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Propagation of delayed lattice differential equations without local quasimonotonicity

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Abstract. This paper is concerned with the traveling wave solutions and asymptotic spreading of delayed lattice differential equations without quasimonotonicity. The spreading speed is obtained by constructing auxiliary equations and using the theory of lattice differential equations without time delay. The minimal wave speed of invasion traveling wave solutions is established by investigating the existence and nonexistence of traveling wave solutions.

1. Introduction. Lattice dynamical systems are very important to describe some evolutionary processes in life sciences [BeC, K] and phase transitions [BC]. For scalar lattice differential equations, a typical example is

$$\frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + f(u_n(t)), \qquad n \in \mathbb{Z},\ t > 0, \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$, and

$$[\mathcal{D}u]_n(x) = D(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t))$$

with $D > 0$. In the past decades, much attention has been paid to its propagation modes indexed by traveling wave solutions and asymptotic spreading: see [AS, BC, BeC, CMV, CG1, CG2, CMS, HL, K, MP1, MP2, MP3, Shen, Ton, Zin1, Zin2]. In particular, to reflect the maturation time of the species under consideration and the time needed for the signals to travel along axons and to cross synapses, time delay was introduced in lattice differential equations, and a delayed version of (1.1) is

$$\frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + f(u_n(t), u_n(t-\tau)), \qquad n \in \mathbb{Z},\ t > 0, \tag{1.2}$$

in which $\tau \geq 0$ is the time delay. Since Wu and Zou [WZ], some results about the existence of traveling wave solutions and the estimation of asymptotic

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spreading of (1.2) have been established: we refer to [FWZ1, FWZ2, HLR, HLZ, LZ, LLP, LW, MZ1, MZ2, TZ, WLW, WHW, Zou].

To better introduce the known results, we assume that

$$f(0,0) = f(K,K) = 0, \quad f(u,u) > 0, \quad u \in (0,K), \tag{1.3}$$

with some $K > 0$, and $f(u,v)$ is continuous for $u, v \in [0,K]$. To apply a comparison principle, the (local) monotonicity of $f(u,v)$ for $v > 0$ is needed; for example, see the scalar models in [FWZ1, FWZ2, LZ, LLP, LW, MZ1, MZ2, TZ, WLW, WHW, Zou]. Moreover, if the time delay $\tau > 0$ is small enough, some results on the existence of traveling wave solutions have been established (see Huang et al. [HLZ]).

The purpose of this paper is to investigate the propagation modes of (1.2) if $f(u,v)$ is not increasing in v near 0. For the sake of convenience, we consider the following special form of (1.2):

$$\frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + u_n(t)g(u_n(t), u_n(t-\tau)), \quad n \in \mathbb{Z},\, t > 0, \tag{1.4}$$

in which $g : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following assumptions:

- (H1) $g(1,0) = 0$, and $g(u,0) > 0$ for $u \in (0,1)$;
- (H2) $g(u,v)$ is Lipschitz continuous and strictly decreasing for $u, v \in [0,1]$, and $g(u,0) \to -\infty$ as $u \to \infty$, more precisely, there exists $I > 0$ such that
$$0 \le |g(u,v) - g(0,0)| \le I(u+v), \quad u, v \in [0,1];$$
- (H3) $g(0,1) > 0$, and there exists $E \in (0,1)$ such that $g(E,E) = 0$;
- (H4) if $1 > \overline{u} \ge \underline{u} > 0$ are such that
$$g(\underline{u}, \overline{u}) \le 0, \quad g(\overline{u}, \underline{u}) \ge 0,$$
then $\underline{u} = \overline{u} = E$.

By (H2), we see that (1.4) does not satisfy the (local) quasimonotonicity condition of [FWZ1, FWZ2, LZ, LLP, LW, MZ1, MZ2, TZ, WLW, WHW, Zou], and a typical example of g satisfying (H1)–(H4) is

$$g(u,v) = 1 - u - av, \quad a \in (0,1),$$

which is a special form of logistic nonlinearity with time delay.

In what follows, by using the spreading speed of undelayed scalar lattice differential equations and constructing auxiliary equations without time delay, we shall investigate the propagation of (1.4). We first prove that the spreading speed of $u_n(t)$ defined by the corresponding initial value problem for (1.4) is the same as that of $g(u,v) = g(u,u)$ by the idea of Lin [Lin], which implies the persistence of spreading speeds of delayed lattice differential equations even if the time delay τ is large and the equation cannot generate monotone semiflows. Furthermore, we establish the minimal wave speed of (1.4) by investigating the existence and nonexistence of traveling wave solutions for all positive wave speeds, which is motivated by the results of Lin and Ruan [LR]. These traveling wave solutions correspond to the successful invasion of one new invader in population dynamics.

