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Boundary values of functions in Cegrell's class $\mathcal{E}_\psi$

by Pham Hoang Hiep (Hanoi)

Abstract. We study boundary values of functions in Cegrell's class $\mathcal{E}_\psi$.

1. Introduction. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Denote by $\mathrm{PSH}(\Omega)$ the plurisubharmonic (psh) functions on Ω. The complex Monge–Ampère operator $(dd^c)^n$ is well defined over the class of locally bounded psh functions, according to the fundamental work of Bedford and Taylor in [BT1], [BT2]. Cegrell introduced a general class $\mathcal{E}$ of psh functions on which the complex Monge–Ampère operator $(dd^c)^n$ can be defined. He obtained many important results of pluripotential theory in the class $\mathcal{E}$, for example, the comparison principle and solvability of the Dirichlet problem (see [Ce1], [Ce2]). Recently, he introduced in [Ce3] a new class $\mathcal{E}_\psi$. The main aim of this note is to study boundary values of functions in the class $\mathcal{E}_\psi$.

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2. Preliminaries. First we recall some elements of pluripotential theory that will be used throughout the paper. All this can be found in [BT2], [Ce1], [Ce2], [Kl], [Ko].

2.1. The following classes of psh functions were introduced by Cegrell in [Ce1] and [Ce2]:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0,\ \int_\Omega (dd^c\varphi)^n < \infty\Big\},$$

$$\mathcal{E}_p = \mathcal{E}_p(\Omega) = \Big\{\varphi \in \mathrm{PSH}(\Omega) : \exists \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi,\ \sup_{j\ge 1} \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n < \infty\Big\},$$

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$$\mathcal{F} = \mathcal{F}(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) : \exists \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi,\ \sup_{j\ge 1}\int_\Omega (dd^c\varphi_j)^n < \infty\Big\},$$

$$\mathcal{E} = \mathcal{E}(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \exists \varphi_K \in \mathcal{F}(\Omega) \text{ such that } \varphi_K = \varphi \text{ on } K,\ \forall K \subset\subset \Omega\}.$$

2.2. For each $\psi \in \mathrm{PSH}^-(\Omega)$, $\psi \not\equiv 0$, Cegrell [Ce3] introduced a new class of psh functions

$$\mathcal{E}_\psi = \mathcal{E}_\psi(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) : \exists \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi,\ \sup_{j\ge 1}\int_\Omega -\psi(dd^c\varphi_j)^n < \infty\Big\}.$$

By Proposition 3.1 in [Ce3] we have $\mathcal{E}_\psi \subset \mathcal{E}$. It is known that if $v \le u$ and $v \in \mathcal{E}_\psi$ then $u \in \mathcal{E}_\psi$. By Theorem 5.5 in [Ce2] we have $u + v \in \mathcal{E}_\psi$ for all $u, v \in \mathcal{E}_\psi$.

2.3. Let $a \in \Omega$. According to Klimek (see [Kl]), the pluricomplex Green function with poles at a is defined by

$$g_{\Omega,a} = g_a(z) = \sup\{u \in \mathrm{PSH}^-(\Omega),\ u(z) - \log|z-a| \le O(1) \text{ as } z \to a\}.$$

Demailly [De] proved that $(dd^c \max(g_a, -\varepsilon))^n$ is weak*-convergent to a measure $\mu_{\Omega,a}$ supported on $\partial\Omega$ as $\varepsilon \to 0$. He discovered the following interesting formula:

$$u(a) = \frac{1}{(2\pi)^n}\int_{\partial\Omega} u\, d\mu_{\Omega,a} + \frac{1}{(2\pi)^n}\int_\Omega g_a\, dd^c u \wedge (dd^c g_a)^{n-1}$$

for all $u \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$.

2.4. Let $u \in \mathrm{PSH}^-(\Omega)$. We set $u^*(\xi) = \limsup_{z\to\xi} u(z)$ for all $\xi \in \overline{\Omega}$. By the comparison principle for the classes $\mathcal{F} \cup \mathcal{E}_p$ we obtain $u^*|_{\partial\Omega} \equiv 0$ for all $u \in \mathcal{F} \cup \mathcal{E}_p$ (see [Åh], [Ce1,2], [ÅCH], [H1,2]). By Theorem 5.8 in [Ce2] we find a function $u \in \mathcal{F}_1$ such that $\liminf_{z\to\xi} u(z) = -\infty$ for all $\xi \in \partial\Omega$.

Next we introduce a result needed for our paper:

2.5. Proposition. *Let $u_j, v_j \in \mathcal{F}$, $u \in \mathrm{PSH}^-(\Omega)$ be such that $u_j \searrow u$, $v_j \searrow u$. Then*

$$\lim_{j\to\infty}\int_\Omega -\varphi(dd^c u_j)^n = \lim_{j\to\infty}\int_\Omega -\varphi(dd^c v_j)^n$$

for all $\varphi \in \mathrm{PSH}^-(\Omega)$.

