ANNALES
POLONICI MATHEMATICI
114.3 (2015)

On Kirchhoff type problems involving critical and singular nonlinearities

by CHUN-YU LEI (Guiyang), CHANG-MU CHU (Guiyang), HONG-MIN SUO (Guiyang) and CHUN-LEI TANG (Chongqing)

Abstract. In this paper, we are interested in multiple positive solutions for the Kirchhoff type problem

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2\,dx)\Delta u = u^5 + \lambda \frac{u^{q-1}}{|x|^\beta} & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $0 \in \Omega$, $1 < q < 2$, λ is a positive parameter and β satisfies some inequalities. We obtain the existence of a positive ground state solution and multiple positive solutions via the Nehari manifold method.

1. Introduction and main results. This paper concerns the positive solutions of the following Kirchhoff type equation:

$$\begin{cases} -\Big(a+b\int\limits_\Omega |\nabla u|^2\,dx\Big)\Delta u = u^5 + \lambda \dfrac{u^{q-1}}{|x|^\beta} & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $a, b > 0$, $0 \in \Omega$, $\lambda > 0$ is a real parameter, $1 < q < 2$ and $0 \le \beta < 2$.

Indeed, (1.1) has its origin in the theory of nonlinear vibration. For example, the following equation describes the nonlinear vibration of a stretched string:

$$\rho\frac{\partial^2 u}{\partial t^2} - \bigg(\frac{\rho_0}{h} + \frac{E}{2L}\int\limits_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx\bigg)\frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, ρ_0, h, E, L are constants, which have the following meaning: ρ is the mass density, ρ_0 is the initial tension, h represents the area of the cross-

2010 *Mathematics Subject Classification*: Primary 35R09; Secondary 35A15, 35B09.
Key words and phrases: Kirchhoff type equation, critical exponents, singular nonlinearity, concentration-compactness principle.

DOI: 10.4064/ap114-3-5

section, E is the Young modulus of the material, and L is the length of the string. The above equation is the first model taking into account the change of the axial tension along the string which is caused by the change of its length during the vibration. It is noteworthy that the model contains a nonlocal term $\int_0^L (\frac{\partial u}{\partial x})^2\,dx$; the above nonlocal equation was first proposed by Kirchhoff in 1876 [13]. In the recent years, the existence and multiplicity of solutions to the Kirchhoff type problem

$$\begin{cases} -\Big(a+b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = h(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has been extensively studied, and some important and interesting results have been found. For example, in [2, 4, 6, 7, 14, 18, 19, 22, 26, 27], the existence of positive solutions has been established by variational methods. The existence of sign-changing solutions for problem (1.2) has been studied via invariant sets of the descent flow (see [20, 21, 28]). If Ω is an unbounded domain, [15–16, 23] established the existence of weak solutions and [12, 24] studied the existence of infinitely many solutions. Recently, there are some papers on the Kirchhoff type problem involving the critical growth (see [1, 9–11, 16, 22, 26] and the references therein).

More recently, Chen et al. [6] considered the Kirchhoff type problem

$$\begin{cases} -\Big(a+b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x)u^{p-2}u + \lambda g(x)|u|^{q-2}u & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

assuming that $1 < q < 2 < p < 6$ and the sign-changing weight functions $f, g \in C(\bar{\Omega})$ satisfy

(h_1) $f^+ = \max\{f, 0\} \neq 0$.
(h_2) $g^+ = \max\{g, 0\} \neq 0$.

We report here one of the main results of [6] for the reader's convenience.

Theorem A (see [6]). *Suppose Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary,* $1 < q < 2$, $4 < p < 6$ *and* (h_1), (h_2) *hold. Then there exists a positive constant* $\lambda_0(a) > 0$ *such that for each* $a > 0$ *and* $\lambda \in (0, \lambda_0(a))$, *problem* (1.3) *has at least two positive solutions.*

Thus, motivated by [6], in equation (1.3), suppose $1 < q < 2$, $p = 6$, $f(x) \equiv 1$, $g(x) = 1/|x|^\beta$; an interesting question now is whether the existence and multiplicity of positive solutions can be established for such Kirchhoff type problems involving critical and singular nonlinearities. We will give a positive answer by applying the Nehari manifold method.

