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Stokes' formula for stratified forms

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Abstract. A stratified form is a collection of forms defined on the strata of a stratification of a subanalytic set and satisfying a continuity property when we pass from one stratum to another. We prove that these forms satisfy Stokes' formula on subanalytic singular simplices.

1. Introduction. In [P], W. Pawłucki establishes that C^1 forms satisfy Stokes' formula for subanalytic (singular) leaves. A variation of this result is given in [Ł] where the author deals with the subanalytic bounded (not necessarily C^1) forms. In this article, we prove a Stokes theorem for a more general class of forms.

We introduce the notion of stratified forms. A stratified form on a subanalytic set X is, roughly speaking, a collection of forms $(\omega_S)_{S\in\Sigma}$, where Σ is a stratification of X, fulfilling a certain continuity property when we pass from one stratum S to an adjacent stratum S'. We define integration of stratified forms and show that they satisfy Stokes' formula (Theorem 3.1). Our differential forms are not assumed to be subanalytic.

Since every smooth differential form and every subanalytic bounded differential form gives rise to a stratified form, our theorem implies the Stokes theorems given in [P, Ł]. This more general approach is useful in showing for instance that pull-backs of differential forms under subanalytic bi-Lipschitz mappings satisfy Stokes' formula. Such a mapping is not smooth everywhere but just almost everywhere. However, it can be stratified in such a way that the pull-back of a stratified form is a stratified form [V]. The novelty is also that our theorem holds not only for leaves but for subanalytic singular simplices.

2. Stratified forms. We shall work with subanalytic sets (see [DS] for their definition and basic properties). Let $\mathcal{S}_n$ denote the set of all subanalytic

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subsets of $\mathbb{R}^n$. A *subanalytic family of subsets of* $\mathbb{R}^n$ is a family $(A_t)_{t\in B}$, $B \in \mathcal{S}_m$, of subanalytic subsets such that $\bigcup_{t\in B}\{t\} \times A_t$ is a subanalytic subset of $\mathbb{R}^{m+n}$.

Given $X \in \mathcal{S}_n$, the singular k-simplices of X will be the subanalytic continuous mappings $\sigma : \Delta_k \to X$, where Δ_k is the oriented standard simplex of $\mathbb{R}^k$. We denote by $C_k(X)$ the group of singular k-chains with coefficients in $\mathbb{R}$.

By X_{reg} and X_{sing}, we respectively denote the regular and singular locus of X. The *regular locus* is the set constituted by the points of X at which X is a C^∞ manifold of dimension $\dim X$. The *singular locus* of X is the complement of X_{reg} in X.

We will write $\mathrm{int}(X)$ for the interior of X, while $\mathrm{cl}(X)$ will stand for its topological closure. We then set $\mathrm{fr}(X) := \mathrm{cl}(X) \setminus X$. Given a point $x \in \mathbb{R}^n$ and $\alpha > 0$, we write $B(x, \alpha)$ for the ball of radius α centered at x (for the Euclidean metric).

If ω is a differential k-form on a submanifold $S \subset \mathbb{R}^n$, we denote by $|\omega(x)|$ the norm of the linear form $\omega(x) : \otimes^k T_x S \to \mathbb{R}$, where S is equipped with the Riemannian metric inherited from the ambient space.

A *stratification* of X is a locally finite partition Σ of X into subanalytic C^∞ submanifolds of $\mathbb{R}^n$, called *strata*. We then denote by $\Sigma^{(k)}$ the collection of all the strata of Σ of dimension k (the union of all the elements of $\Sigma^{(k)}$, denoted $\bigcup \Sigma^{(k)}$, is thus a k-dimensional manifold). A *refinement* of a stratification Σ of a set X is a stratification Σ' of X such that every stratum of Σ is the union of some strata of Σ'. Two stratifications have a common refinement [DS].

Definition 2.1. Let $X \in \mathcal{S}_n$ and let Σ be a stratification of X. A *stratified differential* 0*-form on* (X, Σ) is a collection of functions $\omega_S : S \to \mathbb{R}$, $S \in \Sigma$, that glue together into a continuous function on X. A *stratified differential k-form on* (X, Σ), $k > 0$, is a collection $(\omega_S)_{S\in\Sigma}$ where, for every S, ω_S is a continuous differential k-form on S such that for any $(x_i, \xi_i) \in \otimes^k TS$, with x_i tending to $x \in S' \in \Sigma$ and ξ_i tending to $\xi \in \otimes^k T_x S'$, we have

$$\lim \omega_S(x_i, \xi_i) = \omega_{S'}(x, \xi).$$

We say that $\omega = (\omega_S)_{S\in\Sigma}$ is *differentiable* if ω_S is C^1 for every $S \in \Sigma$ and if $d\omega := (d\omega_S)_{S\in\Sigma}$ is a stratified form.