Proof. For each k we set $w_j = \max(u_k, v_j)$. Integration by parts gives

$$\int_\Omega -\varphi(dd^c w_j)^n \le \int_\Omega -\varphi(dd^c v_j)^n$$

for all $j \geq 1$. Moreover since $w_j \searrow u_k \in \mathcal{F}$ as $j \to \infty$ we obtain

$$\int_{\Omega} -\varphi (dd^c u_k)^n \leq \lim_{j\to\infty} \int_{\Omega} -\varphi (dd^c v_j)^n.$$

Letting $k \to \infty$ we get

$$\lim_{j\to\infty} \int_{\Omega} -\varphi (dd^c u_j)^n \leq \lim_{j\to\infty} \int_{\Omega} -\varphi (dd^c v_j)^n.$$

3. Boundary values of functions in the class $\mathcal{E}_\psi$. The main result of the note is the following

3.1. THEOREM. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ $(n \geq 2)$ and $u \in \mathcal{E}_\psi(\Omega)$ for some $\psi \in \mathrm{PSH}^-(\Omega)$, $\psi \not\equiv 0$. Then $\int_{\partial\Omega} u^* \, d\mu_{\Omega,a} = 0$ for all $a \in \Omega$.*

Proof. By the definition of the class $\mathcal{E}_\psi$ we find $\mathcal{E}_0 \ni u_j \searrow u$ such that

$$\sup_{j\geq 1} \int_{\Omega} -\psi (dd^c u_j)^n < \infty.$$

Let $a \in \Omega$. From

$$\sup_{z\in\Omega} \frac{|\max(g_a(z), -1)|}{|\psi(z)|} < \infty$$

and from Proposition 2.5 we get

$$A = \sup_{j\geq 1} \int_{\Omega} -\max(g_a, -1)(dd^c \max(jg_a, u))^n < \infty.$$

Let K be a compact subset in $\{u^* < 0\} \cap \partial\Omega$. We only have to prove that

$$\int_K d\mu_{\Omega,a} = 0.$$

Let $s > 0$ and U be a neighborhood of K such that $u|_{U\cap\Omega} < -s$. We have

$$\begin{aligned}
&\int_{\Omega} -\max(g_a, -1)(dd^c \max(jg_a, u))^n \\
&\qquad = \int_{\{jg_a \leq u\}} -\max(g_a, -1)(dd^c \max(jg_a, u))^n \\
&\qquad \geq \int_{\{jg_a \leq u\}} -\max(u/j, -1)(dd^c \max(jg_a, u))^n \\
&\qquad = j^{n-1} \int_{\Omega} -\max(u, -j)(dd^c \max(g_a, u/j))^n
\end{aligned}$$

$$\geq j^{n-1} \int\limits_{\Omega} -\max(u,-j)(dd^c \max(g_a, u/j, -s/j))^n$$

$$\geq j^{n-1} s \int\limits_{U} (dd^c \max(g_a, u/j, -s/j))^n = j^{n-1} s \int\limits_{U} (dd^c \max(g_a, -s/j))^n$$

for $j \geq 1$. Therefore

$$\int\limits_{U} (dd^c \max(g_a, -s/j))^n \leq \frac{A}{j^{n-1}s}$$

for $j \geq 1$. Moreover, since $(dd^c \max(g_a, -\varepsilon))^n$ is weak*-convergent to a measure $\mu_{\Omega,a}$ on $\mathbb{C}^n$ as $\varepsilon \to 0$ we obtain

$$\int\limits_{U} d\mu_{\Omega,a} \leq \liminf_{j\to\infty} \int\limits_{U} (dd^c \max(g_a, -s/j))^n \leq \liminf_{j\to\infty} \frac{A}{j^{n-1}s} = 0.$$

Hence

$$\int\limits_{K} d\mu_{\Omega,a} = 0.$$

3.2. COROLLARY. *Let Ω be a bounded B-regular domain in $\mathbb{C}^n$ and $u \in \mathcal{E}_\psi(\Omega)$ for some $\psi \in \mathrm{PSH}^-(\Omega)$, $\psi \not\equiv 0$. Then $u^*|_{\partial\Omega} \equiv 0$.*

