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A symmetry problem

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Abstract. Consider the Newtonian potential of a homogeneous bounded body $D \subset \mathbb{R}^3$ with known constant density and connected complement. If this potential equals $c/|x|$ in a neighborhood of infinity, where $c > 0$ is a constant, then the body is a ball. This known result is now proved by a different simple method. The method can be applied to other problems.

1. Introduction. Consider a bounded domain $D \subset \mathbb{R}^3$ with a connected complement and $C^{1,\lambda}$-smooth boundary S. The smoothness assumptions on S can be weakened, but this is not the point of this paper. Let B_R be a ball of radius R, containing D, and B'_R be its complement in $\mathbb{R}^3$. We denote by S^2 a unit sphere, and by ℓ a unit vector. Let N be the outer unit normal to S. Denote by χ the characteristic function of D, and by $\mathcal{N}$ the set of harmonic functions in B_R. Let the center O of B_R be the origin, and suppose it lies at the center of mass of D.

Consider the Newtonian potential

$$u(x) := \int_D \frac{dy}{|x-y|}, \tag{1}$$

where we have assumed that the density of the mass distribution in D is 1.

Assume that

$$u(x) = c|x|^{-1} \qquad \text{in } B'_R,$$

where $c = \text{const}$. Then the question is:

Does this imply that D is a ball?

It is well known and easy to prove that if D is a ball B_a of radius a, then $u(x) = c|x|^{-1}$ in B'_a, and $c = |B_a|$, where $|B_a|$ is the volume of this ball, $|B_a| = 4\pi a^3/3$, so $a = (3c/4\pi)^{1/3}$. In [1] and [5] one can find different proofs of the fact that the answer to the above question is yes. An especially

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simple proof, due to D. Zagier, is given at the end of this paper (see also review [5], where one can find this proof).

Our goal is to give a simple new proof of this result by a method which can be used in other problems (see, e.g., [4]). The literature on potential theory and inverse potential theory is quite large, and we only mention a few references [1]–[3], [5], where the reader can find additional bibliography, and [4], where an argument similar to the one we use was applied to the study of the Pompeiu problem.

We do not attempt to make the weakest assumption about the smoothness of the boundary of D. In Zagier's proof, given at the end of this paper, no smoothness of the boundary is assumed.

We prove the following theorem:

THEOREM 1. *Under the above assumptions, if* $u(x) = c|x|^{-1}$ *in* B'_R, *then* D *is a ball of radius* $a = (3c/4\pi)^{1/3}$.

This result is proved in Section 2.

2. Proofs

Proof of Theorem 1. We have

$$\Delta u = -4\pi\chi \quad \text{in } \mathbb{R}^3. \tag{2}$$

Multiply (2) by a harmonic function $h \in \mathcal{N}$ and integrate over B_R to get

$$-4\pi \int_D h(x)\,dx = \int_{B_R} h\Delta u\,dx = \int_{B_R} u\Delta h\,dx + I, \tag{3}$$

where we have used Green's formula, and set

$$I := \int_{\partial B_R} (hu_r - uh_r)\,ds, \qquad u_r := \frac{\partial u}{\partial r} = \frac{\partial u}{\partial N}\bigg|_{\partial B_R},$$

where ∂B_R is the boundary of B_R. By our assumption,

$$u_r = -cR^{-2}, \qquad u = cR^{-1} \quad \text{on } \partial B_R.$$

We also have

$$\int_{\partial B_R} h_r\,ds = 0, \qquad \frac{1}{4\pi R^2}\int_{\partial B_R} h\,ds = h(0),$$

where we have used the mean value theorem for harmonic functions and the formula

$$\int_{\partial B_R} h_r\,ds = \int_{B_R} \Delta h\,dx = 0,$$

which is valid for harmonic functions h. Therefore, $I = c_1 h(0)$, where c_1 is a constant, and (3) implies

$$\int_D h(x)\,dx = c_2 h(0) \quad \forall h \in \mathcal{N}, \quad c_2 := -\frac{c_1}{4\pi}. \tag{4}$$

If $h \in \mathcal{N}$, then $h(gx) \in \mathcal{N}$ for any rotation g. Let us check that if $g = g(\phi)$ is the rotation through the angle ϕ about the straight line passing through the origin in the direction ℓ, then

$$\left.\frac{d(g(\phi)x)}{d\phi}\right|_{\phi=0} = [\ell, x], \tag{5}$$

where $[\ell, x]$ is the cross product. To check (5), choose the coordinate system with z-axis along ℓ, and write the matrix g of the ϕ-rotation about the z-axis:

$$g := g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\left.\frac{dg}{d\phi}\right|_{\phi=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} := G.$$

Thus, $Gx = [\ell, x]$ in the chosen coordinate system. This formula does not depend on the choice of coordinate system, so (5) is verified.

A proof of (5), similar to the one used in mechanics for the proof of the conservation of angular momentum, is also possible.

