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On infinitesimal automorphisms of foliated manifolds

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Abstract. Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a product preserving bundle functor on the category $\mathcal{F}ol$ of foliated manifolds $(M, \mathcal{F})$ without singularities and leaf respecting maps. We describe all natural operators C transforming infinitesimal automorphisms $X \in \mathcal{X}(M, \mathcal{F})$ of foliated manifolds $(M, \mathcal{F})$ into vector fields $C(X) \in \mathcal{X}(F(M, \mathcal{F}))$ on $F(M, \mathcal{F})$.

Introduction. The class of product preserving bundle functors on the category $\mathcal{F}ol$ of foliated manifolds without singularities and their leaf respecting maps is a wide class of bundle functors. For example, the normal bundle functor $N : \mathcal{F}ol \to \mathcal{FM}$ sending foliated manifolds $(M, \mathcal{F})$ to their normal bundles $N(M, \mathcal{F})$ and leaf respecting maps $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ to the induced maps $N(f) : N(M_1, \mathcal{F}_1) \to N(M_2, \mathcal{F}_2)$ is product preserving. More generally, for any Weil algebra A the bundle functor $\widetilde{A} : \mathcal{F}ol \to \mathcal{FM}$ of transverse A-points sending foliated manifolds $(M, \mathcal{F})$ to their bundles $\widetilde{A}(M, \mathcal{F})$ of transverse A-points in the sense of [5] and leaf respecting maps $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ to the induced maps $\widetilde{A}(f) : \widetilde{A}(M_1, \mathcal{F}_1) \to \widetilde{A}(M_2, \mathcal{F}_2)$ is product preserving. Also, the usual Weil bundle functor $T^A : \mathcal{M}f \to \mathcal{FM}$ on manifolds can be considered as the product preserving bundle functor $T^A : \mathcal{F}ol \to \mathcal{FM}$ satisfying $T^A(M, \mathcal{F}) = T^A M$ for any foliated manifold $(M, \mathcal{F})$. In particular, the tangent bundle functor on manifolds can be considered as the product preserving bundle functor $T : \mathcal{F}ol \to \mathcal{FM}$ satisfying $T(M, \mathcal{F}) = TM$ for any foliated manifold $(M, \mathcal{F})$. In [3], the second author described all product preserving bundle functors on the category $\mathcal{F}ol$ in terms of Weil algebra homomorphisms. He deduced

Theorem A ([3]). *There is a bijection between the isomorphism classes of product preserving bundle functors on $\mathcal{F}ol$ and the isomorphism classes of Weil algebra homomorphisms.*

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Theorem B ([3]). *Given two product preserving bundle functors on $\mathcal{F}ol$ there is a bijection between natural transformations of them and morphisms of the corresponding Weil algebra homomorphisms.*

Let $(M, \mathcal{F})$ be a foliated manifold (a $\mathcal{F}ol$-object). A vector field X on M is called an *infinitesimal automorphism* of $(M, \mathcal{F})$ if its flow $\mathrm{Exp}(tX)$ is formed by local $\mathcal{F}ol$-isomorphisms. We denote by $\mathcal{X}(M, \mathcal{F})$ the Lie algebra of infinitesimal automorphisms of $(M, \mathcal{F})$.

Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a product preserving bundle functor. Let $\mu : A \to B$ be the Weil algebra homomorphism corresponding to $F : \mathcal{F}ol \to \mathcal{FM}$ (Theorem A). In this paper we study the problem how an infinitesimal automorphism $X \in \mathcal{X}(M, \mathcal{F})$ can induce canonically a vector field $C(X) \in \mathcal{X}(F(M, \mathcal{F}))$ on $F(M, \mathcal{F})$. This problem is reflected in the concept of natural operators $C : \mathcal{X}(M, \mathcal{F}) \to \mathcal{X}(F(M, \mathcal{F}))$ in the sense of [2]. (X is an infinitesimal automorphism of $\mathcal{F}$ if its flow preserves $\mathcal{F}$, or equivalently, $[X, Y]$ is tangent to $\mathcal{F}$ for any vector field Y tangent to $\mathcal{F}$.) In the present note we explicitly describe all natural operators C in question. More precisely, we have the flow operator transforming any infinitesimal automorphism $X \in \mathcal{X}(M, \mathcal{F})$ into a vector field $\mathcal{F}X \in \mathcal{X}(F(M, \mathcal{F}))$ with the flow $\mathrm{Exp}(t\mathcal{F}X) := F(\mathrm{Exp}(tX))$. Given $a \in A$ one can define canonically an affinor $af(a) : TF(M, \mathcal{F}) \to TF(M, \mathcal{F})$ on $F(M, \mathcal{F})$ (see Section 5). Any derivation $D \in \mathcal{D}er(\mu) = \mathcal{L}ie(\mathrm{Aut}(\mu))$ of μ yields the corresponding one-parameter group $D_t : \mu \to \mu$ of automorphisms of μ. Then (see Theorem B) we have the corresponding flow $D_t : F(M, \mathcal{F}) \to F(M, \mathcal{F})$ which defines $\mathrm{op}(D) \in \mathcal{X}(F(M, \mathcal{F}))$. Our main result can be stated as follows.

Theorem C. *Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a product preserving bundle functor. Let $\mu : A \to B$ be the Weil algebra homomorphism corresponding to F. Let $p \geq 1$ and $q \geq 1$ be integers. Let $\mathcal{F}ol_{p,q}$ be the subcategory of foliated $(p+q)$-dimensional manifolds with p-dimensional leaves and their leaf respecting local diffeomorphisms. Any $\mathcal{F}ol_{p,q}$-natural operator C transforming infinitesimal automorphisms $X \in \mathcal{X}(M, \mathcal{F})$ of $\mathcal{F}ol_{p,q}$-objects into vector fields $C(X) \in \mathcal{X}(F(M, \mathcal{F}))$ on $F(M, \mathcal{F})$ is of the form*

$$C(X) = \mathrm{af}(a) \circ \mathcal{F}X + \mathrm{op}(D)$$

for some unique $a \in A$ and $D \in \mathcal{D}er(\mu)$.

We remark that Theorem C is a generalization of the well known result by I. Kolář ([1]) on natural operators lifting vector fields from manifolds to product preserving bundles over manifolds.

