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Weighted composition operators between a weighted-type space and the Hardy space on the unit ball

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Abstract. This paper characterizes the boundedness and compactness of weighted composition operators between a weighted-type space and the Hardy space on the unit ball of $\mathbb{C}^n$.

1. Introduction. Let $\mathbb{B}$ be the open unit ball in the n-dimensional complex vector space $\mathbb{C}^n$, $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$, $H^\infty(\mathbb{B})$ the class of all bounded holomorphic functions with the norm $\|f\|_{H^\infty} = \sup_{z\in\mathbb{B}}|f(z)|$, and $S(\mathbb{B})$ the collection of all the holomorphic self-maps of $\mathbb{B}$. Let $d\sigma$ be the normalized rotation invariant measure on the boundary $S = \partial\mathbb{B}$ of $\mathbb{B}$.

Let $\mathbb{N}$ be the set of positive integers and $k \in \mathbb{N}$. A function $f \in H(\mathbb{B})$ is said to belong to the weighted-type space $H_{\log_k}$ (see [5, p. 3112], and also [2] for the corresponding Bloch-type space) if

$$\|f\|_{H_{\log_k}} = \sup_{z\in\mathbb{B}} (1-|z|^2)\Big(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\Big)|f(z)| < \infty, \tag{1}$$

where $e^{[k]}$ is defined inductively by $e^{[1]} = e$, $e^{[k]} = e^{e^{[k-1]}}$ and $\ln^{[j]} z$ is the j times applied logarithm function. It is easy to show that $H_{\log_k}$ is a Banach space with the norm $\|\cdot\|_{H_{\log_k}}$.

For each p with $1 \le p < \infty$, the Hardy space H^p in $\mathbb{B}$ is defined by

$$H^p(\mathbb{B}) = \Big\{f \in H(\mathbb{B}) : \sup_{0<r<1} \int_S |f(r\zeta)|^p\, d\sigma(\zeta) < \infty\Big\}.$$

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It is well known [28] that H^p is a Banach space under the norm

$$\|f\|_p^p = \sup_{0<r<1} \int_S |f(r\zeta)|^p \, d\sigma(\zeta).$$

If X is a Banach space, we denote by B_X the closed unit ball in X. Let $\varphi \in S(\mathbb{B})$. The *composition operator* C_φ induced by φ is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \qquad f \in H(\mathbb{B}),\ z \in \mathbb{B}.$$

This operator has been intensively studied for four decades (see, for example, [1] and [11]). For some recent results, see [22–25, 27] and the references therein.

Let $u \in H(\mathbb{B})$ and $\varphi \in S(\mathbb{B})$. The *weighted composition* operator uC_φ is defined by

$$uC_\varphi(f) = u(f \circ \varphi), \qquad f \in H(\mathbb{B}),\ z \in \mathbb{B}.$$

It is obvious that when $u = 1$, we have the composition operator C_φ. When $\varphi(z) = z$, we obtain the multiplication operator $M_u f(z) = u(z)f(z)$. Therefore weighted composition operators can be regarded as a generalization of multiplication operators and composition operators.

Recently, there has been an increasing interest in describing the boundedness and compactness of weighted composition operators acting on different spaces of holomorphic functions in terms of the inducing functions; see, for example, [1, 6–10, 12, 17, 19–21, 26] and the references therein. For some product-type operators, containing composition operators, see, for example, [2–4, 14–16, 18], and numerous references therein.

The present paper continues this line of research. The remainder is assembled as follows: In Section 2, we state a couple of lemmas. In Sections 3 and 4, we characterize the boundedness and compactness of weighted composition operators between a weighted-type space and the Hardy space on the unit ball of $\mathbb{C}^n$.

Throughout the paper, C will denote a positive constant, the exact value of which may vary from one appearance to the next. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Some lemmas. To begin, let us state a couple of lemmas, which are used in the proofs of the main results.

Lemma 2.1 (see Lemma 2 in [13]). *The function*

$$h_k(x) = x \prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{x}$$

is increasing on the interval $(0, 1]$.

