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An example of a pseudoconvex domain whose holomorphic sectional curvature of the Bergman metric is unbounded

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Abstract. Let a and m be positive integers such that $2a < m$. We show that in the domain $D := \{z \in \mathbb{C}^3 \mid r(z) := \operatorname{Re} z_1 + |z_1|^2 + |z_2|^{2m} + |z_2 z_3|^{2a} + |z_3|^{2m} < 0\}$ the holomorphic sectional curvature $R_D(z; X)$ of the Bergman metric at z in direction X tends to $-\infty$ when z tends to 0 non-tangentially, and the direction X is suitably chosen. It seems that an example with this feature has not been known so far.

1. Introduction. The unit disc, equipped with the Poincaré metric, is a first example of a domain with a metric of constant negative curvature. The generalization to the unit ball in $\mathbb{C}^n$ with $n \geq 2$ is given by the Bergman metric. Its holomorphic sectional curvature R is also a negative constant. By a result of Lu Qi-Keng [17] the ball is the only simply connected domain (up to biholomorphic equivalence) whose Bergman metric has negative constant holomorphic curvature (see also [9]).

Since the results of [5] and [12] it has become possible to determine, on a bounded strongly pseudoconvex domain D, the boundary behavior of the holomorphic sectional curvature $R_D(z; X)$ for $(z, X) \in D \times \mathbb{C}^n$. For a C^∞-smooth strongly pseudoconvex domain Klembeck [15] has shown, by means of the Fefferman asymptotic formula for the Bergman kernel function, that for any tangent vector $X \neq 0$ the quantity $R_D(z; X)$ tends to $-2/(n + 1)$ when z tends to the boundary. His smoothness assumption was considerably weakened later in [14].

Since the investigations of Bergman [1] it has been known that the holomorphic sectional curvature of the Bergman metric is always less than 2. That it is bounded from below is known in the class of strongly pseudoconvex domains (this is obvious) and also (by [18]) in smooth bounded pseudoconvex domains of finite type in $\mathbb{C}^2$. In [8] the case of smoothly bounded

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Reinhardt domains of finite type in $\mathbb{C}^2$ was treated. The holomorphic sectional curvature of the Bergman metric in such domains, in a neighborhood of the boundary, can be estimated from above by a negative constant.

In the present note we give an example of a smooth bounded pseudoconvex Reinhardt domain D of finite type in $\mathbb{C}^3$ such that the holomorphic sectional curvature $R_D(z;X)$ of the Bergman metric is not bounded from below in certain directions X. The idea and the kind of argument used are completely in the spirit of [10, 11] (see also [7]).

Theorem 1.1. *Let a, m be positive integers such that $2a < m$. Let $D := \{z \in \mathbb{C}^3 \mid r(z) := \operatorname{Re} z_1 + |z_1|^2 + P(z') < 0\}$, where $P(z') := P(z_2, z_3) := |z_2|^{2m} + |z_2 z_3|^{2a} + |z_3|^{2m}$. Then the holomorphic sectional curvature $R_D(-te_1; X)$ tends to $-\infty$ as $t \searrow 0$ if $X = (0, X') \in \{0\} \times \mathbb{C}^2$ and $X' = (X_2, X_3)$ with $X_2, X_3 \neq 0$. Here $e_1 = (1,0,0)$. More precisely,*

$$2 - R_D(-te_1;(0,X')) \approx \frac{1}{\log(1/t)}\left(1 + \frac{1}{t^{1/a-2/m}\log(1/t)}\,\frac{|X_2|^2|X_3|^2}{|X'|^4}\right).$$

The above domain is a Reinhardt domain with center at $\zeta_0 = -\frac{1}{2}e_1$. A phenomenon as described in the theorem is not possible in domains all of whose boundary points are of finite semiregular type (see [2] or [16]). The notion of semiregular type was defined in [6] (see also [19]). A point ζ in a smooth hypersurface M is said to be of (*finite*) *semiregular type* if the D'Angelo type $\Delta_1(\zeta, M)$ of M at ζ is finite, and the n-tuple $(1, \Delta_{n-1}(\zeta, M), \ldots, \Delta_1(\zeta, M))$ of the D'Angelo higher type numbers equals the Catlin multitype $(1, m_2, \ldots, m_n)$ of M at ζ. In dimension 3 a point $\zeta \in M$ is of semiregular type if the D'Angelo type at ζ is finite and equal to the entry m_3 of the Catlin multitype.