All manifolds are assumed to be finite-dimensional. All manifolds and maps are assumed to be smooth, i.e. of class $\mathcal{C}^\infty$. All foliations are assumed to be without singularities.

1. Product preserving bundle functors on foliated manifolds. For the reader's convenience we cite (without proofs) some facts from [3]. We start from the following definitions (see e.g. [2]).

Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a covariant functor. Let $B_{\mathcal{FM}} : \mathcal{FM} \to \mathcal{M}f$ be the base functor and $B_{\mathcal{F}ol} : \mathcal{F}ol \to \mathcal{M}f$ be the forgetful functor.

A *bundle functor* on $\mathcal{F}ol$ is a functor F as above satisfying:

(i) (Base preservation) $B_{\mathcal{FM}} \circ F = B_{\mathcal{F}ol}$. Hence the induced projections form a functor transformation $\pi : F \to B_{\mathcal{F}ol}$.

(ii) (Localization) For every inclusion $i_{(U,\mathcal{F}|U)} : (U, \mathcal{F}|U) \to (M, \mathcal{F})$ of an open subset, $F(U, \mathcal{F}|U)$ is the restriction $\pi^{-1}(U)$ of $\pi : F(M, \mathcal{F}) \to M$ over U and $Fi_{(U,\mathcal{F}|U)}$ is the inclusion $\pi^{-1}(U) \to F(M, \mathcal{F})$.

Given two bundle functors F_1, F_2 on $\mathcal{F}ol$, by a *natural transformation* $\nu : F_1 \to F_2$ we shall mean a system of base preserving fibered maps $\nu : F_1(M, \mathcal{F}) \to F_2(M, \mathcal{F})$ for every foliated manifold $(M, \mathcal{F})$ satisfying $F_2 f \circ \nu = \nu \circ F_1 f$ for every $\mathcal{F}ol$-morphism f.

A bundle functor F on $\mathcal{F}ol$ is *product preserving* if for any product projections $(M_1, \mathcal{F}_1) \xleftarrow{\mathrm{pr}_1} (M_1, \mathcal{F}_1) \times (M_2, \mathcal{F}_2) \xrightarrow{\mathrm{pr}_2} (M_2, \mathcal{F}_2)$ (in the category $\mathcal{F}ol$), $F(M_1, \mathcal{F}_1) \xleftarrow{F\mathrm{pr}_1} F((M_1, \mathcal{F}_1) \times (M_2, \mathcal{F}_2)) \xrightarrow{F\mathrm{pr}_2} F(M_2, \mathcal{F}_2)$ are product projections in the category $\mathcal{FM}$. In other words, $F((M_1, \mathcal{F}_1) \times (M_2, \mathcal{F}_2)) = F(M_1, \mathcal{F}_1) \times F(M_2, \mathcal{F}_2)$ modulo $(F\,\mathrm{pr}_1, F\,\mathrm{pr}_2)$.

Some examples of product preserving bundle functors on $\mathcal{F}ol$ have been mentioned above. Now, we present the most general example of such a functor.

Let A be an associative algebra over the field $\mathbb{R}$ with unit 1. The algebra A is called a *Weil algebra* if it is commutative, of finite dimension over $\mathbb{R}$, and if it admits a unique maximal ideal $\underline{A}$ of codimension 1 such that $\underline{A}^{h+1} = 0$ for some non-negative integer h.

Let $\mu : A \to B$ be a homomorphism of Weil algebras. We are going to construct a product preserving bundle functor $T^\mu : \mathcal{F}ol \to \mathcal{FM}$.

EXAMPLE 1 ([3]). For a foliated manifold $(M, \mathcal{F})$ and $x \in M$ we denote the algebra of germs at x of smooth maps $M \to \mathbb{R}$ by $C^\infty_x(M)$, and the algebra of germs at x of smooth maps $M \to \mathbb{R}$ constant on leaves by $C^\infty_x(M, \mathcal{F})$. Let $T^\mu_x(M, \mathcal{F})$ be the set of pairs (φ, ψ) of algebra homomorphisms $\varphi : C^\infty_x(M, \mathcal{F}) \to A$ and $\psi : C^\infty_x(M) \to B$ such that

$$(*) \qquad \psi(uv) = \mu(\varphi(u))\psi(v)$$

for any $u \in C^\infty_x(M, \mathcal{F})$ and any $v \in C^\infty_x(M)$. Let us define $T^\mu(M, \mathcal{F}) = \bigcup_{x\in M} T^\mu_x(M, \mathcal{F})$. Then the obvious projection $\pi : T^\mu(M, \mathcal{F}) \to M$ is a smooth bundle. More precisely, for an adapted chart $(x^1, \ldots, x^q, y^1, \ldots, y^p)$ on $(M, \mathcal{F})$, where $\mathcal{F}$ is p-dimensional and M is $p+q$-dimensional, we have a set of induced coordinates $(\widetilde{x}^1, \ldots, \widetilde{x}^q, \widetilde{y}^1, \ldots, \widetilde{y}^p) : T^\mu(M, \mathcal{F})|U \to A^q \times B^p$

such that $\widetilde{x}^i(\varphi) = \varphi([x^i]_x) \in A$ and $\widetilde{y}^j(\psi) = \psi([y^j]_x) \in B$ for any $(\varphi, \psi) \in T^\mu_x(M, \mathcal{F})$, $x \in M$. (Condition $(*)$ implies that $(\varphi, \psi) \in T^\mu_x(M, \mathcal{F})$ is uniquely determined by its induced coordinates). Every $\mathcal{F}ol$-map $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ induces a fibered map $T^\mu f : T^\mu(M_1, \mathcal{F}_1) \to T^\mu(M_2, \mathcal{F}_2)$ covering f such that $T^\mu f(\varphi, \psi) = (\overline{\varphi}, \overline{\psi})$ for any $(\varphi, \psi) \in T^\mu_x(M_1, \mathcal{F}_1)$, $x \in M_1$, where $\overline{\varphi} : C^\infty_{f(x)}(M_2, \mathcal{F}_2) \to A$ is defined by $\overline{\varphi}(u) = \varphi(u \circ f)$ and $\overline{\psi} : C^\infty_{f(x)}(M_2) \to B$ is defined by $\overline{\psi}(v) = \psi(v \circ f)$, $u \in C^\infty_{f(x)}(M_2, \mathcal{F}_2)$, $v \in C^\infty_{f(x)}(M)$. If in adapted coordinates a foliated map is of the form $f : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^{\overline{q}} \times \mathbb{R}^{\overline{p}}$, $f(x_i, y_j) = (f_1(x_i), f_2(x_i, y_j))$ for some $f_1 : \mathbb{R}^q \to \mathbb{R}^{\overline{q}}$ and $f_2 : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^{\overline{p}}$, then in the corresponding induced coordinates $T^\mu f : A^q \times B^p \to A^{\overline{q}} \times B^{\overline{p}}$, $T^\mu(a_i, b_j) = (T_A f_1(a_i), T_B f_2(\mu(a_i), b_j))$, where $T_A : \mathcal{M}f \to \mathcal{FM}$ is the Weil functor [2]. The correspondence $T^\mu : \mathcal{F}ol \to \mathcal{FM}$ is a product preserving bundle functor. It is called the product preserving bundle functor on $\mathcal{F}ol$ corresponding to the Weil algebra homomorphism μ.