Replacing $h(x)$ by $h(g(\phi)x)$ in (4), differentiating with respect to ϕ, then setting $\phi = 0$, and using (5), one gets

$$\int_D \nabla h \cdot [\ell, x]\,dx = 0, \tag{6}$$

where $\cdot$ is the dot product of two vectors. Note that

$$\nabla h(x) \cdot [\ell, x] = \nabla \cdot h(x)[\ell, x].$$

Therefore, using the divergence theorem, one rewrites (6) as

$$\int_S h(s) N_s \cdot [\ell, s]\,ds = \ell \cdot \int_S h(s)[s, N_s]\,ds = 0, \quad \forall h \in \mathcal{N},\ \forall \ell \in S^2, \tag{7}$$

where N_s is the outer unit normal to S at the point s. Since ℓ is arbitrary, equation (7) implies

$$\int_S h(s)[s, N_s]\,ds = 0, \quad \forall h \in \mathcal{N}. \tag{8}$$

We claim that the set of restrictions of all harmonic functions $h \in \mathcal{N}$ to S is dense in $L^2(S)$. For the convenience of the reader, this claim is verified after the proof of Theorem 1 is finished. Thus, (8) implies

$$[s, N_s] = 0 \quad \forall s \in S. \tag{9}$$

Let us prove that (9) implies that S is a sphere. Let $\mathbf{r} = \mathbf{r}(p,q)$ be a parametric equation of S. Then $s = \mathbf{r}(p,q)$ and $N_s = [\mathbf{r}_p, \mathbf{r}_q]/|[\mathbf{r}_p, \mathbf{r}_q]|$. Thus, (9) implies

$$0 = [\mathbf{r}, [\mathbf{r}_p, \mathbf{r}_q]] = \mathbf{r}_p \mathbf{r} \cdot \mathbf{r}_q - \mathbf{r}_q \mathbf{r} \cdot \mathbf{r}_p. \tag{10}$$

Since the surface S is assumed $C^{1,\lambda}$-smooth, the normal N_s is well defined at every point $s \in S$, and the vectors $\mathbf{r}_p$ and $\mathbf{r}_q$ are linearly independent. Thus, (10) implies

$$\mathbf{r} \cdot \mathbf{r}_q = \mathbf{r} \cdot \mathbf{r}_p = 0. \tag{11}$$

It follows from (11) that

$$\mathbf{r} \cdot \mathbf{r} = a^2, \tag{12}$$

where $a > 0$ is a constant. This is an equation of a sphere of radius a, centered at the origin, i.e., at the center O of mass of D. So, D is a ball of radius a centered at O. In our argument we do not assume that D is connected.

Theorem 1 is proved. ∎

Let us now verify the claim that the set of restrictions of all harmonic functions $h \in \mathcal{N}$ to S is dense in $L^2(S)$. Assuming the contrary, one concludes that there exists an $f \in L^2(S)$ such that

$$\int_S f(s)h(s)\,ds = 0 \quad \forall h \in \mathcal{N}. \tag{13}$$

Take

$$h(s) = \int_{S_m} \frac{\mu(x)\,dx}{|x-s|},$$

where $m > R$, and $\mu \in L^2(S_m)$ is arbitrary. Then h is harmonic in B_R, and since μ is arbitrary, equation (13) implies

$$v(x) := \int_S f(s)|x-s|^{-1}\,ds = 0 \qquad \forall x \in S_m. \tag{14}$$

The function v is a single-layer potential which vanishes on S_m. Thus, it vanishes everywhere outside S_m, and consequently, everywhere outside S. Since v is continuous in $\mathbb{R}^3$, and vanishes everywhere outside S, it vanishes on S. A function v which is harmonic in D and vanishes on S must vanish in D. So, $v = 0$ in D and in $D' := \mathbb{R}^3 \setminus D$. By the jump relation for the normal derivative of v across the boundary S, one gets

$$0 = (v_N)^+ - (v_N)^- = 4\pi f,$$

where $(v_N)^{\pm}$ is the limiting value of the normal derivative v_N from the inside (resp. outside) of D. So, $f = 0$, which proves the claim. ∎

REMARK. Equation (12) has been derived under the assumption that the origin of the coordinate system is fixed: in this coordinate system $u(x) = c/|x|$ in B'_R. Therefore the surfaces satisfying (12) in which $a = \text{const} > 0$, can be concentric spheres. There cannot be more than two neighboring concentric spheres with different mass densities, because the density of the mass distribution in D is assumed constant. There cannot be two such spheres, i.e., D cannot be a spherical shell, because in this case the domain D' is not connected, contrary to our assumption. Thus, S can only be one sphere, and D can only be a ball.

D. Zagier's proof ([5]). Assume that D is connected. If $u(x) = \int_D dy/|x-y| = c/|x|$ in B'_R, then $u(x) = c/|x|$ in D' by the unique continuation theorem for harmonic functions. Taking gradient, one gets

$$\int_D \frac{dy\,(x-y)}{|x-y|^3} = \frac{cx}{|x|^3}$$

in D'. Taking the dot product with x, one gets

$$\int_D \frac{dy\,(x^2 - y\cdot x)}{|x-y|^3} = \frac{c}{|x|} = \int_D \frac{dy}{|x-y|} = \int_D \frac{dy\,|x-y|^2}{|x-y|^3}, \quad x \in D'.$$

This implies

$$\int_D \frac{dy\,(-y^2 + y\cdot x)}{|x-y|^3} = 0 \quad \text{in } D'.$$

Let $B \subset D$ be the largest ball, inscribed in D and centered at the origin, and $B^c := D \setminus B$. One has

$$\int_B \frac{dy\,(-y^2 + y\cdot x)}{|x-y|^3} = 0 \quad \text{in } B'.$$

Subtract this from the similar formula for D to get

$$\int_{B^c} \frac{dy\,(-y^2 + y\cdot x)}{|x-y|^3} = 0 \quad \text{in } D'.$$

Since $-y^2 + y\cdot x \le 0$ if $y \in B^c$, $x \in D'$, one concludes that $|B^c| = 0$, so $D = B$. Here $|B^c|$ is the Lebesgue measure (volume) of the set B^c. ∎

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