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Existence and asymptotic behavior of positive solutions for elliptic systems with nonstandard growth conditions

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Abstract. Our main purpose is to establish the existence of a positive solution of the system

$$\begin{cases} -\Delta_{p(x)}u = F(x,u,v), & x \in \Omega, \\ -\Delta_{q(x)}v = H(x,u,v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary, $F(x,u,v) = \lambda^{p(x)}[g(x)a(u) + f(v)]$, $H(x,u,v) = \lambda^{q(x)}[g(x)b(v) + h(u)]$, $\lambda > 0$ is a parameter, $p(x), q(x)$ are functions which satisfy some conditions, and $-\Delta_{p(x)}u = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian. We give existence results and consider the asymptotic behavior of solutions near the boundary. We do not assume any symmetry conditions on the system.

1. Introduction. In this paper, our main purpose is to establish the existence of a positive solution of the system

$$(1.1) \qquad \begin{cases} -\Delta_{p(x)}u = F(x,u,v), & x \in \Omega, \\ -\Delta_{q(x)}v = H(x,u,v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary, $F(x,u,v) = \lambda^{p(x)}[g(x)a(u) + f(v)], H(x,u,v) = \lambda^{q(x)}[g(x)b(v) + h(u)]$ and $p(\cdot), q(\cdot) \in C^1(\overline{\Omega})$ are positive functions; the operator $-\Delta_{p(x)}u = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-*Laplacian* and the corresponding equation is called a *variable exponent equation.*

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electro-rheological fluids, etc. (see [R], [Z7]). Many results have already been obtained on this kind of problems (for ex-

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ample [AM1], [AM2], [F1], [FZ1], [FZ2], [FZ], [FWW], [H]). For regularity of weak solutions to differential equations with nonstandard $p(x)$-growth conditions, we refer to [AM1], [AM2], [F1],[FZ1], [FZ2]. For existence results for elliptic systems with variable exponents, we refer to [FWW], [H], [Z1], [Z2].

For the special case $p(x) \equiv p$ (a constant), (1.1) becomes the well known p-Laplacian system, considered in many papers (see [C], [HS], [YY] and the references therein). We point out that elliptic equations involving the $p(x)$-Laplacian are not trivial generalizations of similar problems studied in the constant case since the $p(x)$-Laplacian operator is nonhomogeneous, and some techniques for constant exponent problems, like the Lagrange Multiplier Theorem, are invalid. Another issue is that, if Ω is bounded, then the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx}$$

is zero in general, and is positive only under some special conditions (see [FZZ]). The fact that the first eigenvalue $\lambda_p > 0$ and the existence of the first eigenfunction are very important in the study of p-Laplacian problems (see [C], [HS], [Y]). There are also other difficulties in discussing the existence and asymptotic behavior of solutions of variable exponent problems.

In [HS], the authors studied the existence of positive weak solutions to the problem

$$\begin{cases} -\Delta_p u = \lambda f(v), & x \in \Omega, \\ -\Delta_p v = \lambda g(u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Under the condition

$$\lim_{s\to\infty} \frac{f(M[g(s)]^{1/(p-1)})}{s^{p-1}} = 0, \qquad \forall M > 0, \tag{1.3}$$

they proved the existence of positive solutions for problem (1.2).

In [C], the author considered the existence and nonexistence of positive weak solutions to the p-Laplacian problem

$$\begin{cases} -\Delta_p u = \lambda u^\alpha v^\gamma, & x \in \Omega, \\ -\Delta_q v = \lambda u^\delta v^\beta, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

Recently, in [Y], the authors considered the existence of positive solutions to the following quasilinear elliptic system in a bounded domain $\Omega \subset \mathbb{R}^N$:

$$(1.5)\qquad \begin{cases} -\Delta_p u = \lambda[g(x)a(u) + f(v)], & x \in \Omega, \\ -\Delta_q v = \theta[g_1(x)b(v) + h(u)], & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases}$$

where $\lambda, \theta > 0$ are parameters and $g(x), g_1(x)$ may be negative near $\partial\Omega$.

We note that in order to obtain existence results, the first eigenfunction of $-\Delta_p$ is used to construct a subsolution for problems (1.2), (1.4) and (1.5). But for variable exponent problems, the first eigenvalue and the first eigenfunction of $-\Delta_{p(x)}$ may not exist. Even if the first eigenfunction of $-\Delta_{p(x)}$ exists, because of the nonhomogeneity of $-\Delta_{p(x)}$, we still may not be able to construct a subsolution of a variable exponent problem from the first eigenfunction. In many cases, radial symmetry conditions are effective when dealing with variable exponent problems (see [FWW], [FZZ2], [Z2], [Z4] and references therein). In [Z1], [Z2] and [Z6], with a condition similar to (1.3), the author discussed the existence of positive solutions of the following problems:

$$(1.6)\qquad \begin{cases} -\Delta_{p(x)} u = \lambda f(v), & x \in \Omega, \\ -\Delta_{p(x)} v = \lambda g(u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

and

$$(1.7)\qquad \begin{cases} -\Delta_{p(x)} u = \lambda^{p(x)} f(v), & x \in \Omega, \\ -\Delta_{p(x)} v = \lambda^{p(x)} g(u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