Proposition 2.2. *Let (X, Σ) be a stratified set with X closed. If $\omega = (\omega_S)_{S\in\Sigma}$ is a stratified form then, for every $S \in \Sigma$, $|\omega_S(x)|$ is bounded on every bounded subset of S.*

Proof. If ω is a 0-form, this is clear since ω_S is the restriction of a continuous function on X which is closed. Take a k-form ω with $k > 0$, and assume that the result fails for ω. This means that there is a bounded sequence $(x_i, \xi_i) \in \otimes^k TS$, $S \in \Sigma$, such that $\omega_S(x_i, \xi_i)$ goes to infinity. Since x_i is a bounded sequence, we may assume that it is convergent to some element $x \in S'$, $S' \in \Sigma$. Let

$$\xi_i' := \frac{\xi_i}{\omega_S(x_i, \xi_i)},$$

so that $\omega_S(x_i, \xi_i') = 1$, for all i.

As $|\omega_S(x_i, \xi_i)|$ is going to infinity and ξ_i is bounded away from infinity, the multivector ξ_i' clearly goes to zero. Because ω is a stratified form, this implies that

$$\lim \omega_S(x_i, \xi_i') = \omega_{S'}(x, 0) = 0,$$

contradicting $\omega_S(x_i, \xi_i') \equiv 1$. ■

Definition 2.3. Let $\omega = (\omega_S)_{S \in \Sigma}$ be a stratified form. Let Σ' be a refinement of Σ and take $T \in \Sigma'$. By definition of refinements, there is a unique $S \in \Sigma$ which contains T. Let ω_T denote the differential form induced by ω_S on T. It is a routine to check that $\omega' := (\omega_T)_{T \in \Sigma'}$ is also a stratified form. We then say that ω' is a *refinement* of ω.

2.1. Integration of stratified forms. Let (X, Σ) be a stratified set, $X \in \mathcal{S}_n$ compact. Let $\omega = (\omega_S)_{S \in \Sigma}$ be a stratified k-form on (X, Σ) and let $Y \subset X$ be a subanalytic subset of X of dimension k such that Y_{reg} is oriented. We are going to define the integral of ω on Y, denoted $\int_Y \omega$.

Let Σ' be a refinement of Σ such that Y_{reg} is a union of some strata of Σ' (such a stratification exists because Y_{reg} is subanalytic [DS]). This refinement induces a refinement ω' of ω (as explained in Definition 2.3). We naturally define

$$\int_Y \omega := \sum_{S \in \Sigma'^{(k)}} \int_S \omega_S,$$

where every stratum is endowed with the orientation induced by Y_{reg}. That this integral is finite follows from the fact that ω_S is bounded (by Proposition 2.2) on a set of finite measure (bounded subanalytic manifolds have finite measure [LR]).

Let us check that this definition is independent of the refinement Σ' chosen. Since two stratifications have a common refinement [DS], it is enough to make sure that the integral will be the same if we use a refinement Σ'' of Σ' (instead of Σ'). As Σ'' is a refinement itself, $\dim \bigcup \Sigma'^{(k)} \setminus \bigcup \Sigma''^{(k)} < k$ (which entails that this set is negligible) so that

$$\sum_{S \in \Sigma'^{(k)}} \int_S \omega_S = \sum_{T \in \Sigma''^{(k)}} \int_T \omega_T.$$

This shows that the integral is independent of the refinement chosen.

Integration on singular simplices. We now turn to define the integral of the stratified k-form ω over an oriented singular simplex $\sigma : \varDelta_k \to X$. As σ is subanalytic, there exist stratifications $\mathcal{P}$ of $\varDelta_k$ and $\mathcal{Q}$ of X such that for any S in $\mathcal{P}$ there is $T \in \mathcal{Q}$ such that the mapping $\sigma_{|S} : S \to T$, induced by the restriction of σ, is a C^2 surjective submersion. Since two stratifications have a common refinement, possibly refining ω (see Definition 2.3), we may assume that ω is a stratified form on $\mathcal{P}$. We now set

$$\int_\sigma \omega = \sum_{S \in \mathcal{P}^{(k)}} \int_S \sigma^* \omega_{\sigma(S)}.$$

Again, since the manifold $\bigcup \mathcal{P}^{(k)}$ is independent of the stratification $\mathcal{P}$ up to a negligible set, this definition is clearly independent of the stratifications chosen. The integral over a subanalytic chain $c \in C_k(X)$ is then defined naturally.