In this paper, we shall use the standard ordering and intervals in $\mathbb{R}$. Let

$$C = C(\mathbb{R}, \mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R};\ u \text{ is uniformly continuous and bounded}\}.$$

Then C is a Banach space equipped with the standard supremum norm. When $a < b$, denote

$$C_{[a,b]} = \{u \in C;\ a \le u \le b\}.$$

If $u, u' \in C$, then $u \in C^1(\mathbb{R}, \mathbb{R})$. For any fixed $\mu > 0$, define

$$B_\mu(\mathbb{R}, \mathbb{R}) = \Big\{u \in C(\mathbb{R}, \mathbb{R}); \sup_{t\in\mathbb{R}}\{|u(t)|e^{-\mu|t|}\} < \infty\Big\};$$

then $B_\mu(\mathbb{R}, \mathbb{R})$ is a Banach space with the norm defined by

$$|u|_\mu = \sup_{t\in\mathbb{R}}\{|u(t)|e^{-\mu|t|}\} \quad \text{for } u \in B_\mu(\mathbb{R}, \mathbb{R}).$$

Let l^∞ be the space of bounded doubly infinite sequences $\{u(n)\}_{n=-\infty}^{\infty}$.

We now give the following definition of traveling wave solutions.

Definition 1.1. A *traveling wave solution* of (1.4) is a special solution of the form $u_n(t) = \phi(n+ct)$, where $c > 0$ is the *wave speed* and $\phi \in C^1(\mathbb{R}, \mathbb{R})$ is the *wave profile* that propagates in $\mathbb{Z}$.

From Definition 1.1, ϕ and c must satisfy

(1.5)
$$c\frac{d\phi(\xi)}{d\xi} = D\big(\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)\big) + \phi(\xi)g(\phi(\xi), \phi(\xi - c\tau)), \qquad \xi \in \mathbb{R}.$$

To better reflect the evolutionary processes, we also require the following asymptotic boundary value condition:

$$\lim_{\xi\to-\infty} \phi(\xi) = 0, \qquad \lim_{\xi\to\infty} \phi(\xi) = E. \tag{1.6}$$

To index the asymptotic spreading, we also give the following definition.

Definition 1.2. Assume that $u_n(t) \ge 0$ for all $n \in \mathbb{N}$ and $t > 0$. Then $c_1 > 0$ is the *spreading speed* of $u_n(t)$ if

(1) for any $c > c_1$, $\lim_{t\to\infty} \sup_{|n|>ct} u_n(t) = 0$;
(2) for any $c < c_1$, $\liminf_{t\to\infty} \inf_{|n|<ct} u_n(t) > 0$.

The spreading speed of delayed lattice differential equations has been investigated in [LZ, TZ, WHW]. In particular, if f in (1.1) satisfies

(f1) $f(u) = uh(u)$, $h(M) = 0$ for some $M > 0$,
(f2) $h(u)$ is Lipschitz continuous and decreasing for $u \in [0, M]$,

then we can obtain the spreading speed of $u_n(t)$ defined by the corresponding initial value problem of (1.1) (see [LZ, TZ, WHW]). More precisely, consider the initial value problem

$$(1.7)\qquad \begin{cases} \frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + f(u_n(t)), & n \in \mathbb{Z},\ t > 0, \\ u_n(0) = \psi(n), & n \in \mathbb{Z}. \end{cases}$$

By Ma et al. [MWZ] and Weng et al. [WHW] (see also an abstract form of the comparison principle by Weinberger [Wei]), we have the following conclusions.

Lemma 1.3. *Assume that* (f1)–(f2) *hold. If* $0 \le \psi(n) \le M$ *for all* $n \in \mathbb{Z}$, *then* (1.7) *has a solution* $u_n(t)$ *for all* $n \in \mathbb{Z}$ *and* $t > 0$. *If* $w_n(t)$, $n \in \mathbb{Z}$, $t > 0$, *satisfies*

$$\begin{cases} \frac{dw_n(t)}{dt} \ge (\le)\ [\mathcal{D}w]_n(t) + f(w_n(t)), \\ w_n(0) \ge (\le)\ \psi(n), \end{cases}$$

or

$$w_n(t) \ge (\le)\, e^{-(2D+d)(t-\theta)} w_n(\theta) + \int_\theta^t e^{-(2D+d)(t-s)} \big[d w_n(s) + D(w_{n+1}(s) + w_{n-1}(s)) + f(w_n(s))\big]\, ds$$

with any fixed $d \ge 0$ *and* $\theta \in [0,t)$, *then* $w_n(t) \ge (\le)\, u_n(t)$ *for all* $n \in \mathbb{Z}$ *and* $t > 0$. *In particular,* $w_n(x)$ *is called an* upper (*resp. a* lower) solution *of* (1.7).