Proof. We assume that $u^*(\xi_0) < 0$ for some $\xi_0 \in \partial\Omega$. Let $r > 0$ be such that

$$u^*(\xi) < u^*(\xi_0)/2$$

for all $\xi \in B(\xi_0, r) \cap \partial\Omega$. By Theorem 3.1 we have

$$\int\limits_{B(\xi_0,r)} d\mu_{\Omega,a} = 0$$

for all $a \in \Omega$. Let $f \in C(\partial\Omega)$ be such that $0 \leq f \leq 1$, $f = 1$ on $B(\xi_0, r/2) \cap \partial\Omega$ and $f = 0$ on $\partial\Omega \backslash B(\xi_0, r)$. We find a function $h \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$ such that $(dd^c h)^n = 0$ and $h|_{\partial\Omega} = f$. Let $a \in \Omega$ be such that $h(a) > 0$. By [De] we have

$$h(a) = \frac{1}{(2\pi)^n} \int\limits_{\partial\Omega} f\, d\mu_{\Omega,a} + \frac{1}{(2\pi)^n} \int\limits_{\Omega} g_a\, dd^c h \wedge (dd^c g_a)^{n-1}$$
$$\leq \frac{1}{(2\pi)^n} \int\limits_{\partial\Omega} f\, d\mu_{\Omega,a} \leq \frac{1}{(2\pi)^n} \int\limits_{B(\xi_0,r)} d\mu_{\Omega,a}.$$

Hence

$$\int\limits_{B(\xi_0,r)} d\mu_{\Omega,a} \geq (2\pi)^n h(a) > 0,$$

which contradicts $\int_{B(\xi_0,r)} d\mu_{\Omega,a} = 0$.

3.3. Corollary. *Let $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_1 \subset \mathbb{C}^{n_1}$, $\Omega_2 \subset \mathbb{C}^{n_2}$ are bounded B-regular domains and $u \in \mathcal{E}_\psi(\Omega)$ for some $\psi \in \mathrm{PSH}^-(\Omega)$, $\psi \not\equiv 0$. Then $u^*|_{\partial\Omega_1 \times \partial\Omega_2} \equiv 0$.*

Proof. From $g_{\Omega,(a_1,a_2)} = \max(g_{\Omega_1,a_1}, g_{\Omega_2,a_2})$ and from Theorem 7 in [Bł2] we get

$$\mu_{\Omega,(a_1,a_2)} = \mu_{\Omega_1,a_1} \times \mu_{\Omega_2,a_2}$$

for all $(a_1, a_2) \in \Omega_1 \times \Omega_2$. By this formula and a copy of the proof of Corollary 3.2 we infer that $u^*|_{\partial\Omega_1 \times \partial\Omega_2} \equiv 0$.

Let Ω_1, Ω_2 be bounded hyperconvex domains in $\mathbb{C}$. We construct a function $u \in \mathcal{E}_\psi(\Omega_1 \times \Omega_2)$ for some $\psi \in \mathrm{PSH}^-(\Omega_1 \times \Omega_2)$, $\psi \not\equiv 0$ such that $u^*|_{\partial\Omega_1 \times \Omega_2 \cup \Omega_1 \times \partial\Omega_2} < 0$:

3.4. Proposition. *Let $\Omega = \Omega_1 \times \Omega_2$ where Ω_1, Ω_2 are bounded hyperconvex domains in $\mathbb{C}$. Then $\max(g_{\Omega_1,a_1}, -1) + \max(g_{\Omega_2,a_2}, -1) \in \mathcal{E}_\psi(\Omega)$ with $\psi = \max(g_{\Omega_1,a_1}, g_{\Omega_2,a_2})$ for all $(a_1, a_2) \in \Omega_1 \times \Omega_2$.*

Proof. By Theorem 5.5 in [Ce2] we only have to prove that $u = \max(g_{\Omega_1,a_1}, -1) \in \mathcal{E}_\psi(\Omega)$. Set

$$u_j = \max(g_{\Omega_1,a_1}, j g_{\Omega_2,a_2} - 1).$$

Then $\mathcal{E}_0(\Omega) \ni u_j \searrow u$. By Theorem 7 in [Bł2] we have

$$\begin{aligned}
\int_\Omega -\psi(dd^c u_j)^2 &= \int_\Omega -\psi\, dd^c(g_{\Omega_1,a_1}, -1) \wedge dd^c(j g_{\Omega_2,a_2}, -1) \\
&= \int_{\{g_{\Omega_1,a_1}=-1\}\times\{g_{\Omega_2,a_2}=-1/j\}} -j\psi dd^c(g_{\Omega_1,a_1}, -1) \wedge dd^c(g_{\Omega_2,a_2}, -1/j) \\
&= \int_{\{g_{\Omega_1,a_1}=-1\}\times\{g_{\Omega_2,a_2}=-1/j\}} dd^c(g_{\Omega_1,a_1}, -1) \wedge dd^c(g_{\Omega_2,a_2}, -1/j) \\
&= \int_{\Omega_1} dd^c(g_{\Omega_1,a_1}, -1) \int_{\Omega_1} dd^c(g_{\Omega_2,a_2}, -1/j) = (2\pi)^2.
\end{aligned}$$

Hence $u \in \mathcal{E}_\psi(\Omega)$.

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Department of Mathematics
Hanoi University of Education (Dai Hoc Su Pham HaNoi)
Cau Giay, Hanoi, VietNam
E-mail: phhiep_vn@yahoo.com

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