In our domain the assumption $2a < m$ implies $1/a - 2/m > 0$. It prevents the origin from being a point of (finite) semiregular type. Indeed, the Catlin multitype of ∂D at 0 is $(1, 4a, 4a)$, while the D'Angelo type at this point is $2m$.

Stimulation for this article came from the paper [4], where for the first time an example of a domain Ω was given in which the holomorphic sectional curvature $R_\Omega(z;X)$ of the Bergman metric tends to 2 as z tends to a certain boundary point of D and the direction X is suitably chosen. Also, in [4] it was asked whether there exists a bounded pseudoconvex domain whose holomorphic sectional curvature with respect to the Bergman metric is unbounded.

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2. Proof of Theorem 1.1

The relevant domain functionals. Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain. We denote by $H^2(\Omega)$ the Hilbert space of all holomorphic functions on Ω that are square-integrable with respect to the Lebesgue measure. Put

$$\|f\|_\Omega^2 := \int_\Omega |f|^2 \, d^{2n}z.$$

For $z \in \Omega$ we consider the following subsets of $H^2(\Omega)$:

$$\begin{aligned}
\mathcal{A}_0(\Omega) &:= \{f \in H^2(\Omega) \mid \|f\|_\Omega \le 1\},\\
\mathcal{A}_1(\Omega; z) &:= \{f \in \mathcal{A}_0(\Omega) \mid f(z) = 0\},\\
\mathcal{A}_2(\Omega; z) &:= \left\{f \in \mathcal{A}_1(\Omega) \,\middle|\, \frac{\partial f(z)}{\partial z_j} = 0,\ j = 1, \dots, n\right\}.
\end{aligned}$$

Then we have the well-known relationships between the Bergman kernel $K_\Omega : \Omega \to \mathbb{R}$, the Bergman metric $B_\Omega^2(z; X)$ at z in direction X, and the holomorphic sectional curvature $R_\Omega(z; X)$ of the Bergman metric for $(z, X) \in \Omega \times \mathbb{C}^n$:

(a) $$K_\Omega(z) = \sup\{|f(z)|^2 \mid f \in \mathcal{A}_0(\Omega)\},$$

(b) $$B_\Omega^2(z; X) = \frac{J_{1,\Omega}(z; X)}{K_\Omega(z)},$$

(c) $$2 - R_\Omega(z; X) = \frac{K_\Omega(z) J_{2,\Omega}(z; X)}{J_{1,\Omega}(z; X)^2},$$

where

$$\begin{aligned}
J_{1,\Omega}(z; X) &:= \sup\{|X(f)(z)|^2 \mid f \in \mathcal{A}_1(\Omega; z)\},\\
J_{2,\Omega}(z; X) &:= \sup\{|XX(f)(z)|^2 \mid f \in \mathcal{A}_2(\Omega; z)\}.
\end{aligned}$$

For a vector $X := (X_1, \dots, X_n) \in \mathbb{C}^n$ we denote by $X(f)(z)$ the derivative of f at z in direction X, explicitly $X(f)(z) := \sum_{j=1}^n \frac{\partial f(z)}{\partial z_j} X_j$, and $XX(f)(z) = \sum_{j,k=1}^n \frac{\partial^2 f(z)}{\partial z_j \partial z_k} X_j X_k$.

Splitting off the z_1-direction. For $s > 0$ we put $D_s := \{z' \in \mathbb{C}^2 \mid P(z') < s\}$. Note that for $0 < t < 1/10$ the domain

$$\widetilde{D}_t := \Delta\left(-t, \frac{t - t^2}{2}\right) \times D_{(t-t^2)/4}$$

is contained in D.

The following comparison lemma is needed for the proof of Lemma 2.2 below. Its proof is based on a standard $\overline{\partial}$-argument. The idea is in the spirit of [3, Sec. 6].

LEMMA 2.1. *There exists a constant $C > 0$ such that, for any $0 < t < 1/10$ and any function $f \in H^2(\widetilde{D}_t)$, we can find a function $\widehat{f} \in H^2(D)$ such that $\|\widehat{f}\|_D \le C\|f\|_{\widetilde{D}_t}$, and $\widehat{f} - f$ vanishes to third order at the point $-te_1$.*

Proof. STEP 1: We construct a plurisubharmonic weight function that fits well the size of $\widetilde{D}_t$.