NOTE. The form $(\sigma^* \omega_{\sigma(S)})_{S \in \mathcal{P}}$ is not necessarily a stratified form on $(\varDelta_k, \mathcal{P})$. In particular, $\sigma^* \omega_{\sigma(S)}$ is not necessarily bounded (σ is not assumed to have bounded first derivative).

3. Stokes theorem for stratified forms. In this section we establish the main result of this note:

THEOREM 3.1. *Let $(X, \varSigma)$ be a subanalytic stratified set. If ω is a differentiable stratified $(j-1)$-form on $(X, \varSigma)$, then for all $c \in C_j(X)$,*

$$\int_c d\omega = \int_{\partial c} \omega.$$

The proof of this theorem requires some preliminary lemmas.

DEFINITION 3.2. Let $L \in \mathcal{S}_n$ be a compact set of dimension k. We say that L is a *leaf* if there is a dense subanalytic subset $Z \subset \mathrm{fr}(L_{\mathrm{reg}})$ such that $\mathrm{cl}(L_{\mathrm{reg}})$ is a C^1 submanifold with boundary (of $\mathbb{R}^n$) at every point of Z. We then set $\partial L := \mathrm{fr}(L_{\mathrm{reg}})$. A leaf L is *orientable* if L_{reg} is. Observe that any orientation of L_{reg} induces an orientation of ∂L.

LEMMA 3.3. *Let $L \in \mathcal{S}_n$ be a leaf of dimension k. Any subanalytic closed subset $L' \subset L$ of dimension k is a leaf.*

Proof. Let $L' \subset L$ be as in the statement of the lemma and observe that $L'_{\mathrm{reg}} \subset L_{\mathrm{reg}}$. Let x be a generic regular point of $\mathrm{fr}(L'_{\mathrm{reg}})$. If x lies in L_{reg} then, as $\mathrm{fr}(L'_{\mathrm{reg}})$ is a C^1 submanifold of L_{reg} at x (x is generic), it is clear that $\mathrm{cl}(L'_{\mathrm{reg}})$ is a C^1 manifold with boundary at x (the closure of an open subanalytic set in a manifold is a manifold with boundary at every regular point of the frontier). We thus can suppose that $x \in \partial L$. As x is generic in $\mathrm{fr}(L'_{\mathrm{reg}})$, we can assume that $x \notin \mathrm{fr}(\mathrm{fr}(L'_{\mathrm{reg}}) \cap L_{\mathrm{reg}})$ (this set has dimension strictly less than $k-1$). This means that if x lies in $\mathrm{fr}(L_{\mathrm{reg}})$ then, for x

generic, $\mathrm{fr}(L_{\mathrm{reg}})$ and $\mathrm{fr}(L'_{\mathrm{reg}})$ coincide locally near x, and the result follows from the fact that L is itself a leaf. ∎

We denote by $\mathcal{H}^j$ the j-dimensional Hausdorff measure.

LEMMA 3.4. *Let $(B_{\delta,\varepsilon})_{(\delta,\varepsilon)\in\mathbb{R}^2}$ be a subanalytic family of j-dimensional subsets of $\mathbb{R}^n$. Assume that $\bigcup_{(\delta,\varepsilon)\in\mathbb{R}^2}\{(\delta,\varepsilon)\}\times B_{\delta,\varepsilon}$ is compact and that $\dim B_{0,0} < j$. Then*

$$\lim_{(\delta,\varepsilon)\to(0,0)} \mathcal{H}^j(B_{\delta,\varepsilon}) = 0.$$

