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Linear liftings of symmetric tensor fields of type $(1,2)$ to Weil bundles

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Abstract. This paper contains a classification of all linear liftings of symmetric tensor fields of type $(1,2)$ on n-dimensional manifolds to any tensor fields of type $(1,2)$ on Weil bundles under the condition that $n \geq 3$.

Introduction. Let A be a Weil algebra inducing the Weil functor T^A (see [3]) and let n be a non-negative integer. We will denote by $\mathrm{Te}\, M$ the vector space of all tensor fields of type $(1,2)$ on a manifold M. Of course, $t \in \mathrm{Te}\, M$ is called *symmetric* if $t_x(y_2, y_1) = t_x(y_1, y_2)$ for every $x \in M$ and all $y_1, y_2 \in T_x M$. The vector space of all symmetric tensor fields of type $(1,2)$ on M will be denoted by $\mathrm{SyTe}\, M$.

A *lifting* of symmetric tensor fields of type $(1,2)$ to tensor fields of type $(1,2)$ on T^A is, by definition, a family of maps $L_M : \mathrm{SyTe}\, M \to \mathrm{Te}\, T^A M$ indexed by all n-dimensional manifolds and satisfying

$$L_M(\phi^* t) = (T^A \phi)^*(L_N(t)) \tag{1}$$

for all n-dimensional manifolds M, N, every embedding $\phi : M \to N$ and every $t \in \mathrm{SyTe}\, N$. Of course, here $\phi^* t$ is defined by the formula $(\phi^* t)_x = (T_x\phi)^{-1} \circ t_{\phi(x)} \circ (T_x\phi, T_x\phi)$ for every $x \in M$. Such a lifting is said to be *linear* if L_M is linear for every n-dimensional manifold M. Obviously, all linear liftings form a subspace of the vector space of all liftings in question.

Our purpose is to give a complete description of all these linear liftings. We hope that this result can be applied in the study of affine liftings of torsion-free linear connections to any linear connections on Weil bundles.

Construction of liftings. Our first task is to construct some liftings in question. We will use four known constructions which we now recall.

There is a unique lifting C of tensor fields of type $(1,2)$ to tensor fields of type $(1,2)$ on T^A such that $C^i_U(t)_X(Y_1, Y_2) = (T^A t^i_{jk}(X)) Y_1^j Y_2^k$ for every

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$i \in \{1, \ldots, n\}$, every open subset U of $\mathbb{R}^n$, every $t \in \operatorname{Te} U$, every $X \in T^A U$ and all $Y_1, Y_2 \in A^n$ (see [2]).

For each $a \in A$ there is a unique natural tensor field $\widetilde{a}$ of type $(1,1)$ on T^A such that $(\widetilde{a}^i_U)_X(Y) = aY^i$ for every $i \in \{1, \ldots, n\}$, every open subset U of $\mathbb{R}^n$, every $X \in T^A U$ and every $Y \in A^n$.

Let $C_s A$, where s is a non-negative integer, denote the vector space of all $(s+1)$-linear maps $G : A \times \cdots \times A \to \mathbb{R}$ which are skew-symmetric with respect to the last s variables and such that $G(a, bc, d_2, \ldots, d_s) = G(ab, c, d_2, \ldots, d_s) + G(ac, b, d_2, \ldots, d_s)$ for all $a, b, c, d_2, \ldots, d_s \in A$ whenever $s \geq 1$. Note that elements of $C_0 A$ are nothing but linear maps $A \to \mathbb{R}$. If p, q are two non-negative integers such that $p \geq q$, then for each $G \in C_{p-q} A$ there is a unique lifting $G^{p,q}$ of p-forms to q-forms on T^A such that $G^{p,q}_U(\omega)_X(Y_1, \ldots, Y_q) = G((T^A \omega_{i_1 \ldots i_p}(X))Y_1^{i_1} \ldots Y_q^{i_q}, X^{i_{q+1}}, \ldots, X^{i_p})$ for every open subset U of $\mathbb{R}^n$, every p-form ω on U, every $X \in T^A U$ and all $Y_1, \ldots, Y_q \in A^n$ (see [1]).

An $\mathbb{R}$-linear map $D : A \to A$ is said to be a *derivation* of A if $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. Let us denote by $\operatorname{Der} A$ the vector space of all derivations of A. For each $D \in \operatorname{Der} A$ there is a unique natural vector field $\widetilde{D}$ on T^A such that $(\widetilde{D}^i_U)_X = D(X^i)$ for every $i \in \{1, \ldots, n\}$, every open subset U of $\mathbb{R}^n$ and every $X \in T^A U$ (see [3]).

We can now construct six types of linear liftings of symmetric tensor fields of type $(1,2)$ to tensor fields of type $(1,2)$ on T^A.

Let $P \in A$. Define $\overline{P}_M(t) = \widetilde{P} \circ C_M(t)$ for every n-dimensional manifold M and every $t \in \operatorname{SyTe} M$. Thus $\overline{P}$ is the only lifting such that

$$\overline{P}^i_U(t)_X(Y_1, Y_2) = P(T^A t^i_{jk}(X)) Y_1^j Y_2^k \tag{2}$$

for every $i \in \{1, \ldots, n\}$, every open subset U of $\mathbb{R}^n$, every $t \in \operatorname{SyTe} U$, every $X \in T^A U$ and all $Y_1, Y_2 \in A^n$.

For every n-dimensional manifold M and every $t \in \operatorname{Te} M$ we will denote by $\operatorname{tr} t$ the 1-form on M such that for every $x \in M$ and every $y \in T_x M$, $(\operatorname{tr} t)_x(y)$ is the trace of the endomorphism $T_x M \ni z \mapsto t_x(y, z) \in T_x M$. If $G \in C_0 A$ and $a \in A$, then the formula $L_M(t) = G^{1,1}_M(\operatorname{tr} t) \otimes \widetilde{a}_M$ for every n-dimensional manifold M and every $t \in \operatorname{SyTe} M$ defines a lifting L we want. Since any sum of such liftings is also a lifting, we can carry out a more general construction. Let $Q \in C_0 A \otimes A$. Clearly, Q may be interpreted as an $\mathbb{R}$-linear map $A \to A$ and it is easy to see that there is a unique lifting $\overline{Q}$ such that

$$\overline{Q}^i_U(t)_X(Y_1, Y_2) = Q((T^A t^j_{kj}(X)) Y_1^k) Y_2^i \tag{3}$$

for every $i \in \{1, \ldots, n\}$, every open subset U of $\mathbb{R}^n$, every $t \in \operatorname{SyTe} U$, every $X \in T^A U$ and all $Y_1, Y_2 \in A^n$.