Remark. From Lemma 2.1, it follows that the weight function

$$w(z) = (1-|z|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}$$

is a decreasing function of $|z|$ and $\lim_{|z|\to 1} w(z) = 0$.

The following lemma was proved in [5, Lemma 6]. For related results, see also [16, Lemma 7], [17, Lemma 3] and [18, Lemma 6].

Lemma 2.2. *There exist a positive constant $N = N(n)$ and functions $f_1, \dots, f_N \in H_{\log_k}(\mathbb{B})$ such that*

$$\sum_{m=1}^{N} |f_m(z)| \geq \frac{C}{(1-|z|^2)\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}}, \qquad z \in \mathbb{B}, \tag{2}$$

where C is a positive constant.

Lemma 2.3 (Theorem 4.17 in [28]). *Suppose that $0 < p < \infty$ and $f \in H^p(\mathbb{B})$. Then*

$$|f(z)| \leq \frac{\|f\|_p}{(1-|z|^2)^{n/p}}$$

for all $z \in \mathbb{B}$. Furthermore, the exponent n/p is the best possible.

The following compactness criterion follows from an easy modification of Proposition 3.11 of [1]. Hence we omit the details.

Lemma 2.4. *Assume that $u \in H(\mathbb{B})$ and $\varphi \in S(\mathbb{B})$. Let X or Y be one of the spaces $H_{\log_k}$ and H^p. Then $uC_\varphi : X \to Y$ is compact if and only if $uC_\varphi : X \to Y$ is bounded and for any bounded sequence $\{f_j\}_{j\in\mathbb{N}}$ in X which converges to zero uniformly on compact subsets of $\mathbb{B}$ as $j \to \infty$, we have $\|uC_\varphi f_j\|_Y \to 0$ as $j \to \infty$.*

The proof of the following lemma is well-known, so it is omitted here.

Lemma 2.5. *For $0 < p < \infty$, there is a positive constant C_p, depending on p and N, such that $(\sum_{i=1}^{N} x_i)^p \leq C_p(\sum_{i=1}^{N} x_i^p)$ for all $x_i \in (0,\infty)$, $i \in \{1,\dots,N\}$.*

3. Boundedness and compactness of $uC_\varphi : H_{\log_k} \to H^p$. In this section we characterize the boundedness and compactness of the operator $uC_\varphi : H_{\log_k} \to H^p$.

Theorem 3.1. *Assume that $k \in \mathbb{N}$, $1 \leq p < \infty$, $u \in H(\mathbb{B})$ and $\varphi \in S(\mathbb{B})$. Then $uC_\varphi : H_{\log_k} \to H^p$ is bounded if and only if*

$$M := \sup_{0<r<1} \int_S \frac{|u(r\zeta)|^p}{(1-|\varphi(r\zeta)|^2)^p \big(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(r\zeta)|^2}\big)^p}\, d\sigma(\zeta) < \infty. \tag{3}$$

Proof. Assume that (3) holds. For any $f \in H_{\log_k}$, by (1) and (3) it follows that

$$\begin{aligned}\|uC_\varphi f\|_{H^p}^p &= \sup_{0<r<1} \int_S |uC_\varphi f(r\zeta)|^p \, d\sigma(\zeta) = \sup_{0<r<1} \int_S |u(r\zeta)|^p |f(\varphi(r\zeta))|^p \, d\sigma(\zeta) \\ &\le \|f\|_{H_{\log_k}}^p \sup_{0<r<1} \int_S \frac{|u(r\zeta)|^p}{(1-|\varphi(r\zeta)|^2)^p \big(\prod_{j=1}^k \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(r\zeta)|^2}\big)^p} \, d\sigma(\zeta) \\ &\le M \|f\|_{H_{\log_k}}^p < \infty,\end{aligned}$$

so $uC_\varphi : H_{\log_k} \to H^p$ is bounded.