Proof. Given a vector space $P \subset \mathbb{R}^n$ of dimension j we set

$$K_l^P(B_{\delta,\varepsilon}) := \{x \in P : \operatorname{card} \pi_P^{-1}(x) \cap B_{\delta,\varepsilon} = l\},$$

where π_P is the orthogonal projection onto P and $\operatorname{card}\pi_P^{-1}(x)\cap B_{\delta,\varepsilon}$ stands for the cardinality of this set (this cardinality is finite for almost every P and can only take finitely many values $1,\dots,N$). In view of the Cauchy–Crofton formula [F], we have

$$\mathcal{H}^j(B_{\delta,\varepsilon}) = \sum_{l=1}^{N} l \int_{P\in\mathbb{G}_j^n} \mathcal{H}^j(K_l^P(B_{\delta,\varepsilon}))\, d\gamma_{j,n},$$

where $\mathbb{G}_j^n$ stands for the Grassmannian of j-dimensional vector spaces in $\mathbb{R}^n$ and $\gamma_{j,n}$ is a Radon measure (induced by the Haar measure of the group of orthogonal linear mappings acting on $\mathbb{G}_j^n$). It is therefore enough to show that

$$\lim_{(\delta,\varepsilon)\to(0,0)} \mathcal{H}^j(K_l^P(B_{\delta,\varepsilon})) = 0.$$

Thanks to Lebesgue's Dominated Convergence Theorem, it suffices to show that for almost every $x\in P$, $x\notin\pi_P(B_{\delta,\varepsilon})$ for δ and ε small enough. But if $x\notin\pi_P(B_{0,0})$ (which is $\mathcal{H}^j$-negligible) then, as $\bigcup_{(\delta,\varepsilon)\in\mathbb{R}^2}\{(\delta,\varepsilon)\}\times B_{\delta,\varepsilon}$ is a closed subset, x cannot belong to $\pi_P(B_{\delta,\varepsilon})$ if δ and ε are chosen small enough. ∎

LEMMA 3.5. *Every compact subanalytic set may be decomposed into a finite union of leaves* (*not necessarily disjoint*).

Proof. Let X be a compact subanalytic set and consider a subanalytic triangulation $h : |K| \to X$, K a simplicial complex. For every j-dimensional simplex $\sigma\in K$ ($j=\dim X$), $h(\sigma)$ is a C^0 manifold with boundary, C^1 at the interior points. It is actually a C^1 manifold with boundary at every generic point of the boundary (for instance, the points at which Whitney (b) regularity holds do have this property [P]). ∎

LEMMA 3.6. *Let M be a subanalytic C^1 manifold with boundary that we endow with its canonical stratification $\Sigma := \{M\setminus\partial M, \partial M\}$. Take a subanalytic function $\rho : M\to\mathbb{R}$ which is C^1 and nonnegative with $\rho^{-1}(0)=\partial M$.*

For any differentiable stratified differential form ω on (M, Σ) vanishing outside a compact subset of M we have

$$\lim_{\varepsilon \to 0,\, \varepsilon > 0} \int_{\rho = \varepsilon} \omega = \int_{\partial M} \omega. \tag{3.1}$$

Proof. Up to a partition of unity we may assume that the support of ω fits in one chart of M and, up to a coordinate system, we may identify M with $B(0,\alpha) \cap \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_k \geq 0\}$, $\alpha > 0$. The coefficients of the form $\omega_{x_k} := \omega_{|\mathbb{R}^{k-1} \times \{x_k\}}$ are continuous with respect to x_k. Hence, in the case where ρ is the function given by $\rho(x_1, \ldots, x_k) = x_k$ for all $(x_1, \ldots, x_k)$, the result follows from Lebesgue's Dominated Convergence Theorem.

As a matter of fact, it is enough to check that the limit always exists and is independent of ρ. Let ρ be a function satisfying the assumptions of the lemma. By Stokes' formula we have, for relevant orientations and $0 < \varepsilon < \varepsilon'$,

$$\int_{\rho = \varepsilon} \omega - \int_{\rho = \varepsilon'} \omega = \int_{\rho^{-1}([\varepsilon, \varepsilon'])} d\omega.$$

The measure of $\rho^{-1}([\varepsilon, \varepsilon'])$ tends to zero as $(\varepsilon, \varepsilon')$ goes to zero. As $d\omega$ is bounded, this implies that the right-hand side goes to zero. Consequently, the limit exists for all such functions ρ. That the limit is independent of ρ follows from an analogous argument. ∎

We first establish Stokes' formula for stratified forms on a stratum whose closure is a leaf.

LEMMA 3.7. *Let $X \in \mathcal{S}_n$ be compact and let Σ be a stratification of X. Let $\omega = (\omega_S)_{S \in \Sigma}$ be a differentiable stratified differential $(k-1)$-form on X. Fix a stratum S of dimension k. If $\mathrm{cl}(S)$ is an oriented leaf then*

$$\int_S d\omega_S = \int_{\mathrm{fr}(S)} \omega, \tag{3.2}$$

where $\mathrm{fr}(S)_{\mathrm{reg}}$ is endowed with the induced orientation.