Throughout this paper, we use the following notation:

- The space $H_0^1(\Omega)$ is equipped with the norm $\|u\| = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$, the norm in $L^p(\Omega)$ is denoted by $|u|_p = (\int_\Omega |u|^p\,dx)^{1/p}$.
- $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$.
- $C, C_0, C_1, C_2, \ldots$ denote various positive constants, which may vary from line to line.
- Let S be the best Sobolev constant, that is,

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2\,dx}{(\int_{\mathbb{R}^3} |u|^6\,dx)^{1/3}}. \tag{1.4}$$

The energy functional corresponding to problem (1.1) is given by

$$I_\lambda(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_\Omega (u^+)^6\,dx - \frac{\lambda}{q}\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx.$$

A function u is called a *weak solution* of problem (1.1) if $u \in H_0^1(\Omega)$ and for all $\varphi \in H_0^1(\Omega)$ we have

$$(a + b\|u\|^2)\int_\Omega (\nabla u, \nabla\varphi)\,dx - \int_\Omega (u^+)^5\varphi\,dx - \lambda\int_\Omega \frac{(u^+)^{q-1}}{|x|^\beta}\varphi\,dx = 0.$$

Let $R_0 > 0$ be a constant such that $\Omega \subset B(0, R_0)$, where $B(0, R_0) = \{x \in \mathbb{R}^3 : |x| < R_0\}$. By Hölder's inequality and (1.4), for all $u \in H_0^1(\Omega)$, $1 < q < 2$, $0 \le \beta < 2$, we get

$$\begin{aligned} &\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx \\ &\le \int_\Omega \frac{|u|^q}{|x|^\beta}\,dx \le \Big(\int_\Omega |u|^{q\cdot\frac{6}{q}}\,dx\Big)^{\frac{q}{6}}\Big(\int_\Omega \frac{1}{|x|^{\frac{6\beta}{6-q}}}\,dx\Big)^{\frac{6-q}{6}} \\ &\le S^{-q/2}\|u\|^q\Big(\int_\Omega \frac{1}{|x|^{\frac{6\beta}{6-q}}}\,dx\Big)^{\frac{6-q}{6}} \le S^{-q/2}\|u\|^q\Big(\int_{B(0,R_0)} \frac{1}{|x|^{\frac{6\beta}{6-q}}}\,dx\Big)^{\frac{6-q}{6}} \\ &\le S^{-q/2}\|u\|^q\Big(\int_0^{R_0} \frac{r^2}{r^{\frac{6\beta}{6-q}}}\,dr\Big)^{\frac{6-q}{6}} = \frac{6-q}{18-3q-6\beta}R_0^{(6-2\beta-q)/2}S^{-q/2}\|u\|^q. \end{aligned} \tag{1.5}$$

Furthermore, assume that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and consider an arbitrary subsequence of $\{u_n\}$, still denoted by $\{u_n\}$. By the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}\int_\Omega \frac{(u_n^+)^q}{|x|^\beta}\,dx = \int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx. \tag{1.6}$$

Set

$$
\begin{aligned}
T &= \frac{6-q}{18-3q-6\beta} R_0^{(6-2\beta-q)/2} S^{-q/2}, \\
T_1 &= \frac{1}{T}\left(\frac{2-q}{4}\right)^{(2-q)/4}\left(\frac{4aS}{6-q}\right)^{(6-q)/4}, \\
T_2 &= \frac{aq}{T(6-q)}\left(\frac{2-q}{6-q}aS^3\right)^{(2-q)/4}.
\end{aligned}
\tag{1.7}
$$

Now our main results are as follows:

THEOREM 1.1. *Assume* $1 < q < 2$ *and* $0 \leq \beta < 2$. *Then there exists* $\lambda_* > 0$ *such that for any* $\lambda \in (0, \lambda_*)$, *problem* (1.1) *has a positive ground state solution.*

THEOREM 1.2. *Assume* $1 < q < 2$ *and* $3 - q \leq \beta < 2$. *Then there exists* $\lambda_{**} > 0$ *such that for any* $\lambda \in (0, \lambda_{**})$, *problem* (1.1) *has at least two positive solutions, and one of the solutions is a positive ground state solution.*

REMARK 1.3. Ambrosetti et al. [3] has studied the existence and multiplicity of positive solutions for problem (1.3) with $a = 1$, $b = 0$, $f(x) = g(x) = 1$ and $p = 6$. When $b > 0$, $f(x) = g(x) = 1$ and $p = 6$, (1.3) reduces to a Kirchhoff type problem with concave-convex nonlinearities. However, in that case, to the best of our knowledge, there are no results on multiplicity of positive solutions. The reason is that, in view of $b > 0$, type problem becomes more complicated than in the case $b = 0$, namely, it is difficult to estimate the critical value level.