Remark 1. Let us observe that given a p-dimensional foliation $\mathcal{F}$ on a $p+q$-dimensional manifold M the bundle $T^\mu(M, \mathcal{F})$ admits a canonical $p \dim_{\mathbb{R}} B$-dimensional foliation $\mathcal{F}^\mu$ π-related to $\mathcal{F}$. In the induced coordinates the leaves of $\mathcal{F}^\mu$ are of the form $\{a\} \times B^p \subset A^q \times B^p$.

Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a product preserving bundle functor. We are going to construct a Weil algebra homomorphism $\mu^F : A^F \to B^F$.

Example 2 ([3]). We put $A^F = F(\mathbb{R}, \mathcal{F}')$ and $B^F = F(\mathbb{R}, \mathcal{F}'')$, where $\mathcal{F}'$ is the 0-dimensional foliation on $\mathbb{R}$ and F'' is the foliation on $\mathbb{R}$ with one leaf $\mathbb{R}$. The sum mappings are given by $+_{A^F} = F(+) : A^F \times A^F \to A^F$ and $+_{B^F} = F(+) : B^F \times B^F \to B^F$, where $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the sum map treated as the respective $\mathcal{F}ol$-morphisms. Similarly, the multiplications of A^F and B^F are obtained by applying F to the multiplication of $\mathbb{R}$ being respective $\mathcal{F}ol$-morphisms. The zero maps and the unity maps are obtained by applying F to the zero map and the unity map of $\mathbb{R}$. The homomorphism $\mu^F : A^F \to B^F$ is $F(\mathrm{id}_{\mathbb{R}}) : F(\mathbb{R}, \mathcal{F}') \to F(\mathbb{R}, \mathcal{F}'')$, where $\mathrm{id}_{\mathbb{R}}$ is the identity map of $\mathbb{R}$ treated as the appropriate $\mathcal{F}ol$-morphism. *

For example, the Weil algebra homomorphism corresponding to the normal bundle functor $N : \mathcal{F}ol \to \mathcal{FM}$ (see Introduction) is the unique algebra homomorphism $\kappa_{\mathbb{D}} : \mathbb{D} \to \mathbb{R}$, where $\mathbb{D} = \mathbb{R} \times \mathbb{R}$ is the Weil algebra of dual numbers ($(a, b) + (c, d) = (a + c, b + d)$, $(a, b)(c, d) = (ac, ad + bc)$ for $(a, b), (c, d) \in \mathbb{D}$). The Weil algebra homomorphism corresponding to the bundle functor $\widetilde{A} : \mathcal{F}ol \to \mathcal{FM}$ of transverse A-points (mentioned in Introduction) is the unique algebra homomorphism $\kappa_A : A \to \mathbb{R}$. The Weil algebra homomorphism corresponding to the product preserving bundle functor $T^A : \mathcal{F}ol \to \mathcal{FM}$ of A-near points (see Introduction) is the identity map $\mathrm{id}_A : A \to A$.

The following classification proposition shows that any product preserving bundle functor on $\mathcal{F}ol$ is equivalent to some product preserving bundle functor as in Example 1.

PROPOSITION 1 ([3]). *Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a product preserving bundle functor. Let $\mu^F : A^F \to B^F$ be the corresponding Weil algebra homomorphism. Then we have an explicitly constructed natural equivalence $\Theta^F : F \cong T^{\mu^F}$.*

More precisely, the equivalence $\Theta^F : F(M, \mathcal{F}) \to T^{\mu^F}(M, \mathcal{F})$ is the following. Given a point $y \in F_x(M, \mathcal{F})$, $x \in M$, we define $\Theta^F(y) = (\varphi_y, \psi_y)$, where $\varphi_y : C^\infty_x(M, \mathcal{F}) \to A^F$, $\psi_y : C^\infty_x(M) \to B^F$, $\varphi_y([f]_x) = Ff(y)$, $\psi_y([g]_x) = Fg(y)$ for $[f]_x \in C^\infty_x(M, \mathcal{F})$ and $[g]_x \in C^\infty_x(M)$, where f and g are treated as the corresponding $\mathcal{F}ol$-morphisms. Using the respective definitions, $\Theta^F(y) \in T^{\mu^F}_x(M, \mathcal{F})$. In [3], it is proved that $\Theta^F : F(M, \mathcal{F}) \to T^{\mu^F}(M, \mathcal{F})$ is a diffeomorphism.

REMARK 2. Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a product preserving bundle functor. By Proposition 1 we have the canonical diffeomorphism $\Theta^F : F(M, \mathcal{F}) \to T^{\mu^F}(M, \mathcal{F})$ for any foliated manifold $(M, \mathcal{F})$. It is known that $T^{\mu^F}(M, \mathcal{F})$ admits the foliation $\mathcal{F}^{\mu^F}$ (see Remark 1). Then $F(M, \mathcal{F})$ admits the foliation $\mathcal{F}^F = (\Theta^F)^{-1}\mathcal{F}^{\mu^F}$. Hence F has values in the category $\mathcal{F}ol$. Therefore we can compose two product preserving bundle functors on $\mathcal{F}ol$. This composition is again a product preserving bundle functor on $\mathcal{F}ol$.