First we note that

$$-2t \le r(z) \le -t/4 \quad \text{on } \widetilde{D}_t.$$

Let

$$W_t(z) := \frac{|z_1 + t|^2}{t^2} + \frac{1}{t}\, P(z').$$

If now $z \in \mathbb{C}^3$ and $W_t(z) \le 1/5$, then $|z_1 + t| < t/\sqrt{5} < (t - t^2)/2$ when $t < 1/10$. Because also $P(z') < t/5 < (t - t^2)/4$, we see that $z \in \widetilde{D}_t$. The Levi form of the plurisubharmonic function $\psi_t := e^{r/t}$ can be estimated on the set $\{W_t \le 1/5\}$ by

$$\begin{aligned}
\mathscr{L}_{\psi_t}(z; X) &= e^{r/t}\biggl(\frac{|X_1|^2}{t} + \frac{1}{t}\,\mathscr{L}_P(z'; X') + \biggl|\frac{(1+2\overline{z}_1)X_1}{2t} + \frac{\langle \partial P(z'), X'\rangle}{t}\biggr|^2\biggr) \\
&\ge e^{r/t}\biggl(\frac{1}{t}\,\mathscr{L}_P(z'; X') + \frac{|(1+2\overline{z}_1)X_1|^2}{8t^2} - 2\,\frac{|\langle \partial P(z'), X'\rangle|^2}{t^2}\biggr) \\
&\ge e^{r/t}\biggl(\frac{1}{t}\,\mathscr{L}_P(z'; X') + \frac{|(1-3t)X_1|^2}{8t^2} - 2\,\frac{|\langle \partial P(z'), X'\rangle|^2}{t^2}\biggr) \\
&\ge e^{r/t}\biggl(\frac{1}{t}\,\mathscr{L}_P(z'; X') + \frac{49|X_1|^2}{800t^2} - 2\,\frac{P(z')\mathscr{L}_P(z'; X')}{t^2}\biggr) \\
&\ge e^{r/t}\biggl(\frac{1}{t}\biggl(1 - 2\,\frac{P(z')}{t}\biggr)\mathscr{L}_P(z'; X') + 0.06\,\frac{|X_1|^2}{t^2}\biggr) \\
&\ge e^{-2}\biggl(0.06\,\frac{|X_1|^2}{t^2} + \frac{3}{5t}\,\mathscr{L}_P(z'; X')\biggr) \ge 0.06e^{-2}\mathscr{L}_{W_t}(z; X).
\end{aligned}$$

The estimate $|\langle \partial P(z'), X'\rangle|^2 \le P(z')\mathscr{L}_P(z'; X')$ follows from the fact that $\log P$ is plurisubharmonic.

Next we choose an increasing smooth function $\chi : \mathbb{R} \to (-\infty, 1]$ with

$$\chi(s) = \begin{cases} s & \text{for } s \le 1/10, \\ 1 & \text{for } s \ge 3/20, \end{cases}$$

and put $V_t := M\psi_t + \log \chi \circ W_t$. It is possible to choose M independently of t in such a way that V_t becomes plurisubharmonic throughout D, and moreover,

$$\mathscr{L}_{V_t} \ge \gamma \mathscr{L}_{W_t}$$

on the set $\{W \le 1/5\}$ for some constant $\gamma > 0$ that does not depend on t.

To see this, we note that

$$\mathscr{L}_{V_t}(z;X) = M\mathscr{L}_{\psi_t}(z;X) + (\log\chi)' \circ W_t \cdot \mathscr{L}_{W_t}(z;X) \\ + (\log\chi)'' \circ W_t \cdot |\langle \partial W_t(z), X\rangle|^2$$

for $(z,X) \in D \times \mathbb{C}^3$.