REMARK 1.4. It is of importance to obtain multiple positive solutions for problem (1.1) when $3 - q \leq \beta < 2$. If $\beta = 0$ in (1.1), Figueiredo et al. [10] have obtained infinitely many solutions for (1.1), and the energy functional value level is negative, but they could not get multiple positive solutions. In this paper, the typical difficulty is the lack of compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$. We overcome the difficulty by using the concentration-compactness principle.

This work is organized as follows. In the next section we present some preliminary results. In Section 3, we give the proofs of Theorems 1.1 and 1.2.

2. Some preliminary results. As I_λ is not bounded below on $H_0^1(\Omega)$, we will work on the Nehari manifold

$$\mathcal{N}_\lambda = \{u \in H_0^1(\Omega)\backslash\{0\} : \langle I'_\lambda(u), u\rangle = 0\}.$$

Note that $\mathcal{N}_\lambda$ contains all nonzero solutions of problem (1.1). Moreover,

$u \in \mathcal{N}_\lambda$ if and only if

$$a\|u\|^2 + b\|u\|^4 - \int_\Omega (u^+)^6\,dx - \lambda \int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx = 0.$$

We split $\mathcal{N}_\lambda$ into three parts:

$$\mathcal{N}_\lambda^+ = \Big\{u \in \mathcal{N}_\lambda : (2-q)a\|u\|^2 + (4-q)b\|u\|^4 - (6-q)\int_\Omega (u^+)^6\,dx > 0\Big\},$$

$$\mathcal{N}_\lambda^0 = \Big\{u \in \mathcal{N}_\lambda : (2-q)a\|u\|^2 + (4-q)b\|u\|^4 - (6-q)\int_\Omega (u^+)^6\,dx = 0\Big\},$$

$$\mathcal{N}_\lambda^- = \Big\{u \in \mathcal{N}_\lambda : (2-q)a\|u\|^2 + (4-q)b\|u\|^4 - (6-q)\int_\Omega (u^+)^6\,dx < 0\Big\}.$$

LEMMA 2.1.

(i) *If $\lambda \in (0, T_1)$ (T_1 is as in (1.7)), then $\mathcal{N}_\lambda^\pm \neq \emptyset$.*
(ii) *If $\lambda \in \big(0, \frac{6-q}{2(4-q)}T_1\big)$, then $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. (i) Let $u \in H_0^1(\Omega) \setminus \{0\}$, and define $\Phi, \Phi_1 \in C(\mathbb{R}^+, \mathbb{R})$ by

$$\Phi(t) = at^{-4}\|u\|^2 + bt^{-2}\|u\|^4 - \lambda t^{q-6}\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx,$$

$$\Phi_1(t) = at^{-4}\|u\|^2 - \lambda t^{q-6}\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx.$$

Then

$$\Phi_1'(t) = -4at^{-5}\|u\|^2 - \lambda(q-6)t^{q-7}\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx.$$

Solving $\Phi_1'(t) = 0$, we obtain

$$t_{\max} = \bigg[\frac{\lambda(6-q)\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx}{4a\|u\|^2}\bigg]^{1/(2-q)}.$$

Easy computations show that $\Phi_1'(t) > 0$ for all $0 < t < t_{\max}$ and $\Phi_1'(t) < 0$ for all $t > t_{\max}$. Thus $\Phi_1(t)$ attains its maximum at $t_{\max}$, that is,

$$\Phi_1(t_{\max}) = \frac{2-q}{4}\bigg[\frac{4a}{6-q}\bigg]^{\frac{6-q}{2-q}} \frac{\|u\|^{\frac{2(6-q)}{2-q}}}{\big(\lambda\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx\big)^{\frac{4}{2-q}}}.$$