Similarly to (1.5), we will also consider problem (1.1) when $F(x, u, v) = \lambda[g(x)a(u) + f(v)], H(x, u, v) = \lambda[g(x)b(v) + h(u)]$, i.e. the following system:

$$(1.8)\qquad \begin{cases} -\Delta_{p(x)} u = \lambda[g(x)a(u) + f(v)], & x \in \Omega, \\ -\Delta_{q(x)} v = \lambda[g(x)b(v) + h(u)], & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

It is well known that (1.8) is equal to (1.1) if $p(x) \equiv p \equiv q(x)$ (a constant), but for general functions $p(x), q(x)$, (1.8) is not equal to (1.1) even if $p(x) = q(x)$. We call (1.1) and (1.8) of $(p(x), q(x))$-*type* and refer to (1.6), (1.7) as $(p(x), p(x))$-*type*. There are some differences between the existence of positive solutions of (1.1) and (1.8), and there are some differences between the existence of positive solutions for $(p(x), q(x))$-type and $(p(x), p(x))$-type.

Motivated by the above results, we study problem (1.1) and (1.8) in this paper. Our aim is to establish the existence and asymptotic behavior of positive weak solutions for problem (1.1) and (1.8) without radial symmetry conditions. We prove the existence of positive weak solutions by the sub-supersolution method. Since in problems (1.2) and (1.4), F and H are independent of the variable x, the systems are homogeneous. Our results partially generalize those of [AM2], [HS], [Y], [Z1], [Z2], [Z6].

The paper is organized as follows. In Section 2, we recall some facts that will be needed in the paper. In Section 3, we consider the existence of positive solutions of (1.1) and (1.8). We will discuss the asymptotic behavior of positive solutions of problem (1.1) and (1.8) in the fourth section. In the fifth section, we give an example.

2. Notation and preliminaries. In order to deal with the $p(x)$-Laplacian problem, we need some results on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of the $p(x)$-Laplacian (see [FZ1], [KR], [R], [S]). For any $f(\cdot) \in C(\overline{\Omega})$, we write

$$f^+ = \max_{x\in\overline{\Omega}} f(x), \qquad f^- = \min_{x\in\overline{\Omega}} f(x).$$

Let

$$L^{p(x)}(\Omega) = \Big\{u \;\Big|\; u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\}.$$

We can introduce a norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\bigg\{\lambda > 0 \;\bigg|\; \int_\Omega \bigg|\frac{u(x)}{\lambda}\bigg|^{p(x)} dx \le 1\bigg\},$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space, which we call a *variable exponent Lebesgue space.*

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\},$$

and can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \qquad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, and call it a *variable exponent Sobolev space.* From [FZ1], we know that $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.

We define

$$(L(u), v) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx, \qquad \forall u, v \in W_0^{1,p(x)}(\Omega);$$

then $L : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is a continuous, bounded and strictly monotone operator, and it is a homeomorphism (see [FZ, Theorem 3.1]).

Definition 2.1. (1) $(u,v) \in (W_0^{1,p(x)}(\Omega), W_0^{1,q(x)}(\Omega))$ is called a (*weak*) *solution* of problem (1.1) if it satisfies

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\,dx = \int_\Omega F(x,u,v)\varphi\,dx, \\ \int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla\psi\,dx = \int_\Omega H(x,u,v)\psi\,dx, \end{cases}$$

for any $(\varphi,\psi) \in (W_0^{1,p(x)}(\Omega), W_0^{1,q(x)}(\Omega))$.

(2) $(u,v) \in (W^{1,p(x)}(\Omega), W^{1,q(x)}(\Omega))$ is called a *subsolution* (resp. a *supersolution*) of problem (1.1) if $(u,v) \le (\ge)\ (0,0)$ on $\partial\Omega$ and

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\,dx \le (\ge) \int_\Omega F(x,u,v)\varphi\,dx, \\ \int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla\psi\,dx \le (\ge) \int_\Omega H(x,u,v)\psi\,dx, \end{cases}$$

for any $(\varphi,\psi) \in (W_0^{1,p(x)}(\Omega), W_0^{1,q(x)}(\Omega))$ with $\varphi,\psi \ge 0$.

Define $A : W^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ by

$$\langle Au,\varphi\rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + m(x,u)\varphi)\,dx$$

for $u \in W^{1,p(x)}(\Omega)$, $\varphi \in W_0^{1,p(x)}(\Omega)$, where $m(x,u)$ is continuous on $\overline{\Omega}\times\mathbb{R}$, $m(x,\cdot)$ is increasing, and

$$|m(x,t)| \le C_1 + C_2|t|^{p^*(x)-1},$$

with $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $p(x) \ge N$. Hereafter, we use C_i to denote positive constants. It is easy to check that A is a continuous bounded mapping. From [Z5], we have the following lemma.