Proof. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a C^1 subanalytic positive function such that $\rho^{-1}(0) = \mathrm{fr}(S)$ (the function $x \mapsto d(x, \mathrm{fr}(S))$ has this property; one may approximate this function by a C^1 subanalytic function [S] to get such a function ρ). Set

$$S_\varepsilon := \{x \in \mathrm{cl}(S) : \rho(x) \geq \varepsilon\}.$$

Note that for $\varepsilon > 0$ small enough, by Sard's theorem, S_ε is a smooth manifold with boundary and ω is a smooth form on it. Thus, by the classical Stokes formula,

$$\int_{S_\varepsilon} d\omega_S = \int_{\partial S_\varepsilon} \omega_S.$$

As $d\omega_S$ is bounded, it easily follows from Lebesgue's Dominated Convergence Theorem that

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon} d\omega_S = \int_S d\omega_S.$$

We thus only have to show that

$$\lim_{\varepsilon\to 0} \int_{\partial S_\varepsilon} \omega_S = \int_{\mathrm{fr}(S)} \omega. \tag{3.3}$$

As $\mathrm{cl}(S)$ is a leaf, there is a subanalytic subset $S' \subset \partial S$ such that $\mathrm{cl}(S)$ is a C^1 manifold with boundary at every point of $\mathrm{fr}(S) \setminus S'$ and such that $\dim S' < k-1$. Set, for $\delta \geq 0$,

$$U_\delta := \{x \in \mathbb{R}^n : d(x, S') \leq \delta\},$$

and let $(\varphi_\delta, \psi_\delta)$ denote a partition of unity subordinated to the covering $(\mathrm{int}(U_\delta), \mathbb{R}^n \setminus U_{\delta/2})$.

As (by Proposition 2.2) ω_S is bounded and because φ_δ has support in U_δ, we can write, for some constant C,

$$\int_{\partial S_\varepsilon} \varphi_\delta\, \omega \leq C\mathcal{H}^{k-1}(U_\delta \cap \partial S_\varepsilon). \tag{3.4}$$

Applying Lemma 3.4 to the subanalytic family $B_{\delta,\varepsilon} := U_\delta \cap \partial S_\varepsilon$ we see that

$$\limsup_{\delta\to 0}\, \limsup_{\varepsilon\to 0} \mathcal{H}^{k-1}(U_\delta \cap \partial S_\varepsilon) = 0.$$

By (3.4), this entails that

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \int_{\partial S_\varepsilon} \varphi_\delta\, \omega_S = 0.$$

As a matter of fact, since for each $\delta > 0$ we have $\omega = \varphi_\delta\omega + \psi_\delta\omega$, proving (3.3) reduces to showing that

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \int_{\partial S_\varepsilon} \psi_\delta\, \omega_S = \int_{\mathrm{fr}(S)} \omega. \tag{3.5}$$

It follows from Lemma 3.6 (applied to $\psi_\delta\omega$, which induces a stratified form with compact support on the manifold with boundary $S \setminus U_{\delta/2}$) that

$$\lim_{\varepsilon\to 0} \int_{\partial S_\varepsilon} \psi_\delta\, \omega = \int_{\mathrm{fr}(S)\setminus U_{\delta/2}} \psi_\delta\, \omega = \int_{\mathrm{fr}(S)} \psi_\delta\, \omega.$$

Passing to the limit as $\delta > 0$ tends to zero and applying Lebesgue's Dominated Convergence Theorem, we see that this yields (3.5). ∎

Proof of Theorem 3.1. It is of course enough to carry out the proof for a single simplex $\sigma : \Delta_j \to X$. Denote by Γ the graph of σ and take two Whitney stratifications $\mathcal{S}$ and Σ' of Γ and X respectively such that the mapping $\pi : \Gamma \to X$, $(x,y) \mapsto y$, maps submersively strata onto strata. By Lemmas 3.3 and 3.5, we may suppose that the closures of all the strata of Σ' and $\mathcal{S}$ are leaves. We may also assume that if π induces an immersion on a stratum then it induces an injective mapping on this stratum (this property is clearly preserved under refinement, and it is easy to construct a stratification satisfying this property using cell decompositions). Because Σ and Σ' are Whitney stratifications, they satisfy the *frontier condition* (i.e. the closure of each stratum is a union of strata).