The last construction may be repeated with $\widetilde{a}_M \otimes G_M^{1,1}(\operatorname{tr} t)$ instead of $G_M^{1,1}(\operatorname{tr} t) \otimes \widetilde{a}_M$. Let $Q' \in A \otimes C_0 A$. Clearly, Q' may be interpreted as an $\mathbb{R}$-linear map $A \to A$ and it is easy to see that there is a unique lifting $\overline{Q'}$ such that

$$\overline{Q'}^i_U(t)_X(Y_1, Y_2) = Q'((T^A t^j_{kj}(X))Y_2^k)Y_1^i \tag{4}$$

for every $i \in \{1, \ldots, n\}$, every open subset U of $\mathbb{R}^n$, every $t \in \operatorname{SyTe} U$, every $X \in T^A U$ and all $Y_1, Y_2 \in A^n$.

Let d denote the exterior derivative. If $G \in C_0 A$ and $D \in \operatorname{Der} A$, then the formula $L_M(t) = G_M^{2,2}(d(\operatorname{tr} t)) \otimes \widetilde{D}_M$ for every n-dimensional manifold M and every $t \in \operatorname{SyTe} M$ defines a lifting L we want. More generally, let $R \in C_0 A \otimes \operatorname{Der} A$. Clearly, R may be interpreted as an $\mathbb{R}$-bilinear map $A \times A \to A$ with the property that $R(a, bc) = R(a, b)c + bR(a, c)$ for all $a, b, c \in A$ and it is easy to see that there is a unique lifting $\overline{R}$ such that

$$\overline{R}^i_U(t)_X(Y_1, Y_2) = \frac{1}{2}\, R\left(\left(T^A\left(\frac{\partial t^j_{lj}}{\partial x^k} - \frac{\partial t^j_{kj}}{\partial x^l}\right)(X)\right)Y_1^k Y_2^l, X^i\right) \tag{5}$$

for every $i \in \{1, \ldots, n\}$, every open subset U of $\mathbb{R}^n$, every $t \in \operatorname{SyTe} U$, every $X \in T^A U$ and all $Y_1, Y_2 \in A^n$.

If $G \in C_1 A$ and $a \in A$, then the formula $L_M(t) = G_M^{2,1}(d(\operatorname{tr} t)) \otimes \widetilde{a}_M$ for every n-dimensional manifold M and every $t \in \operatorname{SyTe} M$ defines a lifting L we want. More generally, let $S \in C_1 A \otimes A$. Clearly, S may be interpreted as an $\mathbb{R}$-bilinear map $A \times A \to A$ with the property that $S(a, bc) = S(ab, c) + S(ac, b)$ for all $a, b, c \in A$ and it is easy to see that there is a unique lifting $\overline{S}$ such that

$$\overline{S}^i_U(t)_X(Y_1, Y_2) = \frac{1}{2}\, S\left(\left(T^A\left(\frac{\partial t^j_{lj}}{\partial x^k} - \frac{\partial t^j_{kj}}{\partial x^l}\right)(X)\right)Y_1^k, X^l\right)Y_2^i \tag{6}$$

for every $i \in \{1, \ldots, n\}$, every open subset U of $\mathbb{R}^n$, every $t \in \operatorname{SyTe} U$, every $X \in T^A U$ and all $Y_1, Y_2 \in A^n$.

The last construction may be repeated with $\widetilde{a}_M \otimes G_M^{2,1}(d(\operatorname{tr} t))$ instead of $G_M^{2,1}(d(\operatorname{tr} t)) \otimes \widetilde{a}_M$. Let $S' \in A \otimes C_1 A$. Clearly, S' may be interpreted as an $\mathbb{R}$-bilinear map $A \times A \to A$ with the property that $S'(a, bc) = S'(ab, c) + S'(ac, b)$ for all $a, b, c \in A$ and it is easy to see that there is a unique lifting $\overline{S'}$ such that

$$\overline{S'}^i_U(t)_X(Y_1, Y_2) = \frac{1}{2}\, S'\left(\left(T^A\left(\frac{\partial t^j_{lj}}{\partial x^k} - \frac{\partial t^j_{kj}}{\partial x^l}\right)(X)\right)Y_2^k, X^l\right)Y_1^i \tag{7}$$

for every $i \in \{1, \ldots, n\}$, every open subset U of $\mathbb{R}^n$, every $t \in \operatorname{SyTe} U$, every $X \in T^A U$ and all $Y_1, Y_2 \in A^n$.

Classification theorem. We are now in a position to formulate our main result.

THEOREM. *If $n \geq 3$, then for each linear lifting L of symmetric tensor fields of type $(1,2)$ to tensor fields of type $(1,2)$ on T^A there are uniquely determined $P \in A$, $Q \in C_0A \otimes A$, $Q' \in A \otimes C_0A$, $R \in C_0A \otimes \operatorname{Der} A$, $S \in C_1A \otimes A$ and $S' \in A \otimes C_1A$ such that*

$$L = \overline{P} + \overline{Q} + \overline{Q'} + \overline{R} + \overline{S} + \overline{S'}.$$

The remainder of the paper will be devoted to the proof of this theorem.

Lemma. We first prove an auxiliary result.

LEMMA. *If $n \geq 2$ and $L, \widetilde{L}$ are two linear liftings of symmetric tensor fields of type $(1,2)$ to tensor fields of type $(1,2)$ on T^A, then*

$$L_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right) = \widetilde{L}_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right) \Rightarrow L = \widetilde{L}.$$

Proof. It suffices to show that if

$$L_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right) = 0, \tag{8}$$

then $L = 0$.

From (8) and (1) with $\phi : \mathbb{R}^n \ni x \mapsto (x^1, x^2 + 1, x^3, \ldots, x^n) \in \mathbb{R}^n$ and $t = x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$L_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right) = 0, \tag{9}$$

because $\phi^* t = (x^2 + 1)\frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$.

From (9) and (1) with $\phi : \mathbb{R}^n \ni x \mapsto (x^1 + \lambda x^2, x^2, \ldots, x^n) \in \mathbb{R}^n$, where $\lambda \in \mathbb{R}$, and $t = \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$L_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1} \otimes (dx^1 \otimes dx^2 + dx^2 \otimes dx^1)\right) = 0, \tag{10}$$

$$L_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2\right) = 0, \tag{11}$$

because $\phi^* t = \frac{\partial}{\partial x^1} \otimes (dx^1 + \lambda dx^2) \otimes (dx^1 + \lambda dx^2)$ and because all coefficients of the zero polynomial are zero.