Lemma 2.2 (Comparison Principle). *Let $u,v \in W^{1,p(x)}(\Omega)$. If $Au - Av \le 0$ in $(W_0^{1,p(x)}(\Omega))^*$ and $u \le v$ on $\partial\Omega$ (i.e. $(u-v)^+ \in W_0^{1,p(x)}(\Omega)$), then $u \le v$ a.e. in Ω.*

The following conditions will be required in our results:

(D1) $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^2 boundary $\partial\Omega$;
(D2) $p(\cdot), q(\cdot) \in C^1(\overline{\Omega})$ and $1 < p^- \le p^+$, $1 < q^- \le q^+$;
(D3) $g \in C(\overline{\Omega})$ is nonnegative;
(D4) $f,h \in C^1([0,\infty))$ are nondecreasing, $\lim_{s\to\infty} f(s) = \infty$, $\lim_{s\to\infty} h(s) = \infty$ and

$$\lim_{s\to\infty} \frac{f(M[h(s)]^{\frac{1}{q^--1}})}{s^{p^--1}} = 0, \quad \forall M > 0$$

(a combined sublinear effect at ∞);

(D5) $a, b \in C^1([0,\infty))$ are nonnegative, nondecreasing, and
$$\lim_{s\to\infty} \frac{a(s)}{s^{p^- -1}} = 0, \qquad \lim_{s\to\infty} \frac{b(s)}{s^{q^- -1}} = 0.$$

3. Existence of positive solutions. From now on, we will denote by $d(x)$ the distance of $x \in \Omega$ to the boundary of Ω. Since $\partial\Omega$ is C^2 regular, there exists a constant $\delta > 0$ small enough such that $d(\cdot) \in C^2(\overline{\partial_{3\delta}\Omega})$ and $|\nabla d(x)| \equiv 1$, where $\partial_\varepsilon \Omega = \{x \in \Omega \mid d(x) < \varepsilon\}$.

We now define
$$v_1(x) = \begin{cases} \xi d(x), & d(x) < \delta, \\ \xi\delta + \int\limits_{\delta}^{d(x)} \xi\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- -1}} dt, & \delta \le d(x) < 2\delta, \\ \xi\delta + \int\limits_{\delta}^{2\delta} \xi\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- -1}} dt, & 2\delta \le d(x). \end{cases}$$
Since δ is small enough, we have $0 \le v_1(\cdot) \in C^1(\overline{\Omega})$.

We consider the problem
$$\begin{cases} -\Delta_{p(x)} w(x) = \mu, & x \in \Omega, \\ w = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$
and have the following results.

Lemma 3.1 (see [F2]). *If μ is a large enough positive parameter and w is the unique solution of* (3.1), *then for any $\nu \in (0,1)$, there exist positive constants C_3, C_4 such that*
$$C_3 \mu^{\frac{1}{p^+ -1+\nu}} \le \max_{x\in\overline{\Omega}} w(x) \le C_4 \mu^{\frac{1}{p^- -1}}.$$

Proof. By computation, we have
$$-\Delta_{p(x)} v_1(x) = \begin{cases} -\xi^{p(x)-1}[(\nabla p \nabla d)\ln\xi + \Delta d], & d(x) < \delta, \\ \left\{\frac{2(p(x)-1)}{\delta(p^- -1)} - \frac{2\delta - d}{\delta}\left[\left(\ln \xi\left(\frac{2\delta-d}{\delta}\right)^{\frac{2}{p^- -1}}\right)\nabla p \nabla d + \Delta d\right]\right\} \\ \quad \times \xi^{p(x)-1}\left(\frac{2\delta - d}{\delta}\right)^{\frac{2(p(x)-1)}{p^- -1} - 1}, & \delta < d(x) < 2\delta, \\ 0, \quad 2\delta < d(x). \end{cases}$$
It is easy to see that for any $\nu \in (0,1)$, there exists a positive constant $C = C(\delta, \Omega, p, \nu)$ independent of ξ such that
$$|-\Delta_{p(x)} v_1(x)| \le C\xi^{p(x)-1+\nu} \quad \text{a.e. on } \Omega.$$
If we let $C\xi^{p(x)-1+\nu} = \frac{1}{2}\mu$, then $v_1(x)$ is a subsolution of (3.1). By the definition of $v_1(x)$ and Lemma 2.1, there exists a positive constant C_3 such

that

$$\xi\delta = C_3\mu^{\frac{1}{p^+-1+\nu}} \le \max_{x\in\overline{\Omega}} v_1(x) \le \max_{x\in\overline{\Omega}} w(x).$$

The right hand inequality can be obtained from Lemma 2.1 of [F2]. ■

Now we have the following result.

THEOREM 3.2. *If* (D1)–(D5) *hold, then problem* (1.1) *has a positive solution when* λ *is sufficiently large.*

Proof. According to the sub-supersolution method for $p(x)$-Laplacian equations (see [F2]), we only need to construct a positive subsolution (ϕ_1, ϕ_2) and a supersolution (z_1, z_2) of (1.1) such that $\phi_1 \le z_1$ and $\phi_2 \le z_2$. Then there exists a positive solution (u, v) of (1.1) satisfying $\phi_1 \le u \le z_1$ and $\phi_2 \le v \le z_2$.