There are three cases to be considered:

(i) If $W_t(z) \in [0, 1/10]$, we have ($\log W_t$ is p.s.h.)

$$\mathscr{L}_{V_t}(z;X) = M\mathscr{L}_{\psi_t}(z;X) + \mathscr{L}_{\log W_t}(z;X) \\ \geq M\mathscr{L}_{\psi_t}(z;X) \geq 0.06e^{-2}M\mathscr{L}_{W_t}(z;X).$$

(ii) If $1/5 \geq W_t(z) \geq 3/20$, we have

$$\mathscr{L}_{V_t}(z;X) = M\mathscr{L}_{\psi_t}(z;X) \geq 0.06e^{-2}M\mathscr{L}_{W_t}(z;X)\,.$$

(iii) Assume that $1/10 \leq W_t(z) \leq 3/20$. With some constant $C_3 > 0$, we can estimate $(\log\chi)'(s), (\log\chi)''(s) \geq -C_3$ for $s \in [1/10, 3/20]$. This in conjunction with the log-plurisubharmonicity of P gives

$$\begin{aligned}\mathscr{L}_{V_t}(z;X) &\geq M\mathscr{L}_{\psi_t}(z;X) - C_3\mathscr{L}_{W_t}(z;X) - C_3|\langle \partial W_t(z), X\rangle|^2 \\ &\geq M\mathscr{L}_{\psi_t}(z;X) - C_3(1 + W_t(z))\mathscr{L}_{W_t}(z;X) \\ &\geq M\mathscr{L}_{\psi_t}(z;X) - 2C_3\mathscr{L}_{W_t}(z;X) \\ &\geq (0.06e^{-2}M - 2C_3)\mathscr{L}_{W_t}(z;X).\end{aligned}$$

We can now choose $M > 200C_3e^2$ and $\gamma = C_3$.

Step 2: Let $\xi \in C^\infty(\mathbb{R})$ be a non-negative cut-off function satisfying $\xi(s) = 1$ for $s \leq 1/10$ and $\chi(s) = 0$ if $s \geq 1/5$. Given a function $f \in H^2(\widetilde{D}_t)$ we define the $\overline{\partial}$-closed smooth $(0,1)$-form

$$v = \overline{\partial}(\xi \circ W_t) \cdot f = \xi' \circ W_t \cdot f \cdot \overline{\partial} W_t.$$

This form is defined on D. Measuring its length with respect to the hermitian form $Q := \mathscr{L}_{|z|^2 + 10V_t}$, we will show that

$$(*) \qquad |v|_Q^2 e^{-|z|^2 - 10V_t} \leq C_1|f|^2,$$

with an unimportant constant $C_1 > 0$ (uniformly in t), where

$$|v|_Q^2 := \sum_{j,k=1}^{n} Q^{j\bar{k}} v_j \overline{v}_k$$

and $Q^{j\bar{k}}$ are the coefficients of the inverse of the matrix associated to Q and the v_j are the coefficients of v.

Namely, on $\mathrm{supp}(v) \subset \{1/10 \leq W_t \leq 1/5\}$ we have

$$\mathscr{L}_{W_t} \geq \frac{1}{W_t}\,\partial W_t \otimes \overline{\partial} W_t \geq 5\partial W_t \otimes \overline{\partial} W_t$$

and

$$Q = \mathscr{L}_{|z|^2} + 10\mathscr{L}_{V_t} \geq \mathscr{L}_{|z|^2} + 10\gamma\mathscr{L}_{W_t} \geq \mathscr{L}_{|z|^2} + 50\gamma\partial W_t \otimes \overline{\partial} W_t,$$

hence

$$|v|_Q^2 = (\xi' \circ W_t)^2 |f|^2 |\overline{\partial} W_t|_Q^2 \le \frac{1}{50\gamma} (\xi' \circ W_t)^2 |f|^2.$$

Because of the monotonicity of χ the function V_t is bounded from below on $\operatorname{supp}(v)$ by

$$V_t = M\psi_t + \log \chi \circ W_t \ge \log(1/10).$$

This implies

$$|v|_Q^2 e^{-|z|^2 - 10V_t} \le \frac{10^{10}}{50\gamma} (\xi' \circ W_t)^2 |f|^2,$$

which is $(*)$ with

$$C_1 = \frac{10^{10}}{50\gamma} \max(\xi' \circ W_t)^2.$$

A refinement in the proof of Lemma 4.4.1 from [13] (which is by now standard) gives us a smooth solution u to the equation $\overline{\partial} u = v$ that satisfies

$$(L^2) \qquad \int_D |u|^2 e^{-|z|^2 - 10V_t} \, d^6 z \le \int_D |v|_Q^2 e^{-|z|^2 - 10V_t} \, d^6 z \le C_1 \|f\|_{\widetilde{D}_t}^2.$$