Let $\alpha \in \mathbb{N}^n$. Put $U = \{x \in \mathbb{R}^n : x^1 > 0, x^2 \neq 0, \ldots, x^n \neq 0\}$ and $\psi : U \ni x \mapsto (\psi^1(x), x^2, \ldots, x^n) \in \mathbb{R}^n$, where

$$\psi^1(x) = \begin{cases} \dfrac{(x^1)^{-\alpha^1+1}}{(-\alpha^1+1)(x^2)^{\alpha^2}\ldots(x^n)^{\alpha^n}} & \text{if } \alpha^1 \neq 1, \\[2ex] \dfrac{\ln|x^1|}{(x^2)^{\alpha^2}\ldots(x^n)^{\alpha^n}} & \text{if } \alpha^1 = 1. \end{cases}$$

From (11) and (1) with ψ instead of ϕ and with $t = \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2$ it follows that $L_U\big(x^\alpha \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2|_U\big) = 0$. Replacing U by $V = \{x \in \mathbb{R}^n : x^1 < 0,\, x^2 \neq 0, \dots, x^n \neq 0\}$ in the same manner we can see that $L_V\big(x^\alpha \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2|_V\big) = 0$. Since $U \cup V$ is a dense subset of $\mathbb{R}^n$,

$$L_{\mathbb{R}^n}\left(x^\alpha \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2\right) = 0. \tag{12}$$

If $n \geq 3$, then (10) and (1) with $\phi : \mathbb{R}^n \ni x \mapsto (x^1 + x^3, x^2, \dots, x^n) \in \mathbb{R}^n$ and $t = \frac{\partial}{\partial x^1} \otimes (dx^1 \otimes dx^2 + dx^2 \otimes dx^1)$ imply

$$L_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1} \otimes (dx^3 \otimes dx^2 + dx^2 \otimes dx^3)\right) = 0, \tag{13}$$

because $\phi^* t = \frac{\partial}{\partial x^1} \otimes ((dx^1 + dx^3) \otimes dx^2 + dx^2 \otimes (dx^1 + dx^3))$.

If $n \geq 3$ and $\alpha \in \mathbb{N}^n$, then as in the proof of (12), from (13) and (1) with $\phi = \psi$ and $t = \frac{\partial}{\partial x^1} \otimes (dx^3 \otimes dx^2 + dx^2 \otimes dx^3)$ it follows that

$$L_{\mathbb{R}^n}\left(x^\alpha \frac{\partial}{\partial x^1} \otimes (dx^3 \otimes dx^2 + dx^2 \otimes dx^3)\right) = 0. \tag{14}$$

We now prove that for every $\alpha \in \mathbb{N}^n$,

$$L_{\mathbb{R}^n}\left(x^\alpha \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right) = 0. \tag{15}$$

We consider two cases: $\alpha^2 \neq 0$ and $\alpha^2 = 0$.

If $\alpha^2 \neq 0$, then we obtain (15) from (8), (12) with x^1 and x^2 interchanged and (1) with $\phi : U \ni x \mapsto (x^1, x^\alpha, x^3, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n : x^1 \neq 0,\, x^2 > 0,\, x^3 \neq 0, \dots, x^n \neq 0\}$, and $t = x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$, because $\phi^* t = \big(x^\alpha \frac{\partial}{\partial x^1} - \frac{\alpha^1}{\alpha^2}(x^1)^{\alpha^1 - 1}(x^2)^{\alpha^2+1}(x^3)^{\alpha^3} \dots (x^n)^{\alpha^n} \frac{\partial}{\partial x^2}\big) \otimes dx^1 \otimes dx^1|_U$.

If $\alpha^2 = 0$, then we obtain (15) from (8), (12) with x^1 and x^2 interchanged and (1) with $\phi : \mathbb{R}^n \ni x \mapsto (x^1, x^2 + x^\alpha, x^3, \dots, x^n) \in \mathbb{R}^n$ and $t = x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$, because $\phi^* t = (x^2 + x^\alpha)\big(\frac{\partial}{\partial x^1} - \alpha^1 (x^1)^{\alpha^1 - 1}(x^3)^{\alpha^3} \dots (x^n)^{\alpha^n} \frac{\partial}{\partial x^2}\big) \otimes dx^1 \otimes dx^1$.

From (8), (12) with x^1 and x^2 interchanged and (1) with $\phi : \mathbb{R}^n \ni x \mapsto (x^1, x^1 + x^2, x^3, \dots, x^n) \in \mathbb{R}^n$ and $t = x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$L_{\mathbb{R}^n}\left(x^1 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right) = 0, \tag{16}$$

because $\phi^* t = (x^1 + x^2)\big(\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}\big) \otimes dx^1 \otimes dx^1$.

From (16), (8) and (1) with $\phi : \mathbb{R}^n \ni x \mapsto (x^1 + \lambda x^2, x^2, \dots, x^n) \in \mathbb{R}^n$, where $\lambda \in \mathbb{R}$, and $t = x^1 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$L_{\mathbb{R}^n}\left(x^1 \frac{\partial}{\partial x^1} \otimes (dx^1 \otimes dx^2 + dx^2 \otimes dx^1)\right) = 0, \tag{17}$$

because $\phi^* t = (x^1 + \lambda x^2) \frac{\partial}{\partial x^1} \otimes (dx^1 + \lambda dx^2) \otimes (dx^1 + \lambda dx^2)$.

From (17) and (1) with $\phi : U \ni x \mapsto \big(x^1, \frac{1}{\alpha^2+1}(x^2)^{\alpha^2+1}, x^3, \dots, x^n\big) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n : x^2 > 0\}$ and $\alpha \in \mathbb{N}^n$, and $t = x^1 \frac{\partial}{\partial x^1} \otimes (dx^1 \otimes dx^2 + dx^2 \otimes dx^1)$ it follows that

$$L_{\mathbb{R}^n}\bigg(x^1 (x^2)^{\alpha^2} \frac{\partial}{\partial x^1} \otimes (dx^1 \otimes dx^2 + dx^2 \otimes dx^1)\bigg) = 0. \tag{18}$$

We now prove that for every $\alpha \in \mathbb{N}^n$,

$$L_{\mathbb{R}^n}\bigg(x^\alpha \frac{\partial}{\partial x^1} \otimes (dx^1 \otimes dx^2 + dx^2 \otimes dx^1)\bigg) = 0. \tag{19}$$

We consider two cases: $\alpha^1 \neq 0$ and $\alpha^1 = 0$.