STEP 1. We construct a subsolution of (1.1). By (D3)–(D5), there exists an $M > 2$ such that

$$a(u)g(x) + f(v) \ge 1, \qquad b(v)g(x) + h(u) \ge 1$$

when $u, v \ge M - 1$ and $x \in \Omega$.

Let $\sigma = (\ln M)/k$. For k large enough, we have $\sigma \in (0, \delta)$, and we denote

$$\phi_1(x) = \begin{cases} e^{kd(x)} - 1, & d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} dt, & \sigma \le d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} dt, & 2\delta \le d(x), \end{cases}$$

and

$$\phi_2(x) = \begin{cases} e^{kd(x)} - 1, & d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{q^- - 1}} dt, & \sigma \le d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{q^- - 1}} dt, & 2\delta \le d(x). \end{cases}$$

It is easy to see that $\phi_1, \phi_2 \in C^1(\overline{\Omega})$. Denote

$$\alpha = \min\left\{\frac{\inf p(x) - 1}{4(\sup|\nabla p(x)| + 1)}, \frac{\inf q(x) - 1}{4(\sup|\nabla q(x)| + 1)}, 1\right\},$$
$$\beta = |f(0)| + |h(0)| + [a(M-1) + b(M-1)]\max_{x\in\overline{\Omega}} g(x) + 1.$$

By computation, we have

(3.2)
$$
\begin{aligned}
&- \Delta_{p(x)}\phi_1 \\
&= \begin{cases}
-k(ke^{kd(x)})^{p(x)-1}\Big[p(x)-1 \\
\quad +\Big(d(x)+\dfrac{\ln k}{k}\Big)\nabla p(x)\nabla d(x)+\dfrac{\Delta d(x)}{k}\Big], & d(x)<\sigma,\\
\Big\{\dfrac{2(p(x)-1)}{(2\delta-\sigma)(p^- -1)}-\dfrac{2\delta-d}{2\delta-\sigma} \\
\quad \times\Big[\Big(\ln k\, e^{k\sigma}\Big(\dfrac{2\delta-d}{2\delta-\sigma}\Big)^{\frac{2}{p^- -1}}\Big)\nabla p(x)\nabla d(x)+\Delta d(x)\Big]\Big\} \\
\quad \times (ke^{k\sigma})^{p(x)-1}\Big(\dfrac{2\delta-d}{2\delta-\sigma}\Big)^{\frac{2(p(x)-1)}{p^- -1}-1}, & \sigma<d(x)<2\delta,\\
0, & 2\delta<d(x).
\end{cases}
\end{aligned}
$$

Hence, for k sufficiently large, we have

(3.3)
$$-\Delta_{p(x)}\phi_1 \le -k^{p(x)}\alpha, \qquad d(x)<\sigma.$$

Let $\lambda=\frac{\alpha}{\beta+1}k$. Then $k^{p(x)}\alpha \ge \lambda^{p(x)}\beta$, so

(3.4)
$$-\Delta_{p(x)}\phi_1 \le -\lambda^{p(x)}\beta \le \lambda^{p(x)}[a(\phi_1)g(x)+f(\phi_2)], \qquad d(x)<\sigma.$$

Since $d(\cdot)\in C^2(\overline{\partial_{3\delta}}\Omega)$ and $p(\cdot)\in C^1(\overline{\Omega})$, there exists $C_5>0$ such that

$$
\begin{aligned}
-\Delta_{p(x)}\phi_1 &\le (ke^{k\sigma})^{p(x)-1}\Big(\frac{r-d}{r-\sigma}\Big)^{\frac{2(p(x)-1)}{p^- -1}-1}\Big|\Big\{\frac{2(p(x)-1)}{(r-\sigma)(p^- -1)} \\
&\quad -\frac{r-d}{r-\sigma}\Big[\Big(\ln k\, e^{k\sigma}\Big(\frac{r-d}{r-\sigma}\Big)^{\frac{2}{p^- -1}}\Big)\nabla p(x)\nabla d(x)+\Delta d(x)\Big]\Big\}\Big| \\
&\le C_5(ke^{k\sigma})^{p(x)-1}\ln k.
\end{aligned}
$$

For k sufficiently large and $\lambda=\frac{\alpha}{\beta+1}k$, we have

$$C_5(ke^{k\sigma})^{p(x)-1}\ln k \le \lambda^{p(x)}.$$

When $\sigma<d(x)<2\delta$, we have $\phi_1,\phi_2\ge M-1$. Thus

(3.5)
$$-\Delta_{p(x)}\phi_1 \le \lambda^{p(x)}[a(\phi_1)g(x)+f(\phi_2)], \qquad \sigma<d(x)<2\delta.$$

Obviously

(3.6)
$$-\Delta_{p(x)}\phi_1 = 0 \le \lambda^{p(x)}[a(\phi_1)g(x)+f(\phi_2)], \qquad 2\delta<d(x).$$

From (3.4)–(3.6), we obtain

(3.7)
$$-\Delta_{p(x)}\phi_1 \le \lambda^{p(x)}[a(\phi_1)g(x)+f(\phi_2)] \quad \text{a.e. on } \Omega.$$

Similarly, for k sufficiently large and $\lambda = \frac{\alpha}{\beta+1}k$, we have

$$-\Delta_{q(x)}\phi_2 \leq \lambda^{q(x)}[b(\phi_2)g(x) + h(\phi_1)] \quad \text{a.e. on } \Omega. \tag{3.8}$$

From (3.7) and (3.8), we can see that (ϕ_1, ϕ_2) is a subsolution of (1.1).