Since $V_t \le M$, we have $\int_D |u|^2 \, d^6 z \le C_1 e^{2(\operatorname{diam} D)^2 + 10M} \|f\|_{\widetilde{D}_t}^2$, where $\operatorname{diam} D$ is the diameter of D. Furthermore, because of the term $10V_t$ in the weight that appears on the left-hand side of (L^2), the function u vanishes to third order at the point $-te_1$. Then the holomorphic function

$$\widehat{f} := \xi \circ W_t \cdot f - u$$

satisfies $\|\widehat{f}\|_{L^2(D)} \le C\|f\|_{\widetilde{D}_t}$, with $C := 1 + \sqrt{C_1} e^{(\operatorname{diam} D)^2 + 5M}$, and hence fulfills our requirements.

Lemma 2.1 is proved.

Corollary.

(i) *For any function* $f \in \mathcal{A}_0(\widetilde{D}_t)$ *let* $\widehat{f}$ *be the function from Lemma* 2.1. *Then* $\widehat{f}/C \in \mathcal{A}_0(D)$, *and*

$$K_D(-te_1) \ge C^{-2} K_{\widetilde{D}_t}(-te_1).$$

(ii) *Likewise, for* $k \in \{1,2\}$ *and any function* $f \in \mathcal{A}_k(\widetilde{D}_t; -te_1)$, *the function* $\widehat{f}/C$ *belongs to the family* $\mathcal{A}_k(D; -te_1)$, *and for all* $X \in \mathbb{C}^3$ *we have*

$$J_{k,D}(-te_1; X) \ge C^{-2} J_{k,\widetilde{D}_t}(-te_1; X), \qquad k = 1, 2.$$

This enables us to give lower and upper estimates for the domain functionals in formulas (a), (b), and (c) at the point $-te_1$ by the corresponding domain functionals (evaluated at $0'$) that belong to the domains D_{t-t^2}.

LEMMA 2.2. *With a suitable constant $C_* > 0$, for $0 < t < 1/10$ we have*

(1) $$\frac{1}{C_* t^2} K_{D_{(t-t^2)/4}}(0') \le K_D(-te_1) \le \frac{5}{\pi t^2} K_{D_{(t-t^2)/4}}(0'),$$

and for $X' \in \mathbb{C}^2$ and $k = 1, 2$ we have

(2) $$\frac{1}{C_* t^2} J_{k,D_{(t-t^2)/4}}(0'; X') \le J_{k,D}(-te_1; (0, X')) \le \frac{5}{\pi t^2} J_{k,D_{(t-t^2)/4}}(0'; X')$$

and

(3) $$\frac{1}{C_*} \frac{K_{D_{(t-t^2)/4}}(0') J_{2,D_{(t-t^2)/4}}(0'; X')}{J_{1,D_{(t-t^2)/4}}(0'; X')^2} \le 2 - R_D(-te_1; (0, X')) \le C_* \frac{K_{D_{(t-t^2)/4}}(0') J_{2,D_{(t-t^2)/4}}(0'; X')}{J_{1,D_{(t-t^2)/4}}(0'; X')^2}.$$

Proof. The fact that $\widetilde{D}_t \subset D$ for $0 < t < 1/10$ together with Lemma 2.1 and its corollary gives

$$\frac{1}{C^2} K_{\widetilde{D}_t}(-te_1) \le K_D(-te_1) \le K_{\widetilde{D}_t}(-te_1)$$

and

$$\frac{1}{C^2} J_{k,\widetilde{D}_t}(-te_1; (0, X')) \le J_{k,D}(-te_1; (0, X')) \le J_{k,\widetilde{D}_t}(-te_1; (0, X')), \qquad k = 1, 2,$$

for $0 < t < 1/10$ and $X' \in \mathbb{C}^2$. We use the product formulas

$$K_{\widetilde{D}_t}(-te_1) = K_{\Delta(-t,(t-t^2)/2)}(-t) K_{D_{(t-t^2)}/4}(0')$$

and

$$J_{k,\widetilde{D}_t}(-te_1; (0, X')) = K_{\Delta(-t,(t-t^2)/2)}(-t) J_{k,D_{(t-t^2)/4}}(0'; X'), \qquad k = 1, 2.$$

The desired upper and lower bounds now follow from the estimate

$$\frac{4}{\pi} \frac{1}{t^2} \le K_{\Delta(-t,(t-t^2)/2)}(-t) = \frac{4}{\pi} \frac{1}{(t - t^2)^2} \le \frac{5}{\pi t^2},$$

which holds for $0 < t < 1/10$.