If $\alpha^1 \neq 0$, then (19) can be deduced immediately from (18), (14), (14) with x^3 and x^i, where $i \in \{4, \dots, n\}$, interchanged and (1) with $\phi : U \ni x \mapsto ((x^1)^{\alpha^1} (x^3)^{\alpha^3} \dots (x^n)^{\alpha^n}, x^2, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n : x^1 > 0, x^3 \neq 0, \dots, x^n \neq 0\}$, and $t = x^1 (x^2)^{\alpha^2} \frac{\partial}{\partial x^1} \otimes (dx^1 \otimes dx^2 + dx^2 \otimes dx^1)$, because

$$\phi^* t = \frac{\partial}{\partial x^1} \otimes \bigg(\Big(x^\alpha dx^1 + \sum_{i=3}^n v_i(x) dx^i\Big) \otimes dx^2 + dx^2 \otimes \Big(x^\alpha dx^1 + \sum_{i=3}^n v_i(x) dx^i\Big)\bigg)\bigg|_U,$$

where $v_i(x) = \frac{\alpha^i}{\alpha^1} (x^1)^{\alpha^1+1} (x^2)^{\alpha^2} \dots (x^{i-1})^{\alpha^{i-1}} (x^i)^{\alpha^i - 1} (x^{i+1})^{\alpha^{i+1}} \dots (x^n)^{\alpha^n}$ for every $i \in \{3, \dots, n\}$.

If $\alpha^1 = 0$, then we obtain (19) from (17), (12), (14), (14) with x^3 and x^i, where $i \in \{4, \dots, n\}$, interchanged and (1) with $\phi : \mathbb{R}^n \ni x \mapsto (x^1 + x^\alpha, x^2, \dots, x^n) \in \mathbb{R}^n$ and $t = x^1 \frac{\partial}{\partial x^1} \otimes (dx^1 \otimes dx^2 + dx^2 \otimes dx^1)$, because

$$\phi^* t = (x^1 + x^\alpha) \frac{\partial}{\partial x^1} \otimes \bigg(\Big(dx^1 + \sum_{i=2}^n v_i(x) dx^i\Big) \otimes dx^2 + dx^2 \otimes \Big(dx^1 + \sum_{i=2}^n v_i(x) dx^i\Big)\bigg),$$

where $v_i(x) = \alpha^i (x^2)^{\alpha^2} \dots (x^{i-1})^{\alpha^{i-1}} (x^i)^{\alpha^i - 1} (x^{i+1})^{\alpha^{i+1}} \dots (x^n)^{\alpha^n}$ for every $i \in \{2, \dots, n\}$.

Finally, we can take $\phi : \mathbb{R}^n \ni x \mapsto (x^{\sigma(1)}, \dots, x^{\sigma(n)}) \in \mathbb{R}^n$, where σ is any permutation of the set $\{1, \dots, n\}$, in (1) to conclude from (12), (14), (15) and (19) that $L_{\mathbb{R}^n}\big(x^\alpha \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k\big) = 0$ for every $\alpha \in \mathbb{N}^n$ and all $i, j, k \in \{1, \dots, n\}$. This forces $L_{\mathbb{R}^n} = 0$ according to the Peetre theorem (see [3]), which proves the lemma.

Proof of the classification theorem. Fix a linear lifting L of symmetric tensor fields of type $(1, 2)$ to tensor fields of type $(1, 2)$ on T^A. Clearly, for every $p \in \{1, \dots, n\}$, every open subset U of $\mathbb{R}^n$, every embedding $\phi : U \to \mathbb{R}^n$, every $t \in \operatorname{SyTe} \mathbb{R}^n$, every $X \in T^A U$ and all $Y_1, Y_2 \in A^n$, condition (1) can be rewritten as

$$\text{(20)}\quad \left(T^A \frac{\partial \phi^p}{\partial x^q}(X)\right) L^q_U(\phi^* t)_X(Y_1, Y_2) = L^p_{\mathbb{R}^n}(t)_{T^A\phi(X)}\left(\left(T^A \frac{\partial \phi}{\partial x^q}(X)\right)Y_1^q, \left(T^A \frac{\partial \phi}{\partial x^q}(X)\right)Y_2^q\right).$$

Formula (20) with $\phi : \mathbb{R}^n \ni x \mapsto (\lambda^1 x^1, \ldots, \lambda^n x^n) \in \mathbb{R}^n$, where $\lambda^1, \ldots, \lambda^n \in \mathbb{R} \setminus \{0\}$, and $t = x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ implies

$$\text{(21)}\quad \lambda^1\lambda^2\lambda^p L^p_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = L^p_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_{(\lambda^1 X^1, \ldots, \lambda^n X^n)} ((\lambda^1 Y_1^1, \ldots, \lambda^n Y_1^n), (\lambda^1 Y_2^1, \ldots, \lambda^n Y_2^n)).$$

By continuity, (21) still holds if $\lambda^1, \ldots, \lambda^n \in \mathbb{R}$. Using the homogeneous function theorem (see [3]) together with (21) and keeping in mind that for every $X \in A^n$ the map $L_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X$ is $\mathbb{R}$-bilinear we deduce that there are unique $\mathbb{R}$-trilinear maps $a, b, c : A \times A \times A \to A$ such that

$$\text{(22)}\quad L^1_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = a(X^1, Y_1^1, Y_2^2) + b(X^1, Y_1^2, Y_2^1) + c(X^2, Y_1^1, Y_2^1),$$

there are unique $\mathbb{R}$-trilinear maps $d, e, f : A \times A \times A \to A$ such that

$$\text{(23)}\quad L^2_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = d(X^1, Y_1^2, Y_2^2) + e(X^2, Y_1^1, Y_2^2) + f(X^2, Y_1^2, Y_2^1)$$

and for every $p \in \{3, \ldots, n\}$ there are uniquely determined $\mathbb{R}$-trilinear maps $g^p, h^p, i^p, j^p, k^p, l^p : A \times A \times A \to A$ such that

$$\text{(24)}\quad L^p_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = g^p(X^1, Y_1^2, Y_2^p) + h^p(X^1, Y_1^p, Y_2^2) + i^p(X^2, Y_1^1, Y_2^p) + j^p(X^2, Y_1^p, Y_2^1) + k^p(X^p, Y_1^1, Y_2^2) + l^p(X^p, Y_1^2, Y_2^1).$$