Conversely, suppose that $uC_\varphi : H_{\log_k} \to H^p$ is bounded. Let $f_1, \dots, f_N \in H_{\log_k}$ satisfy (2). Using Lemma 2.5, we get

$$\begin{aligned}\infty > C\|uC_\varphi\|_{H_{\log_k}\to H^p}^p &\ge \sum_{i=1}^N \|f_i\|_{H_{\log_k}}^p \|uC_\varphi\|_{H_{\log_k}\to H^p}^p \ge \sum_{i=1}^N \|uC_\varphi f_i\|_{H^p}^p \\ &= \sum_{i=1}^N \sup_{0<r<1} \int_S |(uC_\varphi f_i)(r\zeta)|^p \, d\sigma(\zeta) = \sum_{i=1}^N \sup_{0<r<1} \int_S |u(r\zeta) f_i(\varphi(r\zeta))|^p \, d\sigma(\zeta) \\ &\ge \sup_{0<r<1} \int_{\mathbb{B}} \Big(\sum_{i=1}^N |f_i(\varphi(r\zeta))|^p \Big) |u(r\zeta)|^p \, d\sigma(\zeta) \\ &\ge C \sup_{0<r<1} \int_S \Big(\sum_{i=1}^N |f_i(\varphi(r\zeta))| \Big)^p |u(r\zeta)|^p \, d\sigma(\zeta) \\ &\ge C \sup_{0<r<1} \int_S \frac{|u(r\zeta)|^p}{(1-|\varphi(r\zeta)|^2)^p \big(\prod_{j=1}^k \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(r\zeta)|^2}\big)^p} \, d\sigma(\zeta).\end{aligned}$$

so (3) holds. ∎

Next we characterize the compactness of $uC_\varphi : H_{\log_k} \to H^p$.

THEOREM 3.2. *Assume that $k \in \mathbb{N}$, $1 \le p < \infty$, $u \in H(\mathbb{B})$ and $\varphi \in S(\mathbb{B})$. Then $uC_\varphi : H_{\log_k} \to H^p$ is compact if and only if $uC_\varphi : H_{\log_k} \to H^p$ is bounded and*

$$(4) \qquad \lim_{\delta\to 1} \sup_{0<r<1} \int_{\{|\varphi(r\zeta)|>\delta\}} \frac{|u(r\zeta)|^p}{(1-|\varphi(r\zeta)|^2)^p \big(\prod_{j=1}^k \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(r\zeta)|^2}\big)^p} \, d\sigma(\zeta) = 0.$$

Proof. Assume $uC_\varphi : H_{\log_k} \to H^p$ is bounded and (4) holds. Let $f(z) = 1 \in H_{\log_k}$. Then we can easily get

$$M_1 := \sup_{0<r<1} \int_S |u(r\zeta)|^p \, d\sigma(\zeta) < \infty.$$

By (4), for any $\varepsilon > 0$ there exists a $\delta_0 \in (0,1)$ such that for every $\delta \in (\delta_0, 1)$, we have

$$\sup_{0<r<1} \int_{\{|\varphi(r\zeta)|>\delta\}} \frac{|u(r\zeta)|^p}{(1-|\varphi(r\zeta)|^2)^p\left(\prod_{j=1}^k \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(r\zeta)|^2}\right)^p} \, d\sigma(\zeta) < \varepsilon. \tag{5}$$

Now let $\{f_m\}_{m\in\mathbb{N}}$ be a sequence of functions with $\sup_{m\in\mathbb{N}} \|f_m\|_{H_{\log_k}} \le 1$, converging to zero uniformly on compact subsets of $\mathbb{B}$ as $m \to \infty$. Let $\delta \in (\delta_0, 1)$. We have

$$\begin{aligned} \|uC_\varphi f_m\|_{H^p}^p &= \sup_{0<r<1} \int_S |f_m(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta) \\ &\le \sup_{0<r<1} \int_{\{|\varphi(r\zeta)|\le\delta\}} |f_m(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta) \\ &\quad + \sup_{0<r<1} \int_{\{|\varphi(r\zeta)|>\delta\}} |f_m(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta) \\ &=: I_1 + I_2. \end{aligned}$$