Estimate (3) follows from (1) and (2) in conjunction with formula (c), stated at the beginning of this section. This proves Lemma 2.2.

Estimation of the single domain functionals of D_s at $0'$. For functions $f, g : (0, \alpha) \to (0, \infty)$ we write $f \approx g$ if there is a constant $C > 0$ such that $C^{-1} g(t) \le f(t) \le C g(t)$ for all $t \in (0, \alpha)$.

LEMMA 2.3. *For $1/10 > s > 0$ we have*

(4) $$K_{D_s}(0') \approx \frac{1}{s^{1/a} \log(1/s)}.$$

For $X' \in \mathbb{C}^2 \setminus \{0\}$ *we have*

$$J_{1,D_s}(0';X') \approx \frac{|X'|^2}{s^{1/m+1/a}}, \tag{5}$$

$$J_{2,D_s}(0';X') \approx \frac{|X'|^4}{s^{2/m+1/a}} + \frac{|X_2|^2|X_3|^2}{s^{2/a}\log(1/s)}. \tag{6}$$

Proof. Since D_s is a Reinhardt domain, we have

$$K_{D_s}(0') = \frac{1}{\mathrm{Vol}(D_s)} \tag{4'}$$

as well as

$$J_{1,D_s}(0';X') = \sum_{j=2}^{3} \frac{|X_j|^2}{\|f_j\|^2_{D_s}} = \frac{|X'|^2}{\|f_2\|^2_{D_s}}, \tag{5'}$$

where $f_j(z') := z_j$ for $j = 2, 3$, and

$$\begin{aligned} J_{2,D_s}(0';X') &= 4\,\frac{|X_2|^4}{\|f_2^2\|^2_{D_s}} + 4\,\frac{|X_3|^4}{\|f_3^2\|^2_{D_s}} + \frac{|X_2|^2|X_3|^2}{\|f_2f_3\|^2_{D_s}} \\ &= 4\,\frac{|X_2|^4+|X_3|^4}{\|f_2^2\|^2_{D_s}} + \frac{|X_2|^2|X_3|^2}{\|f_2f_3\|^2_{D_s}}. \end{aligned} \tag{6'}$$

Here we use the symmetry of D_s.

We have to estimate $\|f_2^\nu\|^2_{D_s}$ for $\nu = 0, 1, 2$ and also $\|f_2f_3\|^2_{D_s}$. Consider the domains

$$\widehat{D}_s := \left\{ z' \in \mathbb{C}^2 \;\middle|\; |z_2| < s^{1/2m},\ |z_3| < \frac{1}{s^{-1/2m} + s^{-1/2a}|z_2|} \right\}.$$

Then for $0 < s < 1/10$ one has

$$\widehat{D}_{s/4} \subset D_s \subset \widehat{D}_{4^m s}. \tag{7}$$

This reduces our task to estimating $\|f_2^\nu\|^2_{\widehat{D}_\sigma}$ for $\nu = 0, 1, 2$ and $\|f_2f_3\|^2_{\widehat{D}_\sigma}$. We have

$$\begin{aligned} \|f_2^\nu\|^2_{\widehat{D}_\sigma} &= \int\limits_{\widehat{D}_\sigma} |z_2|^{2\nu} d^4z' = \pi \int\limits_{|z_2|<\sigma^{1/2m}} \frac{|z_2|^{2\nu} d^2z_2}{(\sigma^{-1/2m} + \sigma^{-1/2a}|z_2|)^2} \\ &= 2\pi^2 \int_0^{\sigma^{1/2m}} \frac{r^{2\nu+1}\,dr}{(\sigma^{-1/2m} + \sigma^{-1/2a}r)^2} \\ &= 2\pi^2\sigma^{1/m} \int_0^{\sigma^{1/2m}} \frac{r^{2\nu+1}\,dr}{(1 + \sigma^{1/2m-1/2a}r)^2} \\ &= 2\pi^2\sigma^{1/m+(\nu+1)(1/a-1/m)} K_\nu(\sigma), \end{aligned}$$