For $\lambda, c > 0$, define

$$\Delta(\lambda, c) = D(e^\lambda + e^{-\lambda} - 2) - c\lambda + h(0).$$

Lemma 1.4. *There exists* $c_2 := \inf_{\lambda>0} \frac{D(e^\lambda+e^{-\lambda}-2)+h(0)}{\lambda} > 0$ *such that:*

(1) *If* $c > c_2$, *then* $\Delta(\lambda,c) = 0$ *has two distinct positive real roots* $\lambda_1(c) < \lambda_2(c)$ *satisfying*

$$\Delta(\lambda, c) \begin{cases} < 0, & \lambda \in (\lambda_1(c), \lambda_2(c)), \\ > 0, & \lambda \in (0, \lambda_1(c)) \text{ or } \lambda > \lambda_2(c). \end{cases}$$

(2) $\Delta(\lambda,c) = 0$ *has no real roots for* $c < c_2$.
(3) *Let* $\varepsilon > 0$ *be any positive constant. Then* c_2 *is continuous and strictly increasing in* $h(0) \ge \varepsilon$.

Lemma 1.5. *Assume that* (f1)–(f2) *hold. If* $0 \le \psi(n) \le M$ *for all* $n \in \mathbb{Z}$, $\psi_n(0) \ne 0$ *for some* $n \in \mathbb{Z}$, *and* $\psi(n) = 0$ *for all large* $|n|$, *then* c_2 *is the spreading speed of* $u_n(t)$ *defined by* (1.7).

2. Asymptotic spreading. In this section, we assume that (H1)–(H3) hold and consider the long time behavior of the initial value problem

$$\begin{cases} \frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + u_n(t)g(u_n(t), u_n(t-\tau)), & n \in \mathbb{Z},\ t > 0, \\ u_n(s) = \varphi_n(s), \quad n \in \mathbb{Z},\ s \in [-\tau, 0], \end{cases} \tag{2.1}$$

in which $\varphi_n : [-\tau, 0] \to \mathbb{R}$ satisfies $0 \le \varphi_n(s) \le 1$, and for each $n \in \mathbb{Z}$ it is uniformly continuous in $s \in [-\tau, 0]$.

In what follows, let $d > 0$ be a constant such that the function

$$[0,1] \ni \omega \mapsto dw + wg(w, 1)$$

is increasing. Consider

$$\frac{dw_n(t)}{dt} = -2Dw_n(t) - dw_n(t), \qquad w_n(0) = \omega(n), \qquad n \in \mathbb{Z}.$$

Then we obtain an analytic and strictly positive semigroup in l^∞ because of the boundedness of $2D + d$. Then the standard semigroup theory implies the following results.

Lemma 2.1. *Assume that* (H1)–(H2) *hold. Then* (2.1) *admits a unique mild solution $u_n(t)$ for all $t > 0$ and $n \in \mathbb{Z}$, which can be written as*

$$u_n(t) = e^{-(2D+d)t}\varphi_n(0) + \int_0^t e^{-(2D+d)(t-s)} H_n(s)\, ds \tag{2.2}$$

with

$$H_n(s) = du_n(s) + D(u_{n+1}(s) + u_{n-1}(s)) + u_n(s)g(u_n(s), u_n(s-\tau)).$$

This lemma is also clear by the variation of constants formula [MZ1]; we omit the proof here. It should be noted that Lemma 2.1 remains true even if $\tau = 0$. By (H1)–(H2), $u_n(t)$ also satisfies the following conclusion.

Lemma 2.2. *Assume that* (H1)–(H2) *hold. If $u_n(t)$ is defined by* (2.1), *then*

$$0 \le u_n(t) \le 1, \qquad t > 0,\ n \in \mathbb{Z}.$$

The positivity of $u_n(t)$ is clear from the quasipositivity of $ug(u, v)$ (see Martin and Smith [MS, p. 7]), and $u_n(t) \le 1$ is clear by (H2) and Lemma 1.3. Furthermore, using Lemmas 2.1–2.2, we obtain the following conclusion.

Lemma 2.3. *Assume that $u_n(t)$ is defined by* (2.1) *and* (H1)–(H2) *hold.*

(1) *For $t > \theta \ge 0$, $n \in \mathbb{Z}$, we have*

$$u_n(t) \le e^{-(2D+d)(t-\theta)} w_n(\theta) + \int_\theta^t e^{-(2D+d)(t-s)} \overline{H}_n(s)\, ds$$

with

$$\overline{H}_n(s) = du_n(s) + D(u_{n+1}(s) + u_{n-1}(s)) + u_n(s)g(u_n(s), 0).$$

(2) *For $t > \theta \geq 0$ and $n \in \mathbb{Z}$, we also have*

$$u_n(t) \geq e^{-(2D+d)(t-\theta)} u_n(\theta) + \int_\theta^t e^{-(2D+d)(t-s)} \underline{H}_n(s)\, ds$$

with

$$\underline{H}_n(s) = du_n(s) + D(u_{n+1}(s) + u_{n-1}(s)) + u_n(s) g(u_n(s), 1).$$

