mathcal{L}_F V \theta) = \mathcal{L}_F H \tilde{\theta} - F(\mathcal{L}_F \theta)\]

From (3.2)–(3.5) we have

**Theorem 3.2.** The cotangent bundle of a Riemannian manifold is para-Kählerian (paraholomorphic Nordenian) with respect to the metric $\mathcal{S}g$ and almost paracomplex structure $F$ defined by (3.1) if and only if the Riemannian manifold is flat.

### 4. A necessary and sufficient condition for the complete lift of a vector field to be paraholomorphic
A vector field $\tilde{X} \in \mathfrak{X}(T^*M^n)$ with respect to which the almost para-Nordenian structure $F$ has a vanishing Lie derivative $(\mathcal{L}_F \theta = 0)$ is said to be almost paraholomorphic (see [4]).

It is well known that [11, p. 277]

\[
\begin{align*}
[C_X, H Y] &= H [X, Y] + V(p(L_X \nabla) Y), \\
[C_X, V \omega] &= V(L_X \omega),
\end{align*}
\]

where $(L_X \nabla)Y = \nabla_Y L_X + R(X, Y)$ and $(L_X \nabla)(Y, Z) = L_X(\nabla_Y X) - \nabla_Y (L_X Z) - \nabla_{[X,Y]} Z$.

A vector field $X \in \mathfrak{X}(M^n)$ is called a Killing vector field (or infinitesimal isometry) if $L_X g = 0$, and $X$ is called an infinitesimal affine transformation if $L_X \nabla g = 0$. A Killing vector field is necessarily an infinitesimal affine transformation, i.e. we have $L_X \nabla g = 0$ as a consequence of $L_X g = 0$.

We now consider the Lie derivative of $F$ with respect to the complete lift $C_X$. Taking account of (3.1) and (4.1), we obtain

\[
\begin{align*}
(\mathcal{L}_{C_X} F)^{V \theta} &= \mathcal{L}_{C_X} F^{V \theta} - F(\mathcal{L}_{C_X} V \theta) = \mathcal{L}_{C_X} H \tilde{\theta} - F(\mathcal{L}_X \theta) \\
&= \mathcal{L}_{C_X} H \tilde{\theta} - H(g^{-1} \circ (L_X \theta)) \\
&= V[X, \tilde{\theta}] + V(p(L_X \nabla) \tilde{\theta}) - H(g^{-1} \circ (L_X \theta)) \\
&= H(L_X (g^{-1} \circ \theta) - g^{-1} \circ (L_X \theta)) + V(p(L_X \nabla) \tilde{\theta}),
\end{align*}
\]

\[
\begin{align*}
(\mathcal{L}_{C_X} F)^{H Y} &= \mathcal{L}_{C_X} F^{H Y} - F(\mathcal{L}_{C_X} H Y) \\
&= \mathcal{L}_{C_X} V \tilde{Y} - F(\mathcal{L}_X V \tilde{Y}) + V(p(L_X \nabla) \tilde{Y})) \\
&= V(L_X (g \circ Y) - g \circ L_X Y) - H(g^{-1} \circ p(L_X \nabla) \tilde{Y}).
\end{align*}
\]

Let now $X$ be a Killing vector field $(L_X g = 0)$. Then by virtue of $L_X \nabla = 0$, from (4.2) and (4.3) we have $\mathcal{L}_{C_X} F = 0$, i.e. $C_X$ is paraholomorphic with respect to $F$. If we assume that $\mathcal{L}_{C_X} F = 0$ and compute the equation (4.3) at $(x^i, 0), p_i = 0$, then we get $L_X (g \circ Y) = g \circ L_X Y$. It follows that $L_X g = 0$. Hence, we have
Theorem 4.1. An infinitesimal transformation $X$ of the Riemannian manifold $(M^n, g)$ is a Killing vector field if and only if its complete lift $C^X$ to the cotangent bundle $T^*M^n$ is an almost paraholomorphic vector field with respect to the almost para-Nordenian structure $(F, S^g)$.

Remark. Let $^R\nabla \in \mathfrak{S}_2^0(T^*M^n)$ be a Riemannian extension of the connection $\nabla_g$ defined by (cf. [11, p. 268])

\[ ^R\nabla(C^X, C^Y) = -p(\nabla_X Y + \nabla_Y X), \quad X, Y \in \mathfrak{g}_1(M_n). \]

The metric $^R\nabla$ has components

\begin{equation}
^R\nabla = \begin{pmatrix} -2p_\alpha \Gamma^\alpha_{ji} & \delta^j_i \\ \delta^i_j & 0 \end{pmatrix}
\end{equation}

with respect to the natural frame $\{\partial_i, \bar{\partial}_i\}$. From (1.2), (1.3) and (4.4) we easily see that

\begin{equation}
^R\nabla(H^X, H^Y) = 0, \quad ^R\nabla(V^\omega, V^\theta) = 0, \quad ^R\nabla(H^X, V^\theta) = V(\theta(X)),
\end{equation}

i.e. the metric $^R\nabla$ is completely determined also by conditions (4.5). Using (1.6), (3.1) and (4.5), we have

\begin{align*}
(^R\nabla \circ F)(H^X, H^Y) &= ^R\nabla(F H^X, H^Y) = ^R\nabla(V \tilde{X}, H^Y) = V(\tilde{X}(Y)) \\
&= V(g(X, Y)) = S^g(H^X, H^Y), \\
(^R\nabla \circ F)(H^X, V^\theta) &= ^R\nabla(F H^X, V^\theta) = ^R\nabla(V \tilde{X}, V^\theta) = S^g(H^X, V^\theta) = 0, \\
(^R\nabla \circ F)(V^\omega, H^Y) &= ^R\nabla(F V^\omega, H^Y) = ^R\nabla(H \bar{\omega}, H^Y) = S^g(V^\omega, H^Y) = 0, \\
(^R\nabla \circ F)(V^\omega, V^\theta) &= ^R\nabla(F V^\omega, V^\theta) = ^R\nabla(H \bar{\omega}, V^\theta) = V(\theta(\bar{\omega})) \\
&= V(g^{-1}(\omega, \theta)) = S^g(V^\omega, V^\theta),
\end{align*}

i.e. $^R\nabla \circ F = S^g$. Thus the almost para-Nordenian structure $F$ determined by the condition (3.1) has an expression of the form $F = (^R\nabla)^{-1} \circ S^g$.

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