If $q \in \{4, \ldots, n\}$, then from (24) and (20) with $p = 3$, $\phi : \mathbb{R}^n \ni x \mapsto (x^1, x^2, x^q, x^4, \ldots, x^{q-1}, x^3, x^{q+1}, \ldots, x^n) \in \mathbb{R}^n$ and $t = x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$\begin{aligned} g^q(X^1, Y_1^2, Y_2^q) &+ h^q(X^1, Y_1^q, Y_2^2) + i^q(X^2, Y_1^1, Y_2^q) \\ &+ j^q(X^2, Y_1^q, Y_2^1) + k^q(X^q, Y_1^1, Y_2^2) + l^q(X^q, Y_1^2, Y_2^1) \\ = g^3(X^1, Y_1^2, Y_2^q) &+ h^3(X^1, Y_1^q, Y_2^2) + i^3(X^2, Y_1^1, Y_2^q) \\ &+ j^3(X^2, Y_1^q, Y_2^1) + k^3(X^q, Y_1^1, Y_2^2) + l^3(X^q, Y_1^2, Y_2^1). \end{aligned}$$

Therefore $g^q = g^3$, $h^q = h^3$, $i^q = i^3$, $j^q = j^3$, $k^q = k^3$, $l^q = l^3$. Define $g = g^3$, $h = h^3$, $i = i^3$, $j = j^3$, $k = k^3$, $l = l^3$. Thus for every $p \in \{3, \dots, n\}$, we can rewrite (24) as

$$(25)\qquad L^p_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = g(X^1, Y_1^2, Y_2^p) + h(X^1, Y_1^p, Y_2^2) + i(X^2, Y_1^1, Y_2^p) + j(X^2, Y_1^p, Y_2^1) + k(X^p, Y_1^1, Y_2^2) + l(X^p, Y_1^2, Y_2^1).$$

From (25) and (20) with $p = 3$, $U = \{x \in \mathbb{R}^n : x^3 > 0\}$, $\phi : U \ni x \mapsto (x^1, x^2, (x^3)^2/2, x^4, \dots, x^n) \in \mathbb{R}^n$ and $t = x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$(26)\qquad \begin{aligned} &X^3(g(X^1, Y_1^2, Y_2^3) + h(X^1, Y_1^3, Y_2^2) + i(X^2, Y_1^1, Y_2^3) \\ &\quad + j(X^2, Y_1^3, Y_2^1) + k(X^3, Y_1^1, Y_2^2) + l(X^3, Y_1^2, Y_2^1)) \\ &= g(X^1, Y_1^2, X^3Y_2^3) + h(X^1, X^3Y_1^3, Y_2^2) + i(X^2, Y_1^1, X^3Y_2^3) \\ &\quad + j(X^2, X^3Y_1^3, Y_2^1) + k((X^3)^2/2, Y_1^1, Y_2^2) + l((X^3)^2/2, Y_1^2, Y_2^1) \end{aligned}$$

for every $X \in T^A U$. Replacing U by $V = \{x \in \mathbb{R}^n : x^3 < 0\}$ in the same manner we can see that (26) holds for every $X \in T^A V$. Since $U \cup V$ is a dense subset of $\mathbb{R}^n$, (26) holds for every $X \in A^n$. Carrying out polarization if necessary we see from (26) that for all $w, x, y, z \in A$,

$$(27)\qquad wg(x, y, z) = g(x, y, wz),$$

$$(28)\qquad wh(x, y, z) = h(x, wy, z),$$

$$(29)\qquad wi(x, y, z) = i(x, y, wz),$$

$$(30)\qquad wj(x, y, z) = j(x, wy, z),$$

$$(31)\qquad wk(x, y, z) + xk(w, y, z) = k(wx, y, z).$$

From (25) and (20) with $p = 3$, $\phi : \mathbb{R}^n \ni x \mapsto (x^1, x^2+1, x^3, \dots, x^n) \in \mathbb{R}^n$ and $t = x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$(32)\qquad L^3_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = i(1, Y_1^1, Y_2^3) + j(1, Y_1^3, Y_2^1),$$

because $\phi^* t = (x^2 + 1) \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$.

From (32) and (20) with $p = 3$, $\phi : \mathbb{R}^n \ni x \mapsto (x^1 + \lambda x^2, x^2, \dots, x^n) \in \mathbb{R}^n$, where $\lambda \in \mathbb{R}$, and $t = \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$(33)\qquad L^3_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2\right)_X (Y_1, Y_2) = 0,$$

because $\phi^* t = \frac{\partial}{\partial x^1} \otimes (dx^1 + \lambda dx^2) \otimes (dx^1 + \lambda dx^2)$ and because the respective coefficients of two equal polynomials are equal.

From (33) and (20) with $p = 3$, $U = \{x \in \mathbb{R}^n : x^1 \neq 0\}$, $\phi : U \ni x \mapsto (-1/x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ and $t = \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2$ it follows that

$$(34) \qquad L^3_{\mathbb{R}^n}\left((x^1)^2 \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2\right)_X (Y_1, Y_2) = 0.$$

From (34) and (20) with $p = 3$, $\phi : \mathbb{R}^n \ni x \mapsto (x^2, x^1, x^3, \ldots, x^n) \in \mathbb{R}^n$ and $t = (x^1)^2 \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2$ it follows that

$$(35) \qquad L^3_{\mathbb{R}^n}\left((x^2)^2 \frac{\partial}{\partial x^2} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = 0.$$

From (25), (35) and (20) with $p = 3$, $U = \{x \in \mathbb{R}^n : x^1 \neq 0\}$, $\phi : U \ni x \mapsto (x^1, x^1 x^2, x^3, \ldots, x^n) \in \mathbb{R}^n$ and $t = x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$(36) \qquad L^3_{\mathbb{R}^n}\left(x^1 x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2)$$
$$= g(X^1, X^2 Y_1^1 + X^1 Y_1^2, Y_2^3) + h(X^1, Y_1^3, X^2 Y_2^1 + X^1 Y_2^2) + i(X^1 X^2, Y_1^1, Y_2^3)$$
$$+ j(X^1 X^2, Y_1^3, Y_2^1) + k(X^3, Y_1^1, X^2 Y_2^1 + X^1 Y_2^2) + l(X^3, X^2 Y_1^1 + X^1 Y_1^2, Y_2^1),$$

because $\phi^* t = \left(x^1 x^2 \frac{\partial}{\partial x^1} - (x^2)^2 \frac{\partial}{\partial x^2}\right) \otimes dx^1 \otimes dx^1|_U$.