Since Lemma 2.1 also holds for $\tau = 0$, u_n is an upper solution of

$$\begin{cases} \frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + u_n(t) g(u_n(t), 1), & n \in \mathbb{Z},\ t > 0, \\ u_n(0) = \varphi_n(0), & n \in \mathbb{Z}, \end{cases}$$

and a lower solution of

$$\begin{cases} \frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + u_n(t) g(u_n(t), 0), & n \in \mathbb{Z},\ t > 0, \\ u_n(0) = \varphi_n(0), & n \in \mathbb{Z}. \end{cases} \tag{2.3}$$

By Lemmas 1.3 and 1.5, we have the following conclusion.

Lemma 2.4. *Assume that* (H1)–(H3) *hold. For any given $\epsilon > 0$, if $\varphi_0(0) \geq \epsilon$ and $0 \leq \varphi_n(0) \leq 1$, $n \in \mathbb{Z}$, and $u_n(t)$ is defined by*

$$\begin{cases} \frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + u_n(t) g(u_n(t), 1), & n \in \mathbb{Z},\ t > 0, \\ u_n(0) = \varphi_n(0), & n \in \mathbb{Z}, \end{cases}$$

then there exists $\delta = \delta(\epsilon) > 0$ such that

$$u_0(t + \tau) > \delta, \qquad t > 0,$$

and δ is independent of $\varphi_n(0)$, $n \neq 0$.

Using Lemma 2.4, we can obtain an auxiliary equation without time delay, which is formulated in the following lemma.

Lemma 2.5. *Assume that $u_n(t)$ is defined by* (2.1) *and* (H1)–(H3) *hold. Then for each $\epsilon \in (0, 1)$, there exists $M = M(\epsilon) \geq 1$ such that*

$$u_n(t) \geq e^{-(2D+d)(t-\theta)} u_n(\theta) + \int_\theta^t e^{-(2D+d)(t-s)} \underline{\underline{H}}_n(s)\, ds$$

for some $\theta \in [0, t)$ and

$$\underline{\underline{H}}_n(s) = du_n(s) + D\big(u_{n+1}(s) + u_{n-1}(s)\big) + u_n(s) g(M u_n(s), \epsilon).$$

Proof. If $u_n(t - \tau) < \epsilon$, then

$$u_n(t) g(u_n(t), u_n(t - \tau)) \geq u_n(t) g(u_n(t), \epsilon)$$

from (H2). If $u_n(t - \tau) > \epsilon$, then (H2)–(H3) and Lemma 2.4 imply that there exists $M > 1$ such that

$$M u_n(t) \geq u_n(t - \tau)$$

and

$$u_n(t)g(u_n(t), u_n(t-\tau)) \geq u_n(t)g(Mu_n(t), \epsilon). \ \blacksquare$$

We now present the main result of this section.

THEOREM 2.6. *Assume that* (H1)–(H3) *hold and set*

$$c^* := \inf_{\lambda>0} \frac{D(e^{\lambda} + e^{-\lambda} - 2) + g(0,0)}{\lambda}.$$

If $\varphi_n(s) = 0$ *for* $|n| > M$ *and* $s \in [-\tau, 0]$ *with some* $M > 0$, *and* $\varphi_n(0) > 0$ *for some* $n \in \mathbb{Z}$, *then* c^* *is the spreading speed of* $u_n(t)$ *which is defined by* (2.1).

Proof. If $c > c^*$, then u_n is a lower solution of (2.3) and

$$\lim_{t\to\infty} \sup_{|n|>ct} u_n(t) = 0$$

by Lemma 1.5. If $c' < c^*$, then there exists $\epsilon > 0$ such that

$$D(e^{\lambda} + e^{-\lambda} - 2) - c\lambda + g(0, \epsilon) > 0$$

for any $2c \leq c' + c^*$ and $\lambda > 0$ by Lemma 1.4. Applying Lemmas 1.5 and 2.5, we further obtain

$$\liminf_{t\to\infty} \inf_{|n|<c't} u_n(t) > 0. \ \blacksquare$$

If (H4) holds, we also have the following convergence.

THEOREM 2.7. *Assume that the assumptions of Theorem* 2.6 *hold and* (H4) *is true. Then*

$$\liminf_{t\to\infty} \inf_{|n|<ct} u_n(t) = \limsup_{t\to\infty} \sup_{|n|<ct} u_n(t) = E$$

for any given $c < c^*$.