Let $K = \{w : |w| \le \delta\}$. Note that it is a compact subset of $\mathbb{B}$. Then

$$\begin{aligned} I_1 &\le \sup_{w\in K} |f_m(w)|^p \sup_{0<r<1} \int_{\{|\varphi(r\zeta)|\le\delta\}} |u(r\zeta)|^p \, d\sigma(\zeta) \\ &\le M_1 \sup_{w\in K} |f_m(w)|^p \to 0 \quad \text{as } m \to \infty, \end{aligned} \tag{6}$$

since $f_m \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $m \to \infty$.

On the other hand, by (1) and (5), we have

$$\begin{aligned} I_2 &\le \|f_m\|_{H_{\log_k}}^p \sup_{0<r<1} \int_{\{|\varphi(r\zeta)|>\delta\}} \frac{|u(r\zeta)|^p}{(1-|\varphi(r\zeta)|^2)^p\left(\prod_{j=1}^k \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(r\zeta)|^2}\right)^p} \, d\sigma(\zeta) \\ &< \varepsilon. \end{aligned} \tag{7}$$

Combining (6) with (7), since ε is an arbitrary positive number, we obtain

$$\lim_{m\to\infty} \|uC_\varphi f_m\|_{H^p} = 0.$$

Hence $uC_\varphi : H_{\log_k} \to H^p$ is compact by Lemma 2.4.

Conversely, suppose that $uC_\varphi : H_{\log_k} \to H^p$ is compact. Then its boundedness is obvious.

Next we prove (4). Let $l \in \{1, \dots, n\}$ and $f_l^{(m)}(z) = z_l^m$, $m \in \mathbb{N}$. By Lemma 2.1, it is easy to check $\|f_l^{(m)}\|_{H_{\log_k}} \le C$, and $f_l^{(m)} \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $m \to \infty$. By Lemma 2.4, $\|uC_\varphi f_l^{(m)}\|_{H^p} \to 0$

as $m \to \infty$. Thus

$$\|uC_\varphi f_l^{(m)}\|_{H^p}^p = \sup_{0<r<1} \int_S |f_l^{(m)}(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta)$$
$$= \sup_{0<r<1} \int_S |\varphi_l(r\zeta)|^{mp} |u(r\zeta)|^p \, d\sigma(\zeta) \to 0 \quad \text{as } m \to \infty.$$

It now follows from Lemma 2.5 and (8) that as $m \to \infty$,

$$\sup_{0<r<1} \int_S |\varphi(r\zeta)|^{mp} |u(r\zeta)|^p \, d\sigma(\zeta)$$
$$\le \sup_{0<r<1} \int_S \Big(\sum_{l=1}^n |\varphi_l(r\zeta)| \Big)^{mp} |u(r\zeta)|^p \, d\sigma(\zeta)$$
$$\le C \sup_{0<r<1} \int_S \Big(\sum_{l=1}^n |\varphi_l(r\zeta)|^{mp} \Big) |u(r\zeta)|^p \, d\sigma(\zeta) \to 0.$$

This means that for every $\varepsilon > 0$, there is an $m_0 \in \mathbb{N}$ such that for every $\delta \in (0,1)$,

$$\delta^{m_0 p} \sup_{0<r<1} \int_{\{|\varphi(r\zeta)|>\delta\}} |u(r\zeta)|^p \, d\sigma(\zeta) \le \sup_{0<r<1} \int_S |\varphi(r\zeta)|^{m_0 p} |u(r\zeta)|^p \, d\sigma(\zeta) < \varepsilon.$$

Let $\delta > 2^{-1/(m_0 p)}$. By the above inequality we obtain

$$\sup_{0<r<1} \int_{\{|\varphi(r\zeta)|>\delta\}} |u(r\zeta)|^p \, d\sigma(\zeta) < 2\varepsilon. \tag{8}$$