(substitute $r =: \sigma^{1/2a-1/2m}\varrho$) with

$$K_\nu(\sigma) := \int\limits_0^{\sigma^{1/m-1/2a}} \frac{\varrho^{2\nu+1}\,d\varrho}{(1+\varrho)^2} = \int\limits_0^1 \frac{\varrho^{2\nu+1}\,d\varrho}{(1+\varrho)^2} + \int\limits_1^{\sigma^{1/m-1/2a}} \frac{\varrho^{2\nu+1}\,d\varrho}{(1+\varrho)^2}$$
$$\approx \int\limits_1^{\sigma^{1/m-1/2a}} \frac{\varrho^{2\nu+1}\,d\varrho}{(1+\varrho)^2}.$$

Note that $\sigma^{1/m-1/2a} > 1$ if $0 < \sigma < 1$, since $2a < m$.

But

$$\int\limits_1^{\sigma^{1/m-1/2a}} \frac{\varrho^{2\nu+1}\,d\varrho}{(1+\varrho)^2} \approx \int\limits_1^{\sigma^{1/m-1/2a}} \varrho^{2\nu-1} d\varrho \approx \begin{cases} \sigma^{2\nu/m-\nu/a} & \text{if } \nu > 0, \\ \log(1/\sigma) & \text{if } \nu = 0. \end{cases}$$

Thus we obtain

$$\|f_2^\nu\|^2_{\widehat{D}_\sigma} \approx \begin{cases} \sigma^{\nu/m+1/a} & \text{if } \nu \geq 1, \\ \sigma^{1/a}\log(1/\sigma) & \text{if } \nu = 0. \end{cases}$$

For $\nu = 0, 1$ this together with (4′) and (5′) implies (4) and (5). If we let $\nu = 2$, we find

$$4\,\frac{|X_2|^4}{\|f_2^2\|^2_{D_s}} + 4\,\frac{|X_3|^4}{\|f_3^2\|^2_{D_s}} \approx \frac{|X'|^4}{s^{2/m+1/a}}.$$

Next we check the estimate for the norm $\|f_2 f_3\|_{\widehat{D}_\sigma}$. Similarly to the above we compute

$$\begin{aligned}\|f_2f_3\|^2_{\widehat{D}_\sigma} &= \int\limits_{|z_2|<\sigma^{1/2m}} |z_2|^2\Big(\int\limits_{|z_3|<(\sigma^{-1/2m}+\sigma^{-1/2a}|z_2|)^{-1}} |z_3|^2\,d^2z_3\Big)d^2z_2 \\ &= 2\pi \int\limits_{|z_2|<\sigma^{1/2m}} |z_2|^2\Big(\int\limits_0^{(\sigma^{-1/2m}+\sigma^{-1/2a}|z_2|)^{-1}} r^3\,dr\Big)d^2z_2 \\ &= \frac{\pi}{2}\int\limits_{|z_2|<\sigma^{1/2m}} |z_2|^2(\sigma^{-1/2m}+\sigma^{-1/2a}|z_2|)^{-4}d^2z_2 \\ &= \pi^2\int\limits_0^{\sigma^{1/2m}} r^3(\sigma^{-1/2m}+\sigma^{-1/2a}r)^{-4}\,dr \\ &= \pi^2\sigma^{2/m}\int\limits_0^{\sigma^{1/2m}} r^3(1+\sigma^{1/2m-1/2a}r)^{-4}\,dr \\ &= \pi^2\sigma^{2/a}\int\limits_0^{\sigma^{1/m-1/2a}} \frac{\varrho^3\,d\varrho}{(1+\varrho)^4} \qquad (r =: \sigma^{1/2a-1/2m}\varrho)\end{aligned}$$

$$\approx \sigma^{2/a} \int_0^{\sigma^{1/m-1/2a}} \frac{\varrho^3\, d\varrho}{1+\varrho^4} = \sigma^{2/a} \int_0^{\sigma^{4/m-2/a}} \frac{d\varrho'}{1+\varrho'}$$
$$\approx \sigma^{2/a} \log(1/\sigma).$$

This in conjunction with (6′) gives (6), and therefore proves the lemma.

End of the proof of Theorem 1.1. We only need to choose $s = (t - t^2)/4$ and combine formulas (3) and (4) through (6).

Theorem 1.1 is proved.

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