Note that $\int_\Omega (u^+)^6\,dx \le \int_\Omega u^6\,dx$. Then from (1.5) one gets

$$\begin{aligned}\Phi(t_{\max}) - \int_\Omega (u^+)^6\,dx &\ge \Phi_1(t_{\max}) - \int_\Omega (u^+)^6\,dx \\ &> \frac{2-q}{4}\bigg[\frac{4a}{6-q}\bigg]^{\frac{6-q}{2-q}} \frac{\|u\|^{\frac{2(6-q)}{2-q}}}{\big(\lambda\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx\big)^{\frac{4}{2-q}}} - \int_\Omega u^6\,dx\end{aligned}$$

$$\geq \left\{\frac{2-q}{4}\left[\frac{4a}{6-q}\right]^{\frac{6-q}{2-q}}\left(\frac{1}{\lambda T}\right)^{\frac{4}{2-q}}\left(\frac{\|u\|^2}{|u|_6^2}\right)^{\frac{6-q}{2-q}}-1\right\}|u|_6^6$$
$$\geq \left\{\frac{2-q}{4}\left[\frac{4aS}{6-q}\right]^{\frac{6-q}{2-q}}\left(\frac{1}{\lambda T}\right)^{\frac{4}{2-q}}-1\right\}|u|_6^6>0,$$

where the last inequality holds for every $0<\lambda<T_1$. It follows that there exist two positive numbers denoted by $t^{\pm}$ such that $0<t^+=t^+(u)<t_{\max}<t^-=t^-(u)$, $t^+u\in\mathcal{N}_\lambda^+$ and $t^-u\in\mathcal{N}_\lambda^-$.

(ii) For contradiction, suppose that there exists $u_0\neq 0$ such that $u_0\in\mathcal{N}_\lambda^0$. It follows that

$$a\|u_0\|^2+b\|u_0\|^4=\int_\Omega (u_0^+)^6\,dx+\lambda\int_\Omega \frac{(u_0^+)^q}{|x|^\beta}\,dx, \tag{2.1}$$

$$4a\|u_0\|^2+2b\|u_0\|^4=\lambda(6-q)\int_\Omega \frac{(u_0^+)^q}{|x|^\beta}\,dx. \tag{2.2}$$

These imply that

$$\lambda\int_\Omega \frac{(u_0^+)^q}{|x|^\beta}\,dx=\frac{2a}{4-q}\|u_0\|^2+\frac{2}{4-q}\int_\Omega (u_0^+)^6\,dx>\frac{2a}{4-q}\|u_0\|^2. \tag{2.3}$$

On the one hand, since $\|u_0\|^2>S|u_0|_6^2$ for $u_0\in\mathcal{N}_\lambda^0$, using (1.5) we get

$$\begin{aligned}\Theta &:= T^{\frac{4}{2-q}}S^{-\frac{6-q}{2-q}}\frac{\|u_0\|^{\frac{2(6-q)}{2-q}}}{\left(\int_\Omega \frac{(u_0^+)^q}{|x|^\beta}\,dx\right)^{\frac{4}{2-q}}}-\int_\Omega (u_0^+)^6\,dx\\
&> T^{\frac{4}{2-q}}S^{-\frac{6-q}{2-q}}\frac{(S|u_0|_6^2)^{\frac{6-q}{2-q}}}{T^{\frac{4}{2-q}}|u_0|_6^{\frac{8}{2-q}}}-\int_\Omega (u_0^+)^6\,dx\\
&=\int_\Omega |u_0|^6\,dx-\int_\Omega (u_0^+)^6\,dx\geq 0.\end{aligned}$$

On the other hand, by (2.3),

$$\begin{aligned}\Theta &= T^{\frac{4}{2-q}}S^{-\frac{6-q}{2-q}}\lambda^{\frac{4}{2-q}}\frac{\|u_0\|^{\frac{2(6-q)}{2-q}}}{\left(\lambda\int_\Omega \frac{(u_0^+)^q}{|x|^\beta}\,dx\right)^{\frac{4}{2-q}}}-\int_\Omega (u_0^+)^6\,dx\\
&\leq T^{\frac{4}{2-q}}S^{-\frac{6-q}{2-q}}\lambda^{\frac{4}{2-q}}\frac{\|u_0\|^{\frac{2(6-q)}{2-q}}}{\left(\frac{2a}{4-q}\right)^{\frac{4}{2-q}}\|u_0\|^{\frac{8}{2-q}}}\\
&= T^{\frac{4}{2-q}}S^{-\frac{6-q}{2-q}}\lambda^{\frac{4}{2-q}}\left(\frac{4-q}{2a}\right)^{\frac{4}{2-q}}\|u_0\|^2-\int_\Omega (u_0^+)^6\,dx.\end{aligned}$$