The following proposition shows that any Weil algebra homomorphism is isomorphic to the Weil algebra homomorphism corresponding to some product preserving bundle functor on $\mathcal{F}ol$.

PROPOSITION 2 ([3]). *Let $\mu : A \to B$ be a Weil algebra homomorphism. Let $F = T^\mu$. Then we have an explicitly constructed isomorphism $\mathcal{O}^\mu : \mu \cong \mu^F$ of Weil algebra homomorphisms.*

We recall that a morphism $\mu_1 \to \mu_2$ of Weil algebra homomorphisms $\mu_1 : A_1 \to B_1$ and $\mu_2 : A_2 \to B_2$ is a pair η of Weil algebra homomorphisms $\eta_1 : A_1 \to A_2$ and $\eta_2 : B_1 \to B_2$ such that $\eta_2 \circ \mu_1 = \mu_2 \circ \eta_1$.

More precisely, $\mathcal{O}^\mu : \mu \to \mu^F$ is the pair of Weil algebra isomorphisms $\mathcal{O}^\mu_1 : A \cong T^\mu(\mathbb{R}, \mathcal{F}') \to F(M, \mathcal{F}') = A^F$ and $\mathcal{O}^\mu_2 : B \cong T^\mu(\mathbb{R}, \mathcal{F}'') \to F(\mathbb{R}, \mathcal{F}'') = B^F$, where $\cong$ is the induced coordinates $\widetilde{x}$ and $\widetilde{y}$ (see Example 1) respectively.

Let $F_1, F_2 : \mathcal{F}ol \to \mathcal{FM}$ be product preserving bundle functors. Let $\mu^{F_1} : A^{F_1} \to B^{F_1}$ and $\mu^{F_2} : A^{F_2} \to B^{F_2}$ be the corresponding Weil algebra homomorphisms. Let $\nu : F_1 \to F_2$ be a natural transformation.

EXAMPLE 3 ([3]). Define a morphism $\eta^\nu = (\eta^\nu_1, \eta^\nu_2) : \mu^{F_1} \to \mu^{F_2}$ of Weil algebra homomorphisms by $\eta^\nu_1 = \nu_{(\mathbb{R}, \mathcal{F}')} : A^{F_1} \to A^{F_2}$ and $\eta^\nu_2 = \nu_{(\mathbb{R}, \mathcal{F}'')} :$

$B^{F_1} \to B^{F_2}$. If ν is an isomorphism, then so is η^ν. We call η^ν the morphism of Weil algebra homomorphisms corresponding to ν.

Let $\mu_1 : A_1 \to B_1$ and $\mu_2 : A_2 \to B_2$ be Weil algebra homomorphisms. Let $\eta = (\eta_1, \eta_2) : \mu_1 \to \mu_2$ be a morphism of Weil algebra homomorphisms.

EXAMPLE 4 ([3]). Given a $\mathcal{F}ol$-object $(M, \mathcal{F})$ define a base preserving fibered map $\nu^\eta : T^{\mu_1}(M, \mathcal{F}) \to T^{\mu_2}(M, \mathcal{F})$ by $\nu^\eta(\varphi, \psi) = (\eta_1 \circ \varphi, \eta_2 \circ \psi)$, $(\varphi, \psi) \in T^{\mu_1}_x(M, \mathcal{F})$, $x \in M$. The family $\nu^\eta : T^{\mu_1} \to T^{\mu_2}$ is a natural transformation. If η is an isomorphism, then so is ν^η. We call ν^η the natural transformation corresponding to ν.

Summing up we have the following object classification theorem corresponding to Theorem A.

THEOREM 1 ([3]). *The correspondence $F \mapsto \mu^F$ induces a bijective correspondence between the equivalence classes of product preserving bundle functors on $\mathcal{F}ol$ and the equivalence classes of Weil algebra homomorphisms. The inverse correspondence is induced by the correspondence $\mu \mapsto T^\mu$.*

Let F_1 and F_2 be two product preserving bundle functors on $\mathcal{F}ol$. Let $\mu^{F_1} : A^{F_1} \to B^{F_1}$ and $\mu^{F_2} : A^{F_2} \to B^{F_2}$ be the corresponding Weil algebra homomorphisms.

LEMMA 1 ([3]). *Let $\eta = (\eta_1, \eta_2) : \mu^{F_1} \to \mu^{F_2}$ be a morphism of Weil algebra homomorphisms. Let $\nu^{[\eta]} : F_1 \to F_2$ be a natural transformation given by the composition $F_1 \xrightarrow{\Theta^{F_1}} T^{\mu^{F_1}} \xrightarrow{\nu^\eta} T^{\mu^{F_2}} \xrightarrow{(\Theta^{F_2})^{-1}} F_2$, where Θ^F is as in Proposition 1 and ν^η is described in Example 4. Then $\nu = \nu^{[\eta]}$ is the unique natural transformation $F_1 \to F_2$ such that $\eta^\nu = \eta$, where η^ν is as in Example 3.*

Summing up we have the following morphism classification theorem corresponding to Theorem B.

THEOREM 2 ([3]). *Let F_1 and F_2 be two product preserving bundle functors on $\mathcal{F}ol$. The correspondence $\nu \mapsto \eta^\nu$ is a bijection between natural transformations $F_1 \to F_2$ and morphisms $\mu^{F_1} \to \mu^{F_2}$ of the corresponding Weil algebra homomorphisms. The inverse correspondence is $\eta \mapsto \nu^{[\eta]}$ (where $\nu^{[\eta]}$ is defined in Lemma 1).*

Let F_1 and F_2 be product preserving bundle functors on $\mathcal{F}ol$. According to Remark 2, the composition $F_1 \circ F_2$ is again a product preserving bundle functor on $\mathcal{F}ol$. Let μ^{F_1}, μ^{F_2} and $\mu^{F_1 \circ F_2}$ be the corresponding Weil algebra homomorphisms. Using the tensor product we get the Weil algebra homomorphism $\mu^{F_1} \otimes \mu^{F_2}$.

PROPOSITION 3 ([3]). $\mu^{F_1 \circ F_2} = \mu^{F_1} \otimes \mu^{F_2}$.