From (25) and (20) with $p = 3$, $U = \{x \in \mathbb{R}^n : x^1 > 0\}$, $\phi : U \ni x \mapsto ((x^1)^2/2, x^2, \ldots, x^n) \in \mathbb{R}^n$ and $t = x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$(37) \qquad L^3_{\mathbb{R}^n}\left(x^1 x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2)$$
$$= g((X^1)^2/2, Y_1^2, Y_2^3) + h((X^1)^2/2, Y_1^3, Y_2^2) + i(X^2, X^1 Y_1^1, Y_2^3)$$
$$+ j(X^2, Y_1^3, X^1 Y_2^1) + k(X^3, X^1 Y_1^1, Y_2^2) + l(X^3, Y_1^2, X^1 Y_2^1).$$

Carrying out polarization if necessary we see from (36) and (37) that for all $w, x, y, z \in A$,

$$(38) \qquad g(w, xy, z) + g(x, wy, z) = g(wx, y, z),$$
$$(39) \qquad h(w, x, yz) + h(y, x, wz) = h(wy, x, z),$$
$$(40) \qquad g(w, xy, z) + i(wx, y, z) = i(x, wy, z),$$
$$(41) \qquad h(w, x, yz) + j(wy, x, z) = j(y, x, wz),$$
$$(42) \qquad k(w, x, yz) = k(w, yx, z),$$
$$(43) \qquad k(w, x, yz) + l(w, yx, z) = 0.$$

Define $R(y, z) = -2k(z, y, 1)$ for all $y, z \in A$. According to (31), we have $xR(y, z) + zR(y, x) = R(y, xz)$ for all $x, y, z \in A$, and so $R \in C_0 A \otimes \operatorname{Der} A$. By (5),

$$(44) \qquad \overline{R}^p_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = \frac{1}{2} R(Y_1^2 Y_2^1 - Y_1^1 Y_2^2, X^p).$$

From (42) it follows that $k(x,y,z) = -\frac{1}{2}R(zy,x)$ for all $x,y,z \in A$ and from (43) it follows that $l = -k$. Combining these with (44) gives, for every $p \in \{1,\dots,n\}$,

$$\overline{R}^p_{\mathbb{R}^n}\left(x^2\,\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X (Y_1,Y_2) = k(X^p,Y_1^1,Y_2^2) + l(X^p,Y_1^2,Y_2^1). \tag{45}$$

Define $S(y,z) = 2g(z,y,1)$ for all $y,z \in A$. According to (38), we have $S(xy,z)+S(zy,x) = S(y,zx)$ for all $x,y,z \in A$, and so $S \in C_1A\otimes A$. By (6),

$$\overline{S}^p_{\mathbb{R}^n}\left(x^2\,\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X (Y_1,Y_2) = \frac{1}{2}\,(S(Y_1^2,X^1) - S(Y_1^1,X^2))Y_2^p. \tag{46}$$

From (27) it follows that $\frac{1}{2}xS(y,z) = g(z,y,x)$ for all $x,y,z \in A$. Combining this with (46) gives, for every $p \in \{1,\dots,n\}$,

$$\overline{S}^p_{\mathbb{R}^n}\left(x^2\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X (Y_1,Y_2) = g(X^1,Y_1^2,Y_2^p) - g(X^2,Y_1^1,Y_2^p). \tag{47}$$

Define $S'(y,z) = 2h(z,1,y)$ for all $y,z \in A$. According to (39), we have $S'(xy,z) + S'(zy,x) = S'(y,zx)$ for all $x,y,z \in A$, and so $S' \in A\otimes C_1A$. By (7),

$$\overline{S'}^p_{\mathbb{R}^n}\left(x^2\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X (Y_1,Y_2) = \frac{1}{2}\,(S'(Y_2^2,X^1) - S'(Y_2^1,X^2))Y_1^p. \tag{48}$$

From (28) it follows that $\frac{1}{2}xS'(y,z) = h(z,x,y)$ for all $x,y,z \in A$. Combining this with (48) gives, for every $p \in \{1,\dots,n\}$,

$$\overline{S'}^p_{\mathbb{R}^n}\left(x^2\,\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X (Y_1,Y_2) = h(X^1,Y_1^p,Y_2^2) - h(X^2,Y_1^p,Y_2^1). \tag{49}$$

Put $L' = L - \overline{R} - \overline{S} - \overline{S'}$ as well as $a' = a-k-h$, $b' = b-l-g$, $c' = c+g+h$, $d' = d-g-h$, $e' = e-k+g$, $f' = f-l+h$, $i' = i+g$ and $j' = j+h$. From (45), (47) and (49) we see that (22), (23) and (25) lead to

$$\begin{aligned} L'^1_{\mathbb{R}^n}&\left(x^2\,\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X (Y_1,Y_2) \\ &= a'(X^1,Y_1^1,Y_2^2) + b'(X^1,Y_1^2,Y_2^1) + c'(X^2,Y_1^1,Y_2^1), \end{aligned} \tag{50}$$

$$\begin{aligned} L'^2_{\mathbb{R}^n}&\left(x^2\,\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X (Y_1,Y_2) \\ &= d'(X^1,Y_1^2,Y_2^2) + e'(X^2,Y_1^1,Y_2^2) + f'(X^2,Y_1^2,Y_2^1), \end{aligned} \tag{51}$$

$$\begin{aligned} L'^p_{\mathbb{R}^n}&\left(x^2\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X (Y_1,Y_2) \\ &= i'(X^2,Y_1^1,Y_2^p) + j'(X^2,Y_1^p,Y_2^1) \end{aligned} \tag{52}$$

for every $p \in \{3,\dots,n\}$. Since (29), (30), (40) and (41) hold for every linear lifting satisfying (22), (23) and (25), upon comparing (22), (23) and (25) with (50), (51) and (52) we can assert that for all $w, x, y, z \in A$,

$$wi'(x, y, z) = i'(x, y, wz), \tag{53}$$

$$wj'(x, y, z) = j'(x, wy, z), \tag{54}$$

$$i'(wx, y, z) = i'(x, wy, z), \tag{55}$$

$$j'(wx, y, z) = j'(x, y, wz). \tag{56}$$

Define $Q(z) = i'(z, 1, 1)$ for every $z \in A$. Of course, $Q \in C_0A \otimes A$. By (3),

$$\overline{Q}^p_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = Q(X^2 Y_1^1) Y_2^p. \tag{57}$$

From (53) and (55) it follows that $xQ(yz) = i'(z, y, x)$ for all $x, y, z \in A$. Combining this with (57) gives, for every $p \in \{1, \dots, n\}$,

$$\overline{Q}^p_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = i'(X^2, Y_1^1, Y_2^p). \tag{58}$$