Since $u_0 \in \mathcal{N}_\lambda^0$, the above equals

$$T^{\frac{4}{2-q}} S^{-\frac{6-q}{2-q}} \lambda^{\frac{4}{2-q}} \left(\frac{4-q}{2a}\right)^{\frac{4}{2-q}} \|u_0\|^2 - \frac{a(2-q)}{6-q}\|u_0\|^2 - \frac{b(4-q)}{6-q}\|u_0\|^4$$
$$= \frac{a(2-q)}{6-q}\|u_0\|^2 \left[T^{\frac{4}{2-q}} S^{-\frac{6-q}{2-q}} \lambda^{\frac{4}{2-q}} \left(\frac{4-q}{2a}\right)^{\frac{4}{2-q}} \frac{6-q}{a(2-q)} - 1\right] - \frac{b(4-q)}{6-q}\|u_0\|^4$$
$$< 0$$

when $\lambda < \frac{6-q}{2(4-q)} T_1$, which is a contradiction. ■

LEMMA 2.2. *I_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. If $u \in \mathcal{N}_\lambda$, then by (1.5) we get

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_\Omega (u^+)^6\,dx - \frac{\lambda}{q}\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx \\ &= \frac{a}{3}\|u\|^2 + \frac{b}{12}\|u\|^4 - \lambda\left(\frac{1}{q} - \frac{1}{6}\right)\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx \\ &\geq \frac{a}{3}\|u\|^2 + \frac{b}{12}\|u\|^4 - \lambda\left(\frac{1}{q} - \frac{1}{6}\right) T\|u\|^q. \end{aligned}$$

Since $1 < q < 2$, the conclusion follows. ■

We remark that by Lemma 2.1 we have $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ for all λ in $\big(0, \frac{6-q}{2(4-q)} T_1\big)$. Moreover, we know that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are nonempty, and by Lemma 2.2 we may define

$$\alpha_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u), \qquad \alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u), \qquad \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} I_\lambda(u).$$

LEMMA 2.3.

(i) $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.

(ii) *If $\lambda \in (0, T_2)$ (T_2 is given in (1.7)), then $\alpha_\lambda^- > \frac{a}{6}\big(\frac{2-q}{6-q} S^3 a\big)^{1/2}$.*

Proof. (i) Suppose $u \in \mathcal{N}_\lambda^+$. Then

$$\int_\Omega (u^+)^6\,dx < \frac{2-q}{6-q} a\|u\|^2 + \frac{4-q}{6-q} b\|u\|^4. \tag{2.4}$$

It follows from (2.4) that

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_\Omega (u^+)^6\,dx - \frac{\lambda}{q}\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) a\|u\|^2 + \left(\frac{1}{4} - \frac{1}{q}\right) b\|u\|^4 + \left(\frac{1}{q} - \frac{1}{6}\right)\int_\Omega (u^+)^6\,dx \end{aligned}$$

$$\begin{aligned}&< \left(\frac{a}{2}-\frac{1}{q}\right)a\|u\|^2+\left(\frac{1}{4}-\frac{1}{q}\right)b\|u\|^4\\&\quad+\left(\frac{1}{q}-\frac{1}{6}\right)\left(\frac{2-q}{6-q}a\|u\|^2+\frac{4-q}{6-q}b\|u\|^4\right)\\&=\frac{1}{3}\left(1-\frac{2}{q}\right)a\|u\|^2+\frac{1}{3}\left(\frac{1}{4}-\frac{1}{q}\right)b\|u\|^4<0.\end{aligned}$$

By the definitions of α_λ and α_λ^+, we obtain $\alpha_\lambda \le \alpha_\lambda^+ < 0$.