Let $f \in B_{H_{\log_k}}$. Define $f_t(z) = f(tz)$, $t \in (0,1)$. It is easy to check that $\|f_t\|_{H_{\log_k}} \le 1$ and f_t converges to f uniformly on compact subsets of $\mathbb{B}$ as $t \to 1$. So by Lemma 2.4,

$$\|uC_\varphi f_t - uC_\varphi f\|_{H^p} = \|uC_\varphi(f_t - f)\|_{H^p} \to 0$$

as $t \to 1$. It follows that for all $\varepsilon > 0$ there is a $t_0 \in (0,1)$ such that for all $t \in (t_0, 1)$,

$$\sup_{0<r<1} \int_S |f(\varphi(r\zeta)) - f_t(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta) < \varepsilon. \tag{9}$$

Now fix t. By Lemma 2.5, (8) and (9), we obtain

$$\sup_{0<r<1} \int_{\{|\varphi(r\zeta)|>\delta\}} |f(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta)$$
$$\le C \sup_{0<r<1} \int_{\{|\varphi(r\zeta)|>\delta\}} |f(\varphi(r\zeta)) - f_t(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta)$$
$$+ C \sup_{0<r<1} \int_{\{|\varphi(r\zeta)|>\delta\}} |f_t(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta)$$

$$\le C\varepsilon + C\|f_t\|_{H^\infty}^p \sup_{0<r<1} \int\limits_{\{|\varphi(r\zeta)|>\delta\}} |u(r\zeta)|^p \, d\sigma(\zeta) \le C\varepsilon(1+\|f_t\|_{H^\infty}).$$

Combining this with (8) shows that for every $f \in B_{H_{\log_k}}$ and $\varepsilon > 0$, there exists $\rho(f,\varepsilon)$, depending on f and ε, such that for $\delta \in [\rho(f,\varepsilon),1)$,

$$\sup_{0<r<1} \int\limits_{\{|\varphi(r\zeta)|>\delta\}} |f(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta) < \varepsilon. \tag{10}$$

Since $uC_\varphi : H_{\log_k} \to H^p$ is compact, $uC_\varphi(B_{H_{\log_k}})$ is a relatively compact subset of H^p. So for each $\varepsilon > 0$, there exists a finite collection of functions $g_1,\dots,g_{N_1}$ in $B_{H_{\log_k}}$ such that for each f in $B_{H_{\log_k}}$, there is a $k \in \{1,\dots,N_1\}$ with $\|uC_\varphi f - uC_\varphi g_k\|_{H^p} < \varepsilon$, which implies that

$$\sup_{0<r<1} \int\limits_{\{|\varphi(r\zeta)|>\delta\}} |f(\varphi(r\zeta)) - g_k(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta) < \varepsilon. \tag{11}$$

By (10), it follows that for $\rho = \max_{1\le k\le N_1} \rho(g_k,\varepsilon)$ and $\delta \in [\rho,1)$,

$$\sup_{0<r<1} \int\limits_{\{|\varphi(r\zeta)|>\delta\}} |g_k(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta) < \varepsilon, \tag{12}$$

for every $k \in \{1,\dots,N_1\}$.

Thus from inequalities (11) and (12), we get

$$\sup_{f\in B_{H_{\log_k}}} \sup_{0<r<1} \int\limits_{\{|\varphi(r\zeta)|>\delta\}} |f(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta) < 2C\varepsilon. \tag{13}$$

Choosing $f_1,\dots,f_N \in H_{\log_k}$ as in Lemma 2.2 with $\|f_i\|_{H_{\log_k}} \le 1$ (if necessary, take $f_i/\|f_i\|$) in (13), we have

$$\sup_{f\in B_{H_{\log_k}}} \sup_{0<r<1} \int\limits_{\{|\varphi(r\zeta)|>\delta\}} |f_i(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta) < 2C\varepsilon.$$