Define $Q'(z) = j'(z, 1, 1)$ for every $z \in A$. Of course, $Q' \in A \otimes C_0A$. By (4),

$$\overline{Q'}^p_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = Q'(X^2 Y_2^1) Y_1^p. \tag{59}$$

From (54) and (56) it follows that $xQ'(yz) = j'(z, x, y)$ for all $x, y, z \in A$. Combining this with (59) gives, for every $p \in \{1, \dots, n\}$,

$$\overline{Q'}^p_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = j'(X^2, Y_1^p, Y_2^1). \tag{60}$$

Put $L'' = L' - \overline{Q} - \overline{Q'}$ as well as $a'' = a'$, $b'' = b'$, $c'' = c' - i' - j'$, $d'' = d'$, $e'' = e' - i'$ and $f'' = f' - j'$. From (58) and (60) we deduce that (50), (51) and (52) lead to

$$\begin{aligned} L''^1_{\mathbb{R}^n}&\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) \\ &= a''(X^1, Y_1^1, Y_2^2) + b''(X^1, Y_1^2, Y_2^1) + c''(X^2, Y_1^1, Y_2^1), \end{aligned} \tag{61}$$

$$\begin{aligned} L''^2_{\mathbb{R}^n}&\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) \\ &= d''(X^1, Y_1^2, Y_2^2) + e''(X^2, Y_1^1, Y_2^2) + f''(X^2, Y_1^2, Y_2^1), \end{aligned} \tag{62}$$

$$L''^p_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = 0 \tag{63}$$

for every $p \in \{3, \dots, n\}$.

From (63) and (20) with $p=3$, $\phi:\mathbb{R}^n\ni x\mapsto(x^1,x^2+x^3,x^3,\dots,x^n)\in\mathbb{R}^n$ and $t=x^2\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1$ it follows that

$$L''^3_{\mathbb{R}^n}\left(x^3\,\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X(Y_1,Y_2)=0, \tag{64}$$

because $\phi^*t=(x^2+x^3)\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1$.

From (62) and (20) with $p=2$, $\phi:\mathbb{R}^n\ni x\mapsto(x^1,x^3,x^2,x^4,\dots,x^n)\in\mathbb{R}^n$ and $t=x^2\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1$ it follows that

$$\begin{aligned}L''^3_{\mathbb{R}^n}&\left(x^3\,\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1)_X(Y_1,Y_2\right)\\&=d''(X^1,Y_1^3,Y_2^3)+e''(X^3,Y_1^1,Y_2^3)+f''(X^3,Y_1^3,Y_2^1).\end{aligned} \tag{65}$$

Comparing (64) with (65) we see that $d''=0$, $e''=0$ and $f''=0$, which enables us to rewrite (62) as

$$L''^2_{\mathbb{R}^n}\left(x^2\,\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X(Y_1,Y_2)=0. \tag{66}$$

If $\phi:\mathbb{R}^n\ni x\mapsto(x^1,x^2+1,x^3,\dots,x^n)\in\mathbb{R}^n$ and $t=x^2\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1$, then $\phi^*t=(x^2+1)\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1$. Taking successively $p=1,2,3$ and these ϕ and t in (20) and combining this with (61), (66) and (63) respectively we get

$$L''^1_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X(Y_1,Y_2)=c''(1,Y_1^1,Y_2^1), \tag{67}$$

$$L''^2_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X(Y_1,Y_2)=0, \tag{68}$$

$$L''^3_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1\right)_X(Y_1,Y_2)=0. \tag{69}$$

If $\phi:\mathbb{R}^n\ni x\mapsto(x^1+\lambda x^2,x^2,\dots,x^n)\in\mathbb{R}^n$, where $\lambda\in\mathbb{R}$, and $t=\frac{\partial}{\partial x^1}\otimes dx^1\otimes dx^1$, then $\phi^*t=\frac{\partial}{\partial x^1}\otimes(dx^1+\lambda dx^2)\otimes(dx^1+\lambda dx^2)$. Upon taking $p=1,2,3$ (2 before 1) and these ϕ and t in (20) and combining this with (67), (68) and (69) respectively we get

$$L''^1_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1}\otimes dx^2\otimes dx^2\right)_X(Y_1,Y_2)=c''(1,Y_1^2,Y_2^2), \tag{70}$$

$$L''^2_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1}\otimes dx^2\otimes dx^2\right)_X(Y_1,Y_2)=0, \tag{71}$$

$$L''^3_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1}\otimes dx^2\otimes dx^2\right)_X(Y_1,Y_2)=0. \tag{72}$$

From (70) and (20) with $p=1$, $U=\{x\in\mathbb{R}^n:x^1>0\}$, $\phi:U\ni x\mapsto(\ln|x^1|,x^2,\dots,x^n)\in\mathbb{R}^n$ and $t=\frac{\partial}{\partial x^1}\otimes dx^2\otimes dx^2$ it follows that

$$L''^1_{\mathbb{R}^n}\left(x^1 \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2\right)_X (Y_1, Y_2) = X^1 c''(1, Y_1^2, Y_2^2). \tag{73}$$

From (73) and (20) with $p = 1$, $\phi : \mathbb{R}^n \ni x \mapsto (x^2, x^1, x^3, \ldots, x^n) \in \mathbb{R}^n$ and $t = x^1 \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2$ it follows that

$$L''^2_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^2} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = X^2 c''(1, Y_1^1, Y_2^1). \tag{74}$$

From (61), (74), (66) and (20) with $p = 2$, $\phi : \mathbb{R}^n \ni x \mapsto (x^1, \lambda x^1 + x^2, x^3, \ldots, x^n) \in \mathbb{R}^n$, where $\lambda \in \mathbb{R}$, and $t = x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$\begin{aligned} L''^2_{\mathbb{R}^n}&\left(x^1 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) \\ &= -a''(X^1, Y_1^1, Y_2^2) - b''(X^1, Y_1^2, Y_2^1) - c''(X^2, Y_1^1, Y_2^1) + X^2 c''(1, Y_1^1, Y_2^1), \end{aligned} \tag{75}$$

because $\phi^* t = (\lambda x^1 + x^2)\big(\frac{\partial}{\partial x^1} - \lambda \frac{\partial}{\partial x^2}\big) \otimes dx^1 \otimes dx^1$.