STEP 2. We construct a supersolution of (1.1). Now we consider the problem

$$\begin{cases} -\Delta_{p(x)}z_1 = \lambda^{p^+}\eta, & x \in \Omega, \\ -\Delta_{q(x)}z_2 = 2\lambda^{q^+}h(\omega), & x \in \Omega, \\ z_1 = z_2 = 0, & x \in \partial\Omega, \end{cases} \tag{3.9}$$

where $\omega = \max_{x\in\overline{\Omega}} z_1(x)$ and η is a positive constant. We will show that (z_1, z_2) is a supersolution of (1.1).

For any $\varphi \in W^{1,p(x)}(\Omega)$ with $\varphi \geq 0$, we have

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \nabla\varphi\, dx = \int_\Omega \lambda^{p^+}\eta\varphi\, dx, \tag{3.10}$$

$$\int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2 \nabla\varphi\, dx = \int_\Omega 2\lambda^{q^+}h(\omega)\varphi\, dx. \tag{3.11}$$

From Lemma 3.1, we know that ω is large when η is large, and by (D3)–(D5), we have

$$\lim_{s\to\infty} \frac{f[C_4(2\lambda^{q^+}h(s))^{\frac{1}{q^- -1}}] + a(s)\max_{x\in\overline{\Omega}} g(x)}{s^{p^- -1}} = 0.$$

Then when η is large enough, by Lemma 3.1, we obtain

$$\lambda^{p^+}\eta \geq \left(\frac{1}{C_4}\omega\right)^{p^- -1} \geq \lambda^{p^+}\Big\{f[C_4(2\lambda^{q^+}h(\omega))^{\frac{1}{q^- -1}}] + a(\omega)\max_{x\in\overline{\Omega}} g(x)\Big\}. \tag{3.12}$$

Since f, a are nondecreasing functions, from (3.10) and (3.12), and using Lemma 3.1 again, we have

$$\begin{aligned} \int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1\nabla\varphi\, dx & \\ &\geq \int_\Omega \lambda^{p^+}\Big\{f[C_4(2\lambda^{q^+}h(\omega))^{\frac{1}{q^- -1}}] + a(\omega)\max_{x\in\overline{\Omega}} g(x)\Big\}\varphi\, dx \\ &\geq \int_\Omega \lambda^{p(x)}[a(z_1)g(x) + f(z_2)]\varphi\, dx. \end{aligned}$$

Since h is nondecreasing, by Lemma 3.1 we have

$$\int_\Omega \lambda^{q^+}h(\omega)\varphi\, dx \geq \int_\Omega \lambda^{q^+}h(z_1)\varphi\, dx. \tag{3.13}$$

From (D4) and (D5), when η large enough,

$$b[C_4(2\lambda^{q^+}h(\omega))^{\frac{1}{q^--1}}]\max_{x\in\overline{\Omega}} g(x) \le h(\omega). \tag{3.14}$$

From (3.11), (3.13), (3.14) and Lemma 3.1, we obtain

$$\begin{aligned}\int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2\nabla\varphi\,dx &\ge \int_\Omega \lambda^{q^+}\Big\{b[C_4(2\lambda^{q^+}h(\omega))^{\frac{1}{q^--1}}]\max_{x\in\overline{\Omega}} g(x)+\lambda^{q^+}h(z_1)\Big\}\varphi\,dx\\ &\ge \int_\Omega \lambda^{q(x)}[b(z_2)g(x)+h(z_1)]\varphi\,dx.\end{aligned}$$

Thus, (z_1, z_2) is a supersolution of (1.1).

STEP 3. We show $\phi_1 \le z_1$ and $\phi_2 \le z_2$. In the definition of $v_1(x)$, let

$$\xi = \frac{2}{\delta}\Big(\max_{x\in\overline{\Omega}}\phi_1(x)+\max_{x\in\overline{\Omega}}|\nabla\phi_1(x)|\Big).$$

From the proof of Lemma 3.1, we know that when η is large enough,

$$v_1(x)\le z_1(x), \quad x\in\overline{\Omega}.$$

So if we prove

$$\phi_1(x)\le v_1(x), \quad x\in\overline{\Omega},$$

the proof will be completed.