Proof. Define

$$\liminf_{t\to\infty} \inf_{|n|<ct} u_n(t) = \underline{E}, \qquad \limsup_{t\to\infty} \sup_{|n|<ct} u_n(t) = \overline{E}.$$

Then

$$\limsup_{t\to\infty} \sup_{|n|<ct} u_n(t-\tau) \geq \overline{E}, \qquad \liminf_{t\to\infty} \inf_{|n|<ct} u_n(t-\tau) \leq \underline{E}.$$

Moreover, what we have proved implies that

$$0 < \underline{E} \leq \overline{E} \leq 1.$$

Note that

$$du_n(t) + u_n(t)g(u_n(t), u_n(t-\tau))$$

is increasing in $u_n(t)$ and decreasing in $u_n(t-\tau)$, so

$$\limsup_{t\to\infty}\sup_{|n|<ct}\{du_n(t)+u_n(t)g(u_n(t),u_n(t-\tau))\}\le d\overline{E}+\overline{E}g(\overline{E},\underline{E}),$$

$$\liminf_{t\to\infty}\inf_{|n|<ct}\{du_n(t)+u_n(t)g(u_n(t),u_n(t-\tau))\}\ge d\underline{E}+\underline{E}g(\underline{E},\overline{E}).$$

Using dominated convergence in (2.2) (see also Thieme [Thi]), we obtain

$$\underline{E}\ge\frac{D(\underline{E}+\underline{E})+d\underline{E}+\underline{E}g(\underline{E},\overline{E})}{2D+d},$$

$$\overline{E}\le\frac{D(\overline{E}+\overline{E})+d\overline{E}+\overline{E}g(\overline{E},\underline{E})}{2D+d}.$$

From (H4), the proof is complete. ∎

Theorem 2.8. *Assume that* (H1)–(H4) *hold. If* $u_n(t)$ *is defined by* (2.1) *and* $\phi_n(0)>0$ *for some* $n\in\mathbb{Z}$, *then*

$$\liminf_{t\to\infty}\inf_{|n|<ct}u_n(t)=\limsup_{t\to\infty}\sup_{|n|<ct}u_n(t)=E$$

for any given $c<c^*$.

The proof is similar to that of Theorem 2.7, and we omit it here. To end this section, we make the following remark.

Remark 2.9. The spreading speed of (1.4) with $\tau>0$ is the same as that of (1.4) with $\tau=0$, and we obtain the persistence of spreading speed of (1.4) with any time delay $\tau>0$.

3. Minimal wave speed. In this part, we shall consider the traveling wave solutions of (1.4), and first present our main conclusion.

Theorem 3.1. *Assume that* (H1)–(H3) *hold. If* $c\ge c^*$ (*resp.* $c<c^*$), *then* (1.4) *has* (*resp. does not have*) *a positive traveling wave solution* ϕ *such that*

$$\lim_{\xi\to-\infty}\phi(\xi)=0,\qquad 0<\liminf_{\xi\to\infty}\phi(\xi)\le\limsup_{\xi\to\infty}\phi(\xi)\le 1. \tag{3.1}$$

Moreover, when (H4) *and* $c\ge c^*$ *are true, then* (1.6) *remains true.*

We prove this result by establishing three lemmas.

Lemma 3.2. *Assume that* (H1)–(H3) *hold. If* $c<c^*$, *then* (1.5) *has no positive solutions satisfying* (3.1).

Proof. If the statement were false, there would exist some $c''<c^*$ such that (1.5) with $c=c''$ has a positive solution satisfying (3.1). Namely, $u(x,t)=\phi(x+c''t)$ also satisfies

$$\begin{cases}\frac{du_n(t)}{dt}=[\mathcal{D}u]_n(x)+u_n(t)g(u_n(t),u_n(t-\tau)), & n\in\mathbb{Z},\ t>0,\\ u_n(s)=\phi(n+c''s), & n\in\mathbb{Z},\ s\in[-\tau,0].\end{cases}$$

Consider $-2n = (c + c'')t$. Then Theorem 2.7 implies that

$$\liminf_{t\to\infty} \inf_{-2n=(c+c'')t} u_n(t) > 0,$$

which contradicts (3.1) because $\xi = n + c''t \to -\infty$ as $t \to \infty$. ∎

LEMMA 3.3. *Assume that* (H1)–(H3) *hold. Then for each fixed* $c > c^*$, (1.4) *with* (3.1) *has a positive solution* ϕ.

Proof. If $\phi, \psi \in C_{[0,1]}$, define an operator F as follows

$$F(\phi,\psi)(\xi) = \frac{1}{c}\int_{-\infty}^{\xi} e^{-(2D+d)(\xi-s)/c} H(\phi,\psi)(s)\,ds$$

with

$$H(\phi,\psi)(s) = d\phi(s) + D\big(\phi(s+c) + \phi(s-c)\big) + \phi(s)g(\phi(s),\psi(s-c\tau)).$$

When $\phi(\xi) = \psi(\xi)$, we also denote

$$F(\phi,\phi)(\xi) := P(\phi)(\xi).$$

Then it is easy to prove that $P : C_{[0,1]} \to C_{[0,1]}$. In fact, since

$$0 \le H(\phi,\psi)(s) \le d + 2D, \qquad s \in \mathbb{R},$$

we have

$$0 \le P(\phi)(\xi) \le 1, \qquad \xi \in \mathbb{R},$$

and the uniform continuity of $P(\phi)(\xi)$ is clear by the boundedness of $H(\phi,\psi)(s)$.