COROLLARY 1 ([3]). *We have the isomorphism* $F_1 \circ F_2 \cong F_2 \circ F_1$ *corresponding to the exchange isomorphism of tensor products.*

COROLLARY 2. *Let* $F : \mathcal{F}ol \to \mathcal{FM}$ *be a product preserving bundle functor. Let* $T : \mathcal{F}ol \to \mathcal{FM}$ *be the tangent bundle functor. For any* $\mathcal{F}ol$-*object* $(M, \mathcal{F})$ *we have the* $\mathcal{F}ol$-*natural isomorphism* $\nu_{(M,\mathcal{F})} : F(T(M,\mathcal{F})) \to T(F(M,\mathcal{F}))$ (*see Corollary* 1) *satisfying the flow property*

$$\pi^{T(F(M,\mathcal{F}))} \circ \nu_{(M,\mathcal{F})} = F\pi^{T(M,\mathcal{F})},$$

where $\pi^{T(F(M,\mathcal{F}))} : T(F(M,\mathcal{F})) \to F(M,\mathcal{F})$ *and* $\pi^{T(M,\mathcal{F})} : T(M,\mathcal{F}) \to (M,\mathcal{F})$ *are the projections of the tangent bundles* (*with the respective foliations as in Remark* 2).

2. The Lie algebra of $\mathrm{Aut}(\mu)$**.** Consider a Weil algebra homomorphism $\mu : A \to B$. We note that the group $\mathrm{Aut}(\mu)$ of all automorphisms of μ is a closed (and hence Lie) subgroup in $\mathrm{GL}(A) \times \mathrm{GL}(B)$.

We have the Lie algebra

$$\mathcal{D}er(\mu) = \{D = (D_1, D_2) \in \mathcal{D}er(A) \times \mathcal{D}er(B) \mid D_2 \circ \mu = \mu \circ D_1\}$$

of derivations of μ. It is a subalgebra in the product $\mathcal{D}er(A) \times \mathcal{D}er(B)$ of the Lie algebras $\mathcal{D}er(A)$ and $\mathcal{D}er(B)$ of derivations of A and B. A *derivation* of A is a linear map $D_1 : A \to A$ such that $D_1(ab) = D_1(a)b + aD_1(b)$ for all $a, b \in A$. The bracket of the Lie algebra $\mathcal{D}er(A)$ is given standardly by $[D_1, D_1'] = D_1 \circ D_1' - D_1' \circ D_1$.

PROPOSITION 4 ([4]). $\mathcal{L}ie(\mathrm{Aut}(\mu)) = \mathcal{D}er(\mu)$.

Proof. By [2], $\mathcal{L}ie(\mathrm{Aut}(A)) = \mathcal{D}er(A)$. Clearly, $(D_1, D_2) \in \mathrm{Aut}(\mu)$ iff $D_1 \in \mathrm{Aut}(A)$, $D_2 \in \mathrm{Aut}(B)$ and $\mu \circ D_1 = D_2 \circ \mu$. ∎

3. Natural automorphisms of $F_{|\mathcal{F}ol_{p,q}} \to \mathcal{FM}$ **into itself.** In this section we prove the following result.

PROPOSITION 5. *Let* $F : \mathcal{F}ol \to \mathcal{FM}$ *be a product preserving bundle functor and* $\mu : A \to B$ *be its algebra homomorphism. Let* $p \geq 1$ *and* $q \geq 1$ *be integers. Every natural automorphism* ν *of* $F_{|\mathcal{F}ol_{p,q}}$ *can be extended uniquely to a natural automorphism of* F. *In particular*, $\mathrm{Aut}(F_{|\mathcal{F}ol_{p,q}}) = \mathrm{Aut}(\mu)$.

Proof. Let $x^1, \dots, x^q, y^1, \dots, y^p$ be the adapted coordinates on the standard $\mathcal{F}ol_{p,q}$-object $(M_0, \mathcal{F}_0) = (\mathbb{R}^q \times \mathbb{R}^p, \{\{a\} \times \mathbb{R}^p\}_{a \in \mathbb{R}^q})$.

Consider a natural automorphism ν of $F_{|\mathcal{F}ol_{p,q}}$ into itself. Since F is product preserving, $F(M_0, \mathcal{F}_0) = A^q \times B^p$. By the $\mathcal{F}ol_{p,q}$-naturality, ν is uniquely determined by $\nu = \nu_{(M_0,\mathcal{F}_0)} : A^q \times B^p \to A^q \times B^p$. Write

$$\nu(a_1, \dots, a_q, a_{q+1}, \dots, a_{q+p}) = (\nu^1(a_1, \dots, a_{q+p}), \dots, \nu^{q+p}(a_1, \dots, a_{q+p}))$$

for $a_1, \dots, a_q \in A$ and $a_{q+1}, \dots, a_{q+p} \in B$.

By the invariance of ν with respect to the $\mathcal{F}ol_{p,q}$-morphisms $(\tau_1 x^1, \dots, \tau_q x^q, \tau_{q+1} y^1, \dots, \tau_{q+p} y^p)$ for $\tau_1, \dots, \tau_{q+p} \in \mathbb{R}_+$ we get the homogeneity conditions $\tau_j \nu^j(a_1, \dots, a_{q+p}) = \nu^j(\tau_1 a_1, \dots, \tau_{q+p} a_{q+p})$ for $j = 1, \dots, q+p$ and any $a_1, \dots, a_q \in A$, $a_{q+1}, \dots, a_{q+p} \in B$ and any $\tau_1, \dots, \tau_{q+p} \in \mathbb{R}_+$. This type of homogeneity implies that ν^j depends linearly only on a_j because of the homogeneous function theorem ([2]).

Using permutations of adapted coordinates we deduce that $\nu = \sigma \times \dots \times \sigma \times \varrho \times \dots \times \varrho : A^q \times B^p \to A^q \times B^p$ for $\sigma = \nu^1 : A \to A$ and $\varrho = \nu^{q+1} : B \to B$. We prove that $(\sigma, \varrho) \in \mathrm{Aut}(\mu)$.