(ii) Suppose $u \in \mathcal{N}_\lambda^-$. Then

$$\int_\Omega (u^+)^6\,dx > \frac{2-q}{6-q}a\|u\|^2+\frac{4-q}{6-q}b\|u\|^4.$$

According to (1.4) and $\int_\Omega (u^+)^6\,dx \le \int_\Omega |u|^6\,dx$, we get

$$\begin{aligned}S^{-3}\|u\|^6 \ge \int_\Omega (u^+)^6\,dx &> \frac{2-q}{6-q}a\|u\|^2+\frac{4-q}{6-q}b\|u\|^4\\&\ge \frac{2-q}{6-q}a\|u\|^2,\end{aligned}$$

and consequently

$$\|u\| \ge \left(\frac{2-q}{6-q}S^3a\right)^{1/4}. \tag{2.5}$$

Assume $\lambda \in (0, T_2)$. Then from $u \in \mathcal{N}_\lambda^-$ and (2.5) one obtains

$$\begin{aligned}I_\lambda(u) &= \frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\frac{1}{6}\int_\Omega (u^+)^6\,dx-\frac{\lambda}{q}\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx\\&=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\frac{1}{6}\left(a\|u\|^2+b\|u\|^4-\lambda\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx\right)-\frac{\lambda}{q}\int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx\\&\ge \frac{a}{3}\|u\|^2-\left(\frac{1}{q}-\frac{1}{6}\right)\lambda T\|u\|^q\\&=\|u\|^q\left\{\frac{a}{3}\|u\|^{2-q}-\left(\frac{1}{q}-\frac{1}{6}\right)\lambda T\right\}\\&\ge \frac{a}{6}\left(\frac{2-q}{6-q}S^3a\right)^{1/2}. \ \blacksquare\end{aligned}$$

Lemma 2.4. *For every $u \in \mathcal{N}_\lambda$, there exist $\varepsilon > 0$ and a continuously differentiable function $f = f(w) > 0$, $w \in H_0^1(\Omega)$, $\|w\| < \varepsilon$, satisfying*

$$f(0)=1, \qquad f(w)(u+w) \in \mathcal{N}_\lambda, \qquad \forall w \in H_0^1(\Omega),\ \|w\|<\varepsilon.$$

Proof. For $u \in \mathcal{N}_\lambda$, define $F : \mathbb{R} \times H_0^1(\Omega) \to \mathbb{R}$ by

$$F(t,w) = t^{2-q} a \int_\Omega |\nabla(u+w)|^2\,dx + t^{4-q} b\Big(\int_\Omega |\nabla(u+w)|^2\,dx\Big)^2 - t^{6-q}\int_\Omega ((u+w)^+)^6\,dx - \lambda \int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx.$$

Since $u \in \mathcal{N}_\lambda$, it is easily seen that $F(1,0) = 0$ and

$$F_t(1,0) = (2-q)a\|u\|^2 + (4-q)b\|u\|^4 - (6-q)\int_\Omega (u^+)^6\,dx.$$

As $u \neq 0$, Lemma 2.1 shows that $F_t(1,0) \neq 0$. Thus, we can apply the implicit function theorem at the point $(0,1)$ to obtain $\varepsilon > 0$ and a continuously differentiable function $f : B(0,\varepsilon) \subset H_0^1(\Omega) \to \mathbb{R}^+$ as in the conclusion of the lemma. ∎

LEMMA 2.5. *For every $u \in \mathcal{N}_\lambda^-$, there exist $\varepsilon > 0$ and a continuously differentiable function $\tilde f = \tilde f(v) > 0$, $v \in H_0^1(\Omega)$, $\|v\| < \varepsilon$, satisfying*

$$\tilde f(0) = 1, \quad \tilde f(v)(u+v) \in \mathcal{N}_\lambda^-, \quad \forall v \in H_0^1(\Omega),\ \|v\| < \varepsilon.$$

Proof. Similar to the argument in Lemma 2.4, for $u \in \mathcal{N}_\lambda^-$, define a function $\tilde F : \mathbb{R} \times H_0^1(\Omega) \to \mathbb{R}$ by

$$\tilde F(t,v) = t^{2-q} a \int_\Omega |\nabla(u+v)|^2\,dx + t^{4-q} b\Big(\int_\Omega |\nabla(u+v)|^2\,dx\Big)^2 - t^{6-q}\int_\Omega ((u+v)^+)^6\,dx - \lambda \int_\Omega \frac{(u^+)^q}{|x|^\beta}\,dx.$$