We now define two continuous functions by

$$\overline{\phi}(\xi) = \min\{e^{\lambda_1(c)\xi}, 1\}, \qquad \underline{\phi}(\xi) = \max\{e^{\lambda_1(c)\xi} - qe^{\eta\lambda_1(c)\xi}, 0\}$$

with $1 < \eta < \min\{2, \lambda_2(c)/\lambda_1(c)\}$ and $q > 1$. Then

$$\underline{\phi}(\xi) \le F(\underline{\phi},\overline{\phi})(\xi) \le P(\phi)(\xi) \le F(\overline{\phi},\underline{\phi})(\xi) \le \overline{\phi}(\xi), \qquad \xi \in \mathbb{R}, \tag{3.2}$$

if $q > 1$ is large enough and

$$\phi \in C_{[0,1]}, \qquad \underline{\phi}(\xi) \le \phi(\xi) \le \overline{\phi}(\xi).$$

In fact, the monotonicity and the definition of d indicate that

$$F(\underline{\phi},\overline{\phi})(\xi) \le P(\phi)(\xi) \le F(\overline{\phi},\underline{\phi})(\xi), \qquad \xi \in \mathbb{R},$$

and it suffices to verify that

$$\underline{\phi}(\xi) \le F(\underline{\phi},\overline{\phi})(\xi), \qquad F(\overline{\phi},\underline{\phi})(\xi) \le \overline{\phi}(\xi), \qquad \xi \in \mathbb{R}.$$

If $\xi > 0$ is such that $\overline{\phi}(\xi) = 1$ and $\overline{\phi}'(\xi) = 0$, then

$$\begin{aligned} D\big(\overline{\phi}(\xi+1) + \overline{\phi}(\xi-1) - 2\overline{\phi}(\xi)\big) + \overline{\phi}(\xi)g(\overline{\phi}(\xi), \underline{\phi}(\xi - c\tau)) \\ \le g(1, \underline{\phi}(\xi - c\tau)) \le 0 = c\overline{\phi}'(\xi). \end{aligned}$$

When $\xi < 0$ is such that $\overline{\phi}(\xi) = e^{\lambda_1(c)\xi}$ and $\overline{\phi}'(\xi) = \lambda_1(c)e^{\lambda_1(c)\xi}$, then

$$\begin{aligned}
&D\big(\overline{\phi}(\xi+1) + \overline{\phi}(\xi-1) - 2\overline{\phi}(\xi)\big) + \overline{\phi}(\xi)g(\overline{\phi}(\xi), \underline{\phi}(\xi - c\tau)) \\
&\qquad \le D\big(e^{\lambda_1(c)(\xi+1)} + e^{\lambda_1(c)(\xi-1)} - 2e^{\lambda_1(c)\xi}\big) + e^{\lambda_1(c)\xi} g(\overline{\phi}(\xi), \underline{\phi}(\xi - c\tau)) \\
&\qquad \le D\big(e^{\lambda_1(c)(\xi+1)} + e^{\lambda_1(c)(\xi-1)} - 2e^{\lambda_1(c)\xi}\big) + e^{\lambda_1(c)\xi} g(0,0) \\
&\qquad = e^{\lambda_1(c)\xi}[D(e^{\lambda_1(c)} + e^{-\lambda_1(c)} - 2) + g(0,0)] \\
&\qquad = c\lambda_1(c)e^{\lambda_1(c)\xi} = c\overline{\phi}'(\xi).
\end{aligned}$$

Thus, we have verified that

$$\begin{aligned}
&c\overline{\phi}'(\xi) + (d+2D)\overline{\phi}(\xi) \\
&\qquad\qquad \ge D\big(\overline{\phi}(\xi+1) + \overline{\phi}(\xi-1)\big) + \overline{\phi}(\xi)g(\overline{\phi}(\xi), \underline{\phi}(\xi - c\tau)), \qquad \xi \ne 0.
\end{aligned}$$