STEP 1: *ϱ is an algebra isomorphism.* We know that ϱ is $\mathbb{R}$-linear. Using the invariance of ν with respect to the local $\mathcal{F}ol_{p,q}$-morphism $(x^1, \dots, x^q, y^1 + (y^1)^2, y^2, \dots, y^p)$ we derive that $\varrho(b + b^2) = \varrho(b) + (\varrho(b))^2$, i.e. $\varrho(b^2) = (\varrho(b))^2$ for any $b \in B$. Then $\varrho((b_1 + b_2)^2) = (\varrho(b_1 + b_2))^2$, i.e. $\varrho(b_1 b_2) = \varrho(b_1)\varrho(b_2)$ for any $b_1, b_2 \in B$. So, ϱ is multiplicative. Using the invariance of ν with respect to the $\mathcal{F}ol_{p,q}$-map $(x^1, \dots, x^q, y^1 + 1, y^2, \dots, y^p)$ we derive that $\varrho(b+1) = \varrho(b) + 1$, i.e. $\varrho(1) = 1$. These facts show that ϱ is an algebra homomorphism. Since η is an isomorphism, so is ϱ.

STEP 2: *σ is an algebra homomorphism.* The proof is quite similar to Step 1. (Now, we use the $\mathcal{F}ol_{p,q}$-map $(x^1 + (x^1)^2, x^1, \dots, x^q, y^1, \dots, y^p)$.)

STEP 3: *$\varrho \circ \mu = \mu \circ \sigma$.* We use the invariance of ν with respect to the $\mathcal{F}ol_{p,q}$-map $\varphi = (x^1, \dots, x^q, y^1 + x^1, y^2, \dots, y^p)$. Then

$$\begin{aligned}(\sigma(a), 0, \dots, 0, \mu(\sigma(a)), 0, \dots, 0) &= F\varphi \circ \nu(a, 0, \dots, 0, 0, \dots, 0)\\ &= \nu \circ F\varphi(a, 0, \dots, 0, 0, \dots, 0)\\ &= (\sigma(a), 0, \dots, 0, \varrho(\mu(a)), 0, \dots, 0).\end{aligned}$$

Hence $\varrho(\mu(a)) = \mu(\sigma(a))$ for $a \in A$.

We have proved that $\eta = (\sigma, \varrho) \in \mathrm{Aut}(\mu)$. By Theorem 2 we have the natural transformation $\nu^{[\eta]} : F \to F$ corresponding to η. Clearly, ν is the restriction of $\nu^{[\eta]}$. If $\widetilde{\nu} : F \to F$ is another such transformation, then $\widetilde{\nu} = \nu^{[\eta]}$ because $\widetilde{\nu}$ coincides with ν on $A^p \times B^q$. ∎

4. Canonical vector fields on product preserving bundles over foliated manifolds. Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a product preserving bundle functor and let $\mu : A \to B$ be its corresponding Weil algebra homomorphism. Let $p \geq 1$ and $q \geq 1$ be integers.

A *$\mathcal{F}ol_{p,q}$-canonical* vector field on $F_{|\mathcal{F}ol_{p,q}}$ is a family of vector fields $V = V_{F(M,\mathcal{F})}$ on $F(M,\mathcal{F})$ for any $\mathcal{F}ol_{p,q}$-object $(M,\mathcal{F})$ such that for any $\mathcal{F}ol_{p,q}$-objects $(M_1,\mathcal{F}_1)$ and $(M_2,\mathcal{F}_2)$ and any $\mathcal{F}ol_{p,q}$-map $\varphi : (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$ the vector fields V on $F(M_1,\mathcal{F}_1)$ and V on $F(M_2,\mathcal{F}_2)$ are $F\varphi$-related.

We have the following example of canonical vector fields on $F_{|\mathcal{F}ol_{p,q}}$.

Example 5 (The operators $\mathrm{op}(D)$). Let $D \in \mathcal{D}er(\mu) = \mathcal{L}ie(\mathrm{Aut}(\mu))$ be an element from the Lie algebra of the Lie group of all automorphisms of μ (see Proposition 2). Let D_t be the one-parameter subgroup in $\mathrm{Aut}(\mu)$ corresponding to D. By Theorem 2 we have the corresponding one-parameter subgroup D_t of natural equivalences of F. So, for every $\mathcal{F}ol_{p,q}$-object $(M,\mathcal{F})$ we have the flow D_t on $F(M,\mathcal{F})$. This flow defines a vector field $\mathrm{op}(D)$ on $F(M,\mathcal{F})$. Clearly, $\mathrm{op}(D)$ is canonical.

For example, let $F = N : \mathcal{F}ol_{p,q} \to \mathcal{FM}$ be the normal bundle functor. As we mentioned, the Weil algebra homomorphism μ corresponding to N is the unique algebra homomorphism $\kappa_{\mathbb{D}} : \mathbb{D} \to \mathbb{R}$, where $\mathbb{D} = \mathbb{R}\times\mathbb{R}$ is the algebra of dual numbers, $(a,b)+(c,d) = (a+c, b+d)$, $(a,b)(c,d) = (ac, ad+bc)$. Then (it is easy to see that) $\mathrm{Aut}(\kappa_{\mathbb{D}}) = \{(\mathrm{id}_{\mathbb{R}} \times \tau\, \mathrm{id}_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}}) \mid \tau \in \mathbb{R}\setminus\{0\}\} \cong \mathbb{R}\setminus\{0\}$. Then $\mathcal{D}er(\kappa_D) \cong \mathbb{R}$. For $D = 1 \in \mathbb{R} = \mathcal{D}er(\kappa_{\mathbb{D}})$, the one-parameter group corresponding to $D = 1$ is $D_t = (\mathrm{id}_{\mathbb{R}} \times \exp(t)\,\mathrm{id}_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}}) \in \mathrm{Aut}(\kappa_{\mathbb{D}})$. Then $\mathrm{op}(D) = L$ is the Liouville vector field on the vector bundle $N(M,\mathcal{F})$ for any foliated manifold $(M,\mathcal{F})$ (here $L(v) = [v+tv] \in T_vN_x(M,\mathcal{F}) \subset T_vN(M,\mathcal{F})$, $v \in N_x(M,\mathcal{F})$, $x \in M$).