Obviously,

$$\phi_1(x)\le 2\max_{x\in\overline{\Omega}}\phi_1(x)\le v_1(x), \quad d(x)\ge\delta.$$

Since $\phi_1 - v_1 \in C^1(\overline{\partial_\delta\Omega})$, there exists $x_0\in\overline{\partial_\delta\Omega}$ such that

$$\phi_1(x_0)-v_1(x_0)=\max_{x\in\overline{\partial_\delta\Omega}}[\phi_1(x)-v_1(x)].$$

If $\phi_1(x_0)-v_1(x_0)>0$, then $0<d(x_0)<\delta$, so we have

$$\nabla\phi_1(x_0)-\nabla v_1(x_0)=0. \tag{3.15}$$

By the definition of $v_1(x)$ and ξ,

$$|\nabla v_1(x)|\equiv\xi>|\nabla\phi_1(x)|, \quad 0<d(x)<\delta.$$

This contradicts (3.15), so

$$\max_{x\in\overline{\partial_\delta\Omega}}[\phi_1(x)-v_1(x)]\le 0,$$

i.e.

$$\phi_1(x)\le v_1(x), \quad 0\le d(x)<\delta.$$

Consequently,

$$\phi_1(x)\le v_1(x), \quad x\in\Omega.$$

Thus

$$\phi_1(x) \le z_1(x), \qquad x \in \Omega.$$

From Lemma 3.1, we can see that ω is large enough when η is large enough, and so $h(\omega)$ is large enough. Similarly,

$$\phi_2(x) \le z_2(x), \qquad x \in \Omega. \ \blacksquare$$

For problem (1.8), we have the following result.

THEOREM 3.3. *Suppose* (D1)–(D5) *hold. If*

$$\operatorname*{osc}_{x\in\partial\Omega} p(x),\ \operatorname*{osc}_{x\in\partial\Omega} q(x) < 1$$

and

$$\Big(\sup_{x\in\partial\Omega}(p(x)-1),\ \inf_{x\in\partial\Omega} p(x)\Big) \cap \Big(\sup_{x\in\partial\Omega}(q(x)-1),\ \inf_{x\in\partial\Omega} q(x)\Big) \ne \emptyset. \tag{3.16}$$

then problem (1.8) *has a positive solution when λ is sufficiently large.*

Proof. Since $\operatorname{osc}_{x\in\partial\Omega} p(x), \operatorname{osc}_{x\in\partial\Omega} q(x) < 1$, by the continuity of $p(x)$, $q(x)$, without loss of generality, we may assume that $\operatorname{osc}_{x\in\overline{\partial_{3\delta}\Omega}} p(x) < 1$ and $\operatorname{osc}_{x\in\overline{\partial_{3\delta}\Omega}} q(x) < 1$. Then

$$\Big(\sup_{x\in\overline{\partial_{3\delta}\Omega}}(p(x)-1),\ \inf_{x\in\overline{\partial_{3\delta}\Omega}} p(x)\Big) \ne \emptyset$$

and

$$\Big(\sup_{x\in\overline{\partial_{3\delta}\Omega}}(q(x)-1),\ \inf_{x\in\overline{\partial_{3\delta}\Omega}} q(x)\Big) \ne \emptyset.$$

By (3.16) and the continuity of $p(x), q(x)$, for δ small enough we have

$$\begin{aligned} \Gamma = &\Big(\sup_{x\in\overline{\partial_{3\delta}\Omega}}(p(x)-1),\ \inf_{x\in\overline{\partial_{3\delta}\Omega}} p(x)\Big) \\ &\cap \Big(\sup_{x\in\overline{\partial_{3\delta}\Omega}}(q(x)-1),\ \inf_{x\in\overline{\partial_{3\delta}\Omega}} q(x)\Big) \ne \emptyset. \end{aligned}$$

Now for any $\gamma \in \Gamma$ and $\lambda > 0$, there exists a $k \in \mathbb{R}$ such that $k^\gamma = \lambda$. Then the estimates $k^{p(x)}\alpha \ge \lambda\beta$, $k^{q(x)}\alpha \ge \lambda\beta$, $C_5(ke^{k\sigma})^{p(x)-1}\ln k \le \lambda$ and $C_5(ke^{k\sigma})^{q(x)-1}\ln k \le \lambda$ can be satisfied simultaneously when λ is large enough.

By the argument of Theorem 3.2, (ϕ_1, ϕ_2) is also a subsolution of (1.8).

Consider the problem

$$\begin{cases} -\Delta_{p(x)} z_1 = \lambda\eta, & x \in \Omega, \\ -\Delta_{q(x)} z_2 = 2\lambda h(\omega), & x \in \Omega, \\ z_1 = z_2 = 0, & x \in \partial\Omega. \end{cases} \tag{3.17}$$

When η is large enough, the solution (z_1, z_2) of (3.17) is a supersolution of (1.8).