Since the projection $(x,y) \mapsto x$ induces a subanalytic homeomorphism on Γ, we will identify Δ_j with Γ, and $\mathcal{S}$ with Σ, and work as if σ were the mapping π. Refining the stratifications Σ and Σ', we may assume that ω is a stratified form on Σ'. We endow the strata of Σ of dimension j with the orientation induced by Δ_j.

Let $S \in \Sigma$ and let $i := \dim S$. Either $\sigma_{|S}$ is a diffeomorphism onto its image or $\dim \sigma(S) < i$. In the former case, if we endow S and $\sigma(S)$ with coherent orientations, we obviously have, for any continuous bounded i-form α on S,

$$\int_S \sigma_{|S}^* \alpha = \int_{\sigma(S)} \alpha. \tag{3.6}$$

If dim $\sigma(S) < i$, then both sides are zero (since $\sigma_{|S}^*\alpha$ is identically zero) and this equality continues to hold.

Observe also that if $\dim S = j$ and if $\sigma_{|S}$ is a diffeomorphism then $\sigma(\mathrm{fr}(S)) = \mathrm{fr}(\sigma(S))$, which entails that (putting on $\mathrm{fr}(S)_{\mathrm{reg}}$ the orientation induced by the leaf $\mathrm{cl}(S)$)

$$\int_{\mathrm{fr}(\sigma(S))} \omega = \int_{\sigma(\mathrm{fr}(S))} \omega. \tag{3.7}$$

We claim that this formula is true even if $\sigma_{|S}$ fails to be a diffeomorphism. Indeed, assume $\dim \sigma(S) < j$ and take a stratum $T \subset \mathrm{fr}(\sigma(S))$ of dimension $j-1$, on which we choose an orientation. As the left-hand side of (3.7) vanishes in this case (since $\dim \mathrm{fr}(\sigma(S)) < j-1$), it is enough to check that so does the right-hand side. For any stratum $S' \subset \mathrm{fr}(S) \cap \sigma^{-1}(T)$ of dimension $j-1$, as $\sigma_{|S'} : S' \to T$ is a diffeomorphism, we have $\sigma(S') = \pm T$ (here $-T$ means T with the opposite orientation). As $\dim \sigma(S) < j$, we must have $\sum_{S' \in \Sigma^{(j-1)},\, S' \subset \mathrm{fr}(S)} \sigma(S') = 0$ (as formal sums of oriented manifolds). This shows that the right-hand side of (3.7) vanishes, as claimed.

By Lemma 3.7, for any stratum S of dimension j, (since the closure of every stratum is a leaf) we have

$$\int_{\sigma(S)} d\omega = \int_{\mathrm{fr}(\sigma(S))} \omega \overset{(3.7)}{=} \int_{\sigma(\mathrm{fr}(S))} \omega. \tag{3.8}$$

We thus obtain

$$\begin{aligned}\int_{\sigma} d\omega &= \sum_{S\in\Sigma^{(j)}} \int_S \sigma_{|S}^{*}\, d\omega_{\sigma(S)} \\ &\overset{(3.6)}{=} \sum_{S\in\Sigma^{(j)}} \int_{\sigma(S)} d\omega_{\sigma(S)} \overset{(3.8)}{=} \sum_{S\in\Sigma^{(j)}} \int_{\sigma(\mathrm{fr}(S))} \omega.\end{aligned} \tag{3.9}$$

Observe that every stratum $T \in \Sigma^{(j-1)}$ lies in the frontier of exactly two strata of Σ (inducing on T opposite orientations) if $T \cap \partial\Delta_j = \emptyset$ and one such stratum whenever $T \subset \partial\Delta_j$ (we may assume that $\partial\Delta_j$ is a union of strata). Therefore

$$\sum_{S\in\Sigma^{(j)}} \int_{\sigma(\mathrm{fr}(S))} \omega = \sum_{S\in\Sigma^{(j-1)},\, S\subset\partial\Delta_j} \int_{\sigma(S)} \omega \overset{(3.6)}{=} \int_{\partial\sigma} \omega.$$

Together with (3.9), this yields the desired formula. ∎

REMARK 3.8. If (X,Σ) is a stratified set, any smooth differential form ω which is defined in a neighborhood of X gives rise to a stratified form $(\omega_S)_{S\in\Sigma}$ on (X,Σ) obtained by considering the respective restrictions of ω to the strata. The Stokes formula that we have proved is therefore a generalization of the Stokes formula for smooth forms on singular varieties. For the same reasons, it also implies the generalized Stokes formulas given in [Ł, P].

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