Proposition 6. *Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a product preserving bundle functor and $\mu : A \to B$ be its Weil algebra homomorphism. Let $p \geq 1$ and $q \geq 1$ be integers. Every $\mathcal{F}ol_{p,q}$-canonical vector field V on $F_{|\mathcal{F}ol_{p,q}}$ is of the form $V = \mathrm{op}(D)$ for some $D \in \mathcal{D}er(\mu)$.*

Proof. For every $\mathcal{F}ol_{p,q}$-object $(M,\mathcal{F})$ we have some vector field V on $F(M,\mathcal{F})$ invariant with respect to $\mathcal{F}ol_{p,q}$-maps. The flow $\mathrm{Exp}(tV)$ of V is $\mathcal{F}ol_{p,q}$-invariant. Using Theorem 1 we can easily show that there exists $v \in F(M,\mathcal{F})$ such that $F(M,\mathcal{F})$ is the orbit of U with respect to $\mathcal{F}ol_{p,q}$-maps for any open neighborhood $U \subset F(M,\mathcal{F})$ of v. This implies that V is complete. Thus $\mathrm{Exp}(tV)$ corresponds to some one-parameter subgroup in $\mathrm{Aut}(F_{\mathcal{F}ol_{p,q}})$. By Proposition 4 and Theorem 2 it corresponds to some $D \in \mathcal{D}er(\mu)$. Then $V = \mathrm{op}(D)$. ∎

5. Some canonical affinors on product preserving bundle functors. Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a product preserving bundle functor and $\mu : A \to B$ be its Weil algebra homomorphism.

A *$\mathcal{F}ol$-natural affinor* on F is a family of affinors (tensor fields of type $(1,1)$) $t = t_{F(M,\mathcal{F})}$ on $F(M,\mathcal{F})$ for any $\mathcal{F}ol$-object $(M,\mathcal{F})$ such that for any $\mathcal{F}ol$-map $\varphi : (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$ the tensor fields t on $F(M_1,\mathcal{F}_1)$ and t on $F(M_2,\mathcal{F}_2)$ are $F\varphi$-related.

Example 6. Let (M,F) be a $\mathcal{F}ol$-object. By Corollary 2 we have the natural isomorphism $\nu_{(M,\mathcal{F})} : F(T(M,\mathcal{F})) \to T(F(M,\mathcal{F}))$ satisfying the flow property. The fiber multiplication $m : \mathbb{R}\times T(M,\mathcal{F}) \to T(M,\mathcal{F})$ can be treated as a $\mathcal{F}ol$-map, where $\mathbb{R}$ is foliated by points and $T(M,\mathcal{F})$ is a foliated manifold (see Remark 2). Applying the functor F we obtain $Fm : F\mathbb{R} \times F(T(M,\mathcal{F})) \to F(T(M,\mathcal{F}))$. Using the above flow isomorphism and observing that $F\mathbb{R} = A$ (see Example 2) from Fm we obtain $\widetilde{Fm} : A \times T(F(M,\mathcal{F})) \to T(F(M,\mathcal{F}))$. Thus given $a \in A$ we have

$$\mathrm{af}(a) := \widetilde{Fm}(a, \cdot) : T(F(M,\mathcal{F})) \to T(F(M,\mathcal{F})).$$

Using the flow property of $\nu_{(M,\mathcal{F})}$ we observe that $\mathrm{af}(a)$ is a $\mathcal{F}ol$-natural affinor on F.

6. Natural operators lifting infinitesimal automorphisms of foliated manifolds to vector fields on product preserving bundles. Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a product preserving bundle functor and $\mu : A \to B$ be its Weil algebra homomorphism. Let $p \geq 1$ and $q \geq 1$ be integers.

A *$\mathcal{F}ol_{p,q}$-natural operator* C transforming infinitesimal automorphisms X of $\mathcal{F}ol_{p,q}$-objects $(M,\mathcal{F})$ into vector fields $C(X)$ on $F(M,\mathcal{F})$ is a family of regular $\mathcal{F}ol_{p,q}$-invariant operators

$$C_{(M,\mathcal{F})} : \mathcal{X}(M,\mathcal{F}) \to \mathcal{X}(F(M,\mathcal{F}))$$

from the space $\mathcal{X}(M,\mathcal{F})$ of all infinitesimal automorphisms of $(M,\mathcal{F})$ into the space $\mathcal{X}(F(M,\mathcal{F}))$ of all vector fields on $F(M,\mathcal{F})$ for any $\mathcal{F}ol_{p,q}$-objects $(M,\mathcal{F})$. The $\mathcal{F}ol_{p,q}$-invariance means that for any $\mathcal{F}ol_{p,q}$-map $\varphi : (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$ and any infinitesimal automorphisms $X_1 \in \mathcal{X}(M_1,\mathcal{F}_1)$ and $X_2 \in \mathcal{X}(M_2,\mathcal{F}_2)$, if X_1 and X_2 are φ-related then $C(X_1)$ and $C(X_2)$ are $F\varphi$-related. The regularity means that C transforms smoothly parametrized families of infinitesimal automorphisms into smoothly parametrized families of vector fields.

EXAMPLE 7 (The flow operator). Let $(M,\mathcal{F})$ be a $\mathcal{F}ol_{p,q}$-object. Let $X \in \mathcal{X}(M,\mathcal{F})$. Then the flow $\mathrm{Exp}(tX)$ of X is formed by $\mathcal{F}ol_{p,q}$-maps. So we can apply a functor $F : \mathcal{F}ol \to \mathcal{FM}$ to $\mathrm{Exp}(tX)$ and obtain the flow $F(\mathrm{Exp}(tX))$ on $F(M,\mathcal{F})$. Then we have the vector field $\mathcal{F}X$ on $F(M,\mathcal{F})$ corresponding to the flow $F(\mathrm{Exp}(tX))$. Clearly, the correspondence $\mathcal{F} : X \to \mathcal{F}X$ is a $\mathcal{F}ol_{p,q}$-natural operator in question.

The main result of this note is the following classification theorem.

THEOREM 3. *Let $F : \mathcal{F}ol \to \mathcal{FM}$ be a product preserving bundle functor and $\mu : A \to B$ be its Weil algebra homomorphism. Let $p \geq 1$ and $q \geq 1$ be integers. Any $\mathcal{F}ol_{p,q}$-natural operator C transforming infinitesimal automorphisms $X \in \mathcal{X}(M,\mathcal{F})$ into vector fields $C(X) \in \mathcal{X}(F(M,\mathcal{F}))$ is of the form*

$$C(X) = \mathrm{af}(a) \circ \mathcal{F}X + \mathrm{op}(D)$$

for some unique $a \in A$ and $D \in \mathcal{D}er(\mu)$.