If $\xi < 0$, then

$$\begin{aligned}
F(\overline{\phi}, \underline{\phi})(\xi) &= \frac{1}{c}\int_{-\infty}^{\xi} e^{-(2D+d)(\xi - s)/c} H(\overline{\phi}, \underline{\phi})(s)\, ds \\
&\le \frac{1}{c}\int_{-\infty}^{\xi} e^{-(2D+d)(\xi - s)/c} [c\overline{\phi}'(s) + (d+2D)\overline{\phi}(s)]\, ds = \overline{\phi}(\xi),
\end{aligned}$$

and when $\xi > 0$, we have

$$\begin{aligned}
F(\overline{\phi}, \underline{\phi})(\xi) &= \frac{1}{c}\int_{-\infty}^{\xi} e^{-(2D+d)(\xi - s)/c} H(\overline{\phi}, \underline{\phi})(s)\, ds \\
&= \frac{1}{c}\Big[\int_{-\infty}^{0} + \int_{0}^{\xi}\Big] e^{-(2D+d)(\xi - s)/c} H(\overline{\phi}, \underline{\phi})(s)\, ds \\
&\le \frac{1}{c}\Big[\int_{-\infty}^{0} + \int_{0}^{\xi}\Big] e^{-(2D+d)(\xi - s)/c} [c\overline{\phi}'(s) + (d+2D)\overline{\phi}(s)]\, ds = \overline{\phi}(\xi),
\end{aligned}$$

which completes the proof of $F(\overline{\phi}, \underline{\phi})(\xi) \le \overline{\phi}(\xi)$ for $\xi \in \mathbb{R}$.

If $\underline{\phi}(\xi) = e^{\lambda_1(c)\xi} - qe^{\eta\lambda_1(c)\xi} > 0$ with $\underline{\phi}'(\xi) = \lambda_1(c)e^{\lambda_1(c)\xi} - q\eta\lambda_1(c)e^{\eta\lambda_1(c)\xi}$, then

$$\begin{aligned}
&D\big(\underline{\phi}(\xi+1) + \underline{\phi}(\xi-1) - 2\underline{\phi}(\xi)\big) + \underline{\phi}(\xi)g(\underline{\phi}(\xi), \overline{\phi}(\xi - c\tau)) \\
&\quad \ge D\big(\underline{\phi}(\xi+1) + \underline{\phi}(\xi-1) - 2\underline{\phi}(\xi)\big) + \underline{\phi}(\xi)g(0,0) - I\underline{\phi}(\xi)\big(\underline{\phi}(\xi) + \overline{\phi}(\xi - c\tau)\big) \\
&\quad \ge e^{\lambda_1(c)\xi}[D(e^{\lambda_1(c)} + e^{-\lambda_1(c)} - 2) + g(0,0)] \\
&\qquad - qe^{\eta\lambda_1(c)\xi}[D(e^{\eta\lambda_1(c)} + e^{-\eta\lambda_1(c)} - 2) + g(0,0)] - 2Ie^{2\lambda_1(c)\xi} \\
&\quad \ge c\lambda_1(c)e^{\lambda_1(c)\xi} - cq\eta\lambda_1(c)e^{\eta\lambda_1(c)\xi} = c\underline{\phi}'(\xi)
\end{aligned}$$

provided that

$$q > \frac{2I}{c\eta\lambda_1(c) - D(e^{\eta\lambda_1(c)} + e^{-\eta\lambda_1(c)} - 2) - g(0,0)} + 1 > 1.$$

When $\underline{\phi}(\xi) = 0$, it is clear that

$$c\underline{\phi}'(\xi) \le D\big(\underline{\phi}(\xi+1) + \underline{\phi}(\xi-1) - 2\underline{\phi}(\xi)\big) + \underline{\phi}(\xi) g(\underline{\phi}(\xi), \overline{\phi}(\xi - c\tau)).$$

Therefore, if $\underline{\phi}(\xi) > 0$, then

$$\begin{aligned} F(\underline{\phi}, \overline{\phi})(\xi) &= \frac{1}{c}\int_{-\infty}^{\xi} e^{-(2D+d)(\xi-s)/c} H(\underline{\phi}, \overline{\phi})(s)\, ds \\ &\ge \frac{1}{c}\int_{-\infty}^{\xi} e^{-(2D+d)(\xi-s)/c}[c\underline{\phi}'(s) + (d+2D)\underline{\phi}(s)]\, ds = \underline{\phi}(\xi), \end{aligned}$$

and when $\underline{\phi}(\xi) = 0$, we have

$$F(\underline{\phi}, \overline{\phi})(\xi) = \frac{1}{c}\int_{-\infty}^{\xi} e^{-(2D+d)(\xi-s)/c} H(\underline{\phi}, \overline{\phi})(s)\, ds$$

$$\ge \frac{1}{c}\Big[\int_{-\infty}^{\frac{-\ln q}{(\eta-1)\lambda_1(c)}} + \int_{\frac{-\ln q}{(\eta-1)\lambda_1(c)}}^{\xi}\Big] e^{-(2D+d)(\xi-s)/c}[c\underline{\phi}'(s) + (d+2D)\underline{\phi}(s)]\, ds = \underline{\phi}(\xi),$$

which implies (3.2).