From (68) and (20) with $p = 2$, $U = \{x \in \mathbb{R}^n : x^1 > 0\}$, $\phi : U \ni x \mapsto ((x^1)^2/2, x^2, \ldots, x^n) \in \mathbb{R}^n$ and $t = \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1$ it follows that

$$L''^2_{\mathbb{R}^n}\left(x^1 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = 0. \tag{76}$$

Comparing (75) with (76) we see that $a'' = 0$ and $b'' = 0$, which enables us to rewrite (61) as

$$L''^1_{\mathbb{R}^n}\left(x^2 \frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = c''(X^2, Y_1^1, Y_2^1), \tag{77}$$

and that for all $x, y, z \in A$,

$$xc''(1, y, z) = c''(x, y, z). \tag{78}$$

If $\phi : \mathbb{R}^n \ni x \mapsto (x^1, x^2 + \lambda x^3, x^3, \ldots, x^n) \in \mathbb{R}^n$, where $\lambda \in \mathbb{R}$, and $t = \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2$, then $\phi^* t = \frac{\partial}{\partial x^1} \otimes (dx^2 + \lambda dx^3) \otimes (dx^2 + \lambda dx^3)$. Upon taking $p = 1, 2, 3$ (3 before 2) and these ϕ and t in (20) and combining this with (70), (71) and (72) respectively we get

$$\begin{aligned} L''^1_{\mathbb{R}^n}&\left(\frac{\partial}{\partial x^1} \otimes (dx^2 \otimes dx^3 + dx^3 \otimes dx^2)\right)_X (Y_1, Y_2) \\ &= c''(1, Y_1^2, Y_2^3) + c''(1, Y_1^3, Y_2^2), \end{aligned} \tag{79}$$

$$L''^2_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1} \otimes (dx^2 \otimes dx^3 + dx^3 \otimes dx^2)\right)_X (Y_1, Y_2) = 0, \tag{80}$$

$$L''^3_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^1} \otimes (dx^2 \otimes dx^3 + dx^3 \otimes dx^2)\right)_X (Y_1, Y_2) = 0. \tag{81}$$

If $U = \{x \in \mathbb{R}^n : 1 + x^2 x^3 \neq 0\}$, $\phi : U \ni x \mapsto \big(\frac{x^1}{1 + x^2 x^3}, x^2, \ldots, x^n\big) \in \mathbb{R}^n$ and $t = \frac{\partial}{\partial x^1} \otimes (dx^2 \otimes dx^3 + dx^3 \otimes dx^2)$, then $\phi^* t = (1 + x^2 x^3) \frac{\partial}{\partial x^1} \otimes (dx^2 \otimes dx^3$

$+ dx^3 \otimes dx^2)|_U$. Taking $p = 1, 2, 3$ (2 and 3 before 1) and these ϕ and t in (20) and combining this with (79), (80) and (81) respectively we get

$$\text{(82)} \quad L''^1_{\mathbb{R}^n}\left(x^2x^3\,\frac{\partial}{\partial x^1} \otimes (dx^2 \otimes dx^3 + dx^3 \otimes dx^2)\right)_X (Y_1, Y_2) = X^2X^3(c''(1, Y_1^2, Y_2^3) + c''(1, Y_1^3, Y_2^2)).$$

If $\phi : \mathbb{R}^n \ni x \mapsto \left(\frac{x^1}{1+(x^3)^2}, x^2, \ldots, x^n\right) \in \mathbb{R}^n$ and $t = \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2$, then $\phi^* t = (1 + (x^3)^2)\frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2$. Taking $p = 1, 3$ (3 before 1) and these ϕ and t in (20) and combining this with (70) and (72) respectively we get

$$\text{(83)} \quad L''^1_{\mathbb{R}^n}\left((x^3)^2\,\frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2\right)_X (Y_1, Y_2) = (X^3)^2 c''(1, Y_1^2, Y_2^2).$$

From (83) and (20) with $p = 1$, $\phi : \mathbb{R}^n \ni x \mapsto (x^1, x^3, x^2, x^4, \ldots, x^n) \in \mathbb{R}^n$ and $t = (x^3)^2 \otimes \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2$ it follows that

$$\text{(84)} \quad L''^1_{\mathbb{R}^n}\left((x^2)^2\,\frac{\partial}{\partial x^1} \otimes dx^3 \otimes dx^3\right)_X (Y_1, Y_2) = (X^2)^2 c''(1, Y_1^3, Y_2^3).$$

From (83), (82), (84), (70) and (20) with $p = 1$, $U = \{x \in \mathbb{R}^n : x^3 \neq 0\}$, $\phi : U \ni x \mapsto (x^1, x^2x^3, x^3, \ldots, x^n) \in \mathbb{R}^n$ and $t = \frac{\partial}{\partial x^1} \otimes dx^2 \otimes dx^2$ it follows that

$$\text{(85)} \quad (X^3)^2 c''(1, Y_1^2, Y_2^2) + X^2X^3(c''(1, Y_1^2, Y_2^3) + c''(1, Y_1^3, Y_2^2)) + (X^2)^2 c''(1, Y_1^3, Y_2^3) = c''(1, X^3Y_1^2 + X^2Y_1^3, X^3Y_2^2 + X^2Y_2^3),$$

because $\phi^* t = \frac{\partial}{\partial x^1} \otimes (x^3dx^2 + x^2dx^3) \otimes (x^3dx^2 + x^2dx^3)|_U$. From (85) we see that for all $w, x, y, z \in A$,

$$\text{(86)} \quad wxc''(1, y, z) = c''(1, wy, xz).$$

Define $P = c''(1, 1, 1)$. Of course, $P \in A$. By (2),

$$\text{(87)} \quad \overline{P}^p_{\mathbb{R}^n}\left(x^2\,\frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = \begin{cases} PX^2Y_1^1Y_2^1 & \text{if } p = 1, \\ 0 & \text{if } p \in \{2, \ldots, n\}. \end{cases}$$

From (86) and (78) it follows that $xyzP = c''(x, y, z)$ for all $x, y, z \in A$. Combining this with (87) gives

$$\text{(88)} \quad \overline{P}^p_{\mathbb{R}^n}\left(x^2\,\frac{\partial}{\partial x^1} \otimes dx^1 \otimes dx^1\right)_X (Y_1, Y_2) = \begin{cases} c''(X^2, Y_1^1, Y_2^1) & \text{if } p = 1, \\ 0 & \text{if } p \in \{2, \ldots, n\}. \end{cases}$$

From (88) and the lemma we see that (77), (66) and (63) lead to $L'' = \overline{P}$. Analyzing (44), (46), (48), (57), (59) and (87), one easily finds that R, S, S', Q, Q' and P are uniquely determined, which completes the proof.

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