Proof. Let C be an operator in question. Let $(M,\mathcal{F})$ be a $\mathcal{F}ol_{p,q}$-object. Let 0 be the zero infinitesimal automorphism on $(M,\mathcal{F})$ for any $\mathcal{F}ol_{p,q}$-object. Then $C(0)$ is a $\mathcal{F}ol_{p,q}$-canonical vector field on $F_{|\mathcal{F}ol_{p,q}}$. Thus replacing C by $C - C(0)$ and applying Proposition 6 we can assume that $C(0) = 0$.

Since any automorphism X on $(M,\mathcal{F})$ with non-vanishing transversal vector field is $\partial/\partial x^1$ on $(M_0,\mathcal{F}_0) = (\mathbb{R}^q \times \mathbb{R}^p, \{\{a\} \times \mathbb{R}^p\}_{a\in\mathbb{R}^q})$ in some adapted coordinates, C is uniquely determined by $C(\varrho\partial/\partial x^1) : A^q \times B^p \to A^q \times B^p$, $\varrho \in \mathbb{R}$. Using the invariance with respect to the homotheties being $\mathcal{F}ol_{p,q}$-morphisms $(M_0,\mathcal{F}_0) \to (M_0,\mathcal{F}_0)$ and the homogeneous function theorem ([2]) and $C(0) = 0$ we deduce that for any ϱ the map $C\big(\varrho\frac{\partial}{\partial x^1}\big)$: $A^q \times B^p \to A^q \times B^p$ is constant and linearly dependent on ϱ. Then using the invariance with respect to $\mathcal{F}ol_{p,q}$-maps $(x^1, tx^2, \ldots, tx^q, ty^1, \ldots, ty^p)$ for $t \neq 0$ we deduce that the map $C(\varrho\partial/\partial x^1) : A^q \times B^p \to A \times \{0\} \times \{0\}$ is constant and linearly dependent on ϱ. Hence the vector space of all natural operators C in question with $C(0) = 0$ is at most $\dim_{\mathbb{R}} A$-dimensional. But all natural operators $\mathrm{af}(a)\circ\mathcal{F}$ form a $\dim_{\mathbb{R}} A$-dimensional vector space. Thus the proof is complete by a dimension argument. ∎

Remark 3. Let $\mathcal{FM}_{q,p}$ be the category of fibered manifolds with q-dimensional bases and p-dimensional fibers and their local fibered diffeomorphisms. The category $\mathcal{FM}_{q,p}$ is in an obvious way a subcategory in $\mathcal{F}ol_{p,q}$. The categories $\mathcal{FM}_{q,p}$ and $\mathcal{F}ol_{p,q}$ have the same skeleton. Any projectable vector field X on an $\mathcal{FM}_{q,p}$-object Y is in an obvious way an infinitesimal automorphism of the $\mathcal{F}ol_{p,q}$-object Y. By [3], any product preserving bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ can be uniquely extended to a product preserving bundle functor $F : \mathcal{F}ol \to \mathcal{FM}$. In [4], J. Tomáš classified all $\mathcal{FM}_{q,p}$-natural operators C transforming projectable vector fields X on an $\mathcal{FM}_{q,p}$-object Y into vector fields $C(X)$ on FY for any product preserving bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ and deduced the corresponding formula $C(X) = af(a) \circ \mathcal{F}X + \mathrm{op}(D)$. Therefore it seems that Theorem 3 can also be deduced from the result of [4].

7. An application to the normal bundle. Let $N : \mathcal{F}ol_{p,q} \to \mathcal{FM}$ be the normal bundle functor (see Introduction). Its corresponding Weil algebra homomorphism is the unique algebra homomorphism $\kappa_{\mathbb{D}} : \mathbb{D} \to \mathbb{R}$, where $\mathbb{D}$ is the algebra of dual numbers (see the text after Example 2). It is easily seen (see [2]) that $\dim_{\mathbb{R}} \mathbb{D} = 2$ and $\dim_{\mathbb{R}} \mathcal{D}er(\kappa_{\mathbb{D}}) = 1$. Therefore (because of Theorem 3) the vector space of all $\mathcal{F}ol_{p,q}$-natural operators transforming infinitesimal automorphisms $X \in \mathcal{X}(M,\mathcal{F})$ into vector fields $C(X) \in \mathcal{X}(N(M,\mathcal{F}))$ is of dimension 3. On the other hand, given $X \in \mathcal{X}(M,\mathcal{F})$ we have the following vector fields on the (vector) normal bundle $N(M,\mathcal{F})$:

1. The flow vector field $\mathcal{N}X$ of X on $N(M,\mathcal{F})$.

2. The vertical lifting X^V of X to $N(M,\mathcal{F})$ defined as follows. Given $v \in N_x(M,\mathcal{F})$, $x \in M$, we have $[X(x)] \in N_x(M,\mathcal{F}) = T_xM/T_x\mathcal{F}$. Then we put $X^V(v) := [X(x)] \in N_x(M,\mathcal{F}) = T_v(N_x(M,\mathcal{F})) \subset T_v(N(M,\mathcal{F}))$.
3. The Liouville vector field L on the vector bundle $N(M,\mathcal{F})$, $L(v) = [v+tv] \in T_v(N_x(M,F))$, $v \in N_x(M,\mathcal{F})$, $x \in M$.

Thus we have the corresponding natural operators: $\mathcal{N}$, $(\ )^V$ and L. These operators are linearly independent. So (by a dimension argument) they form a basis in the vector space of all $\mathcal{F}ol_{p,q}$-natural operators C transforming infinitesimal automorphisms $X \in \mathcal{X}(M,\mathcal{F})$ into vector fields $C(X) \in \mathcal{X}(N(M,\mathcal{F}))$. Then we have

COROLLARY 3. *Any $\mathcal{F}ol_{p,q}$-natural operator C transforming infinitesimal automorphisms $X \in \mathcal{X}(M,\mathcal{F})$ into vector fields $C(X)$ on the normal bundle $N(M,\mathcal{F})$ is of the form*

$$C(X) = a\mathcal{N}X + bX^V + cL$$

for some real numbers a, b, c.

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