Let $\mu \in (0, d/(4c))$ be a constant and define

$$\Gamma = \{\phi \in C_{[0,1]};\ \underline{\phi}(\xi) \le \phi(\xi) \le \overline{\phi}(\xi) \text{ for all } \xi\}.$$

Then Γ is convex and nonempty, and is bounded and closed in the sense of $|\cdot|_\mu$. From (3.2), we also obtain $P : \Gamma \to \Gamma$. Moreover, the mapping is completely continuous in the sense of the decay norm $|\cdot|_\mu$, the proof of which is independent of monotonicity (see Huang et al. [HLZ, Lemmas 3.3 and 3.5] and Ma et al. [MWZ, Theorem 3.1]).

By Schauder's fixed point theorem, there is $\phi \in \Gamma$ satisfying

$$P(\phi)(\xi) = \phi(\xi), \quad \underline{\phi}(\xi) \le \phi(\xi) \le \overline{\phi}(\xi), \quad \xi \in \mathbb{R},$$

which is also a solution of (1.4).

Since ϕ is a special positive solution to (2.1), the asymptotic boundary condition is clear by Section 3. ∎

LEMMA 3.4. *Assume that* (H1)–(H3) *hold. If* $c = c^*$, *then* (1.4) *has a positive solution* ϕ *satisfying* (3.1).

Proof. We prove the result by passing to a limit function [BrC, LR]. Let $c_i \to c^*$, $i \in \mathbb{N}$, be strictly decreasing. Then for each fixed c_i, the operator

P with $c = c_i$ has a positive fixed point ϕ_i such that

$$0 < \phi_i(\xi) < 1, \quad \liminf_{\xi\to\infty} \phi_i(\xi) > 0, \quad \lim_{\xi\to-\infty} \phi_i(\xi) = 0, \quad i \in \mathbb{N}.$$

Without loss of generality, we assume that

$$\phi_i(0) = \delta, \quad \phi_i(\xi) < \delta, \quad \xi < 0$$

with $g(4\delta, 1) > 0$. Due to the uniform boundedness of ϕ_i', $\{\phi_i\}$ is an equicontinuous and uniformly bounded sequence. By the Ascoli–Arzelà lemma and a nested subsequence argument, $\{\phi_i\}$ has a subsequence, still denoted by $\{\phi_i\}$, and there exists $\phi \in C_{[0,1]}$ such that

$$\phi_i(\xi) \to \phi(\xi), \quad i \to \infty,$$

in which the limit is pointwise and locally uniform on any bounded interval of $\xi \in \mathbb{R}$. Clearly, we also have

$$\phi(0) = \delta, \quad \phi(\xi) \le \delta, \quad \xi < 0.$$

Note that

$$e^{-(2D+d)(\xi-s)/c_i} \to e^{-(2D+d)(\xi-s)/c^*}, \quad i \to \infty,$$

and the convergence is uniform for $\xi \in \mathbb{R}$, $s \le \xi$. Therefore, ϕ is a fixed point of P with $c = c^*$. By the properties of P, ϕ is a positive solution to (1.4).

Due to the results of Section 3, the limit behavior as $\xi \to \infty$ is clear. We now consider the limit behavior when $\xi \to -\infty$. If $\limsup_{\xi\to-\infty} \phi(\xi) > 0$, then there exists $\varepsilon_0 > 0$ such that there exists $\xi_i < -i$ such that

$$\phi(\xi_i) > \varepsilon_0, \quad i \in \mathbb{N}.$$

Since ϕ is a special positive solution to (2.1), Theorem 2.8 implies that there exists T independent of i such that

$$\phi(\xi_i + T) > 3\delta,$$

and a contradiction occurs when $i \to \infty$. ∎

To illustrate our main results, we consider the following example.

EXAMPLE 3.5. Assume that $r > 0$ and $a \in [0, 1)$. Let

$$c_* = \inf_{\lambda>0} \frac{D(e^{\lambda} + e^{-\lambda} - 2) + r}{\lambda}.$$

Then c_* is the minimal wave speed of traveling wave solutions connecting 0 with $1/(1 + a)$ of

$$(3.3) \quad \frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + ru_n(t)[1 - u_n(t) - au_n(t - \tau)], \quad n \in \mathbb{Z},\ t > 0.$$

Moreover, c_* is the spreading speed of $u_n(t)$ which is defined by the corresponding initial value problem of (3.3) if $u_n(s) \ge 0$, $n \in \mathbb{N}$, $s \in [-\tau, 0]$, satisfies:

(I1) for each $n \in \mathbb{N}$, $u_n(s)$ is continuous in $s \in [-\tau, 0]$;
(I2) $u_n(s) = 0$ for all $|n| > M$ and $s \in [-\tau, 0]$ with some $M > 0$;
(I3) $u_n(0) > 0$ for some $n \in \mathbb{Z}$.

To end this paper, we make the following remark.

REMARK 3.6. Although the delayed term reflects the intraspecific competition in population dynamics, the delay may be harmless to the propagation if the instantaneous competition dominates the delayed one (see (H3)).

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