As $u \in \mathcal{N}_\lambda^-$, we get $\tilde F(1,0) = 0$ and $\tilde F_t(1,0) < 0$. Therefore, we can apply the implicit function theorem at $(0,1)$ to get the result. ∎

LEMMA 2.6. *If $\{u_n\} \subset \mathcal{N}_\lambda$ is a minimizing sequence of I_λ, then for any $\varphi \in H_0^1(\Omega)$,*

$$\langle I_\lambda'(u_n), \varphi\rangle = \frac{|f_n'(0)|\,\|u_n\| + \|\varphi\|}{n}. \tag{2.6}$$

Proof. By Lemma 2.2, I_λ is coercive on $\mathcal{N}_\lambda$. Then by Ekeland's variational principle [8], there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ for I_λ such that

$$I_\lambda(u_n) < \alpha_\lambda + \frac{1}{n}, \qquad I_\lambda(v) - I_\lambda(u_n) \ge -\frac{1}{n}\|v - u_n\|, \qquad \forall v \in \mathcal{N}_\lambda. \tag{2.7}$$

Obviously, Lemma 2.2 shows that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. So there exist a subsequence (still denoted $\{u_n\}$) and u_* in $H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_* & \text{weakly in } H_0^1(\Omega), \\ u_n \to u_* & \text{strongly in } L^p(\Omega)\ (1 \le p < 6), \\ u_n(x) \to u_*(x) & \text{a.e. in } \Omega. \end{cases}$$

Pick $t > 0$ small enough and $\varphi \in H_0^1(\Omega)$, and set $u = u_n$, $w = t\varphi \in H_0^1(\Omega)$. By Lemma 2.4 there exists $f_n(t) = f_n(t\varphi)$ satisfying $f_n(0) = 1$, $f_n(t)(u_n + t\varphi) \in \mathcal{N}_\lambda$. Note that

$$a\|u_n\|^2 + b\|u_n\|^4 - \int_\Omega (u_n^+)^6\,dx - \lambda \int_\Omega \frac{(u_n^+)^q}{|x|^\beta}\,dx = 0. \tag{2.8}$$

Then (2.7) implies that

$$\begin{aligned} \frac{1}{n}[|f_n(t) - 1| \cdot \|u_n\| + t f_n(t)\|\varphi\| t] &\ge \frac{1}{n}\|f_n(t)(u_n + t\varphi) - u_n\| \\ &\ge I_\lambda(u_n) - I_\lambda[f_n(t)(u_n + t\varphi)] \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &I_\lambda(u_n) - I_\lambda[f_n(t)(u_n + t\varphi)] \\ &\quad = \frac{1 - f_n^2(t)}{2} a\|u_n\|^2 + \frac{1 - f_n^4(t)}{4} b\|u_n\|^4 \\ &\qquad + \frac{f_n^6(t) - 1}{6} \int_\Omega ((u_n + t\varphi)^+)^6\,dx + \lambda \frac{f_n^q(t) - 1}{q} \int_\Omega \frac{((u_n + t\varphi)^+)^q}{|x|^\beta}\,dx \\ &\qquad + \frac{f_n^2(t)}{2}\left(a + \frac{f_n^2(t)}{2} b(\|u_n\|^2 + \|u_n + t\varphi\|^2)\right)(\|u_n\|^2 - \|u_n + t\varphi\|^2) \\ &\qquad + \frac{1}{6}\left(\int_\Omega ((u_n + t\varphi)^+)^6\,dx - \int_\Omega (u_n^+)^6\,dx\right) \\ &\qquad + \frac{\lambda}{q} \int_\Omega \frac{((u_n + t\varphi)^+)^q - (u_n^+)^q}{|x|^\beta}\,dx. \end{aligned}$$

Combining this with (2.8) and (2.9), dividing by t and letting $t \to 0$, we obtain

$$\begin{aligned} &\frac{|f_n'(0)|\,\|u_n\| + \|\varphi\|}{n} \\ &\quad \ge -f_n'(0) a\|u_n\|^2 + f_n'(0) b\|u_n\|^4 + f_n'(0) \int_\Omega (u_n^+)^6\,dx + \lambda f_n'(0) \int_\Omega \frac{(u_n^+)^q}{|x|^\beta}\,dx \\ &\qquad - (a + b\|u