By the same argument of Theorem 3.2 (Step 3), we have

$$\phi_1(x) \le z_1(x), \quad \phi_2(x) \le z_2(x), \quad x \in \Omega. \ \blacksquare$$

REMARK 3.4. We note that if we replace (D3) with

(D3′) $g \in C(\overline{\Omega})$ and g is nonnegative away from $\partial\Omega$,

and take

$$\beta = |f(0)| + |h(0)| + [a(M-1) + b(M-1)] \max_{x\in\overline{\Omega}} |g(x)| + 1$$

in the proof of Theorem 3.2, then the conclusions of Theorems 3.2 and 3.3 still hold. Since we do not assume any sign-changing conditions on $f(0)$ or $h(0)$, in our system (1.1) or (1.8), $F(x,0,0)$ or $H(x,0,0)$ could be negative for some $x \in \Omega$. In fact, it is usually assumed that $F(x,u,v), H(x,u,v)$ are nonnegative (see [ACR], [YY], [Z1]) and it is well known that the study of positive solutions with a sign-changing weight is mathematically challenging (see [L], [OSS], [Y]).

REMARK 3.5. From Corollary 5 in [Y], we note that when $p(x) = q(x) \equiv p$ (a constant), then problem (1.5) has at least one positive solution when $\lambda = \theta$ is large enough. Thus, our results partially generalize those in [Y].

4. Asymptotic behavior of positive solutions. In this section, we will discuss the asymptotic behavior of positive solutions near the boundary. We establish the following theorems.

THEOREM 4.1. *Under conditions* (D1)–(D5), *if* (u,v) *is a solution of* (1.1) *which has been obtained in Theorem* 3.2, *then for any* $\nu \in (0,1)$, *there exist positive constants* C_6, C_7 *such that*

$$C_6\lambda d(x) \le u(x) \le C_7(\lambda^{p^+}\eta)^{\frac{1}{p^- -1}}(d(x))^\nu, \tag{4.1}$$

$$C_6\lambda d(x) \le v(x) \le C_7\{2\lambda^{q^+} h[C_4(\lambda^{p^+}\eta)^{\frac{1}{p^- -1}}]\}^{\frac{1}{q^- -1}}(d(x))^\nu, \tag{4.2}$$

as $d(x) \to 0$, *where* η *is a large constant satisfying* (3.12).

Proof. Obviously, when $d(x) < \sigma$ is small enough, there exists $C_6 > 0$ such that

$$u(x), v(x) \ge \phi_1(x) = e^{kd(x)} - 1 \ge C_6\lambda d(x). \tag{4.3}$$

Define

$$v_3(x) = \kappa(d(x))^\nu, \quad x \in \overline{\partial_\varsigma\Omega},$$

where $0 < \varsigma < \delta$ is small enough and $\nu \in (0,1)$ is a constant.

By computation, we have

$$\begin{aligned} -\Delta_{p(x)} v_3(x) = &- (\kappa\nu)^{p(x)-1}(\nu-1)(p(x)-1)(d(x))^{(\nu-1)(p(x)-1)-1} \\ &\times (1+\Pi(x)), \quad x \in \partial_\varsigma\Omega, \end{aligned} \tag{4.4}$$

where

$$\Pi(x) = \frac{d(x)\nabla p \nabla d \ln \kappa\nu}{(\nu-1)(p(x)-1)} + \frac{d(x)\nabla p \nabla d \ln d}{p(x)-1} + \frac{d(x)\Delta d}{(\nu-1)(p(x)-1)}$$

and it is easy to see that $\Pi(x) \to 0$ as $d(x) \to 0$. Let $\kappa = C_4(\lambda^{p^+}\eta)^{\frac{1}{p^- -1}}/\varsigma$. When ς is small enough, from (4.4) we have

$$-\Delta_{p(x)} v_3(x) \geq \kappa^{p(x)-1} \geq \lambda^{p^+}\eta.$$

Obviously $v_3(x) \geq z_1(x)$ when $d(x) = 0$ or $d(x) = \varsigma$ for ς small enough.

On the other hand, when $\max\{(1-\nu)p^+, (1-\nu)q^+\} < 1$, we have $v_3 \in W^{1,p(x)}(\partial_\varsigma \Omega) \cap W^{1,q(x)}(\partial_\varsigma \Omega)$. According to Lemma 2.2, we have $v_3(x) \geq z_1(x)$ on $\overline{\partial_\varsigma \Omega}$. Thus

$$u(x) \leq C_7(\lambda^{p^+}\eta)^{\frac{1}{p^- -1}}(d(x))^\nu \quad \text{as } d(x) \to 0. \tag{4.5}$$

If we let $\kappa = C_4\{2\lambda^{q^+}h[C_4(\lambda^{p^+}\eta)^{\frac{1}{p^- -1}}]\}^{\frac{1}{q^- -1}}/\varsigma$, then for ς small enough we obtain

$$-\Delta_{q(x)} v_3(x) \geq \kappa^{q(x)-1} \geq 2\lambda^{q^+}h[C_4(\lambda^{p^+}\eta)^{\frac{1}{p^- -1}}].$$

Similarly, we have

$$v(x) \leq C_7\{2\lambda^{q^+}h[C_4(\lambda^{p^+}\eta)^{\frac{1}{p^- -1}}]\}^{\frac{1}{q^- -1}}(d(x))^\nu \quad \text{as } d(x) \to 0. \tag{4.6}$$