Therefore

$$\begin{aligned}
\sup_{0<r<1} \int\limits_{\{|\varphi(r\zeta)|>\delta\}} & \frac{|u(r\zeta)|^p}{(1-|\varphi(r\zeta)|^2)^p \big(\prod_{j=1}^k \ln^{[j]} \frac{e^{[k]}}{1-|\varphi(r\zeta)|^2}\big)^p} \, d\sigma(\zeta) \\
&\le \sup_{0<r<1} \int\limits_{\{|\varphi(r\zeta)|>\delta\}} \Big(\sum_{i=1}^N |f_i(\varphi(r\zeta))|\Big)^p |u(r\zeta)|^p \, d\sigma(\zeta) \\
&\le C \sum_{i=1}^N \sup_{0<r<1} \int\limits_{\{|\varphi(r\zeta)|>\delta\}} |f_i(\varphi(r\zeta))|^p |u(r\zeta)|^p \, d\sigma(\zeta) \le C\varepsilon.
\end{aligned}$$

Hence (4) holds. ∎

4. Boundedness and compactness of $uC_\varphi : H^p \to H_{\log_k}$. In this section we characterize the boundedness and compactness of the operator $uC_\varphi : H^p \to H_{\log_k}$. The results in this section are somewhat easier to obtain than those in Section 3, but we will give complete proofs for the benefit of the reader.

THEOREM 4.1. *Assume that $k \in \mathbb{N}$, $1 \le p < \infty$, $u \in H(\mathbb{B})$ and $\varphi \in S(\mathbb{B})$. Then $uC_\varphi : H^p \to H_{\log_k}$ is bounded if and only if*

$$\sup_{z\in\mathbb{B}}(1-|z|^2)\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\Big)\frac{|u(z)|}{(1-|\varphi(z)|^2)^{n/p}} < \infty. \tag{14}$$

Proof. Assume that $uC_\varphi : H^p \to H_{\log_k}$ is bounded. For any $w \in \mathbb{B}$, choose the test function

$$f_w(z) = \frac{(1-|\varphi(w)|^2)^a}{(1-\langle z,\varphi(w)\rangle)^{n/p+a}}, \qquad a>0.$$

By Theorem 1.12 in [28] we get

$$\begin{aligned}\|f_w\|_{H^p}^p &= \sup_{0<r<1}\int_S |f_w(r\zeta)|^p\,d\sigma(\zeta) = \sup_{0<r<1}\int_S \frac{(1-|\varphi(w)|^2)^{pa}}{|1-r\langle\zeta,\varphi(w)\rangle|^{n+pa}}\,d\sigma(\zeta)\\ &\le C\sup_{0<r<1}\frac{(1-|\varphi(w)|^2)^{ap}}{(1-r^2|\varphi(w)|^2)^{ap}} \le C.\end{aligned} \tag{15}$$

It follows that

$$\begin{aligned}\|f_w\|_{H^p}\|uC_\varphi\|_{H^p\to H_{\log_k}} &\ge \|uC_\varphi f_w(z)\|_{H_{\log_k}}\\ &= \sup_{z\in\mathbb{B}}(1-|z|^2)\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|z|^2}\Big)|u(z)f_w(\varphi(z))|\\ &\ge (1-|w|^2)\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|w|^2}\Big)|u(w)f_w(\varphi(w))|\\ &= (1-|w|^2)\Big(\prod_{j=1}^{k}\ln^{[j]}\frac{e^{[k]}}{1-|w|^2}\Big)\frac{|u(w)|}{(1-|\varphi(w)|^2)^{n/p}}.\end{aligned} \tag{16}$$

Since w is an arbitrary element in $\mathbb{B}$, condition (14) follows.

Conversely, suppose that (14) holds. Then for any $f \in H^p$, by Lemma 2.3,

$$\begin{aligned}\|uC_\varphi f\|_{H_{\log_k}} &= \sup_{z\in\mathbb{B}}(1-|z|^2)\Big(\prod_{j=1}^{k}\frac{e^{[k]}}{1-|z|^2}\Big)|u(z)f(\varphi(z))|\\ &\le \|f\|_{H^p}\sup_{z\in\mathbb{B}}(1-|z|^2)\Big(\prod_{j=1}^{k}\frac{e^{[k]}}{1-|z|^2}\Big)\frac{|u(z)|}{(1-|\varphi(z)|^2)^{n/p}} < \infty,\end{aligned}$$

proving the boundedness of $uC_\varphi : H^p \to H_{\log_k}$. ■

THEOREM 4.2. *Assume that* $k \in \mathbb{N}$, $1 \le p < \infty$, $u \in H(\mathbb{B})$ *and* $\varphi \in S(\mathbb{B})$. *Then* $uC_\varphi : H^p \to H_{\log_k}$ *is compact if and only if* $uC_\varphi : H^p \to H_{\log_k}$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)\bigg(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\bigg) \frac{|u(z)|}{(1-|\varphi(z)|^2)^{n/p}} = 0. \tag{17}$$

Proof. First, assume that $uC_\varphi : H^p \to H_{\log_k}$ is bounded and (17) holds. Taking $f(z) = 1$, it is easy to show that

$$M_2 := (1-|z|^2)\bigg(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\bigg)|u(z)| < \infty.$$

It follows from (17) that for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that

$$(1-|z|^2)\bigg(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\bigg) \frac{|u(z)|}{(1-|\varphi(z)|^2)^{1/p}} < \varepsilon,$$

when $\delta < |\varphi(z)| < 1$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in H^p, say by L, converging to zero uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$. Then from the above condition we get

$$\begin{aligned}
\|uC_\varphi f_k\|_{H_{\log_k}} &= \sup_{z\in\mathbb{B}} (1-|z|^2)\bigg(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\bigg)|u(z)f_k(\varphi(z))| \\
&= \sup_{|\varphi(z)|\le\delta} (1-|z|^2)\bigg(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\bigg)|u(z)f_k(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)|>\delta} (1-|z|^2)\bigg(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\bigg)|u(z)f_k(\varphi(z))| \\
&\le M_2 \sup_{|w|\le\delta} |f_k(w)| \\
&\quad + \sup_{|\varphi(z)|>\delta} (1-|z|^2)\bigg(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z|^2}\bigg)\frac{|u(z)|}{(1-|\varphi(z)|^2)^{n/p}}\|f_k\|_{H^p} \\
&\le M_2 \sup_{|w|\le\delta} |f_k(w)| + \varepsilon L^p \to \varepsilon L^p, \qquad k\to\infty.
\end{aligned}$$

Since ε is an arbitrary number, we obtain

$$\lim_{k\to\infty} \|uC_\varphi f_k\|_{H_{\log_k}} = 0.$$

Lemma 2.4 now yields the compactness of $uC_\varphi : H^p \to H_{\log_k}$.

Conversely, suppose that $uC_\varphi : H^p \to H_{\log_k}$ is compact. Then it is bounded. Let $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{B}$ be such that $\lim_{k\to\infty} |\varphi(z_k)| = 1$ (if such a sequence does not exist then (17) obviously holds). Let

$$f_k(z) = \frac{(1-|\varphi(z_k)|^2)^a}{(1-\langle z, \varphi(z_k)\rangle)^{n/p+a}}, \qquad a > 0,\ k \in \mathbb{N}.$$

Then from (15) we know that $f_k \in H^p$ with $\sup_{k\in\mathbb{N}} \|f_k\|_{H^p} \le C$, and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$. Lemma 2.4 yields

$$\lim_{k\to\infty} \|uC_\varphi f_k\|_{H_{\log_k}} = 0. \tag{18}$$

From (16) we easily see that

$$\|uC_\varphi f_k\|_{H_{\log_k}} \ge (1-|z_k|^2)\Bigg(\prod_{j=1}^{k} \ln^{[j]} \frac{e^{[k]}}{1-|z_k|^2}\Bigg) \frac{|u(z_k)|}{(1-|\varphi(z_k)|^2)^{n/p}}.$$

Letting $k \to \infty$ and using (18) we obtain (17). ∎

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