From (4.3), (4.5) and (4.6), we obtain (4.1) and (4.2). ■

Similarly, we have

THEOREM 4.2. *Under conditions* (D1)–(D5), *if* (u,v) *is a solution of* (1.8) *which has been obtained in Theorem* 3.3, *then for any* $\nu \in (0,1)$, *there exist positive constants* C_8, C_9 *such that*

$$C_8\lambda^{\frac{1}{p^+}} d(x) \leq u(x) \leq C_9(\lambda\eta)^{\frac{1}{p^- -1}}(d(x))^\nu,$$
$$C_8\lambda^{\frac{1}{q^+}} d(x) \leq v(x) \leq C_9\{2\lambda h[C_4(\lambda\eta)^{\frac{1}{p^- -1}}]\}^{\frac{1}{q^- -1}}(d(x))^\nu,$$

as $d(x) \to 0$, *where* η *is a large constant satisfying*

$$\lambda\eta \geq \left(\frac{1}{C_4}\omega\right)^{p^- -1} \geq \lambda\Big\{f[C_4(2\lambda h(\omega))^{\frac{1}{q^- -1}}] + a(\omega)\max_{x\in\overline{\Omega}} g(x)\Big\},$$

where $\omega = \max_{x\in\overline{\Omega}} z_1(x)$ *and* z_1 *satisfies* (3.17).

5. An example. We consider the problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda^{p(x)}[e^{-|x|}u^s + v^m], & x \in \Omega, \\ -\Delta_{q(x)} v = \lambda^{q(x)}[e^{-|x|}v^t + u^n], & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

We assume:

(D6) $0 \le s < p^- - 1$, $0 \le t < q^- - 1$, $0 < m, n$ and $mn < (p^- - 1)(q^- - 1)$.

If we set $g(x) = e^{-|x|}$, $a(u) = u^s$, $b(v) = v^t$, $f(v) = v^m$ and $h(u) = u^n$, then (D3)–(D5) are satisfied. We obtain

THEOREM 5.1. *If* (D1), (D2) *and* (D6) *hold, then* (5.1) *has a positive solution when λ is sufficiently large.*

THEOREM 5.2. *Under conditions* (D1), (D2) *and* (D6), *if (u, v) is a solution of* (5.1) *which has been obtained in Theorem* 5.1, *then for any $\nu \in (0,1)$, there exist positive constants C_6, C_7 such that*

$$C_6 \lambda d(x) \le u(x) \le C_7 (\lambda^{p^+} \eta)^{\frac{1}{p^- - 1}} (d(x))^{\nu}, \tag{5.2}$$

$$C_6 \lambda d(x) \le v(x) \le C_7 \{2\lambda^{q^+} [C_4 (\lambda^{p^+} \eta)^{\frac{1}{p^- - 1}}]^n\}^{\frac{1}{q^- - 1}} (d(x))^{\nu}, \tag{5.3}$$

as $d(x) \to 0$, where $\eta = \lambda^{\frac{(p^+)^2 (q^- - 1) + m q^+ p^+}{(q^- - 1)(p^- - 1 - \theta)} - p^+}$ is a constant and $\theta = \max\{mn/(q^- - 1), s\}$.

Proof. According to Theorem 4.1, we only need η to satisfy (3.12), i.e.

$$\left(\frac{1}{C_4}\omega\right)^{p^- - 1} \ge \lambda^{p^+} \Big\{ [C_4 (2\lambda^{q^+} \omega^n)^{\frac{1}{q^- - 1}}]^m + \omega^s \max_{x \in \overline{\Omega}} e^{-|x|} \Big\}. \tag{5.4}$$

We can assume $\omega > 1$ and take $\theta = \max\{mn/(q^- - 1), s\}$. Since λ is large, we only need to show that

$$\left(\frac{1}{C_4}\omega\right)^{p^- - 1 - \theta} \ge 2\lambda^{p^+} [C_4 (2\lambda^{q^+})^{\frac{1}{q^- - 1}}]^m,$$

that is,

$$\omega \ge C_4 \{2\lambda^{p^+} [C_4 (2\lambda^{q^+})^{\frac{1}{q^- - 1}}]^m\}^{\frac{1}{p^- - 1 - \theta}}.$$

From Lemma 3.1, for any $\nu \in (0,1)$, we have

$$C_3 (\lambda^{p^+} \eta)^{\frac{1}{p^+ - 1 + \nu}} \le \omega,$$

and we only need

$$\eta \ge \frac{\{\frac{C_4}{C_3}\{2\lambda^{p^+} [C_4 (2\lambda^{q^+})^{\frac{1}{q^- - 1}}]^m\}^{\frac{1}{p^- - 1 - \theta}}\}^{p^+ - 1 + \nu}}{\lambda^{p^+}}. \tag{5.5}$$

Assuming λ is large enough, if we let

$$\eta = \lambda^{\frac{(p^+)^2 (q^- - 1) + m q^+ p^+}{(q^- - 1)(p^- - 1 - \theta)} - p^+},$$

then (5.5) holds, so (5.4) does. By Theorem 4.1, we obtain (5.2) and (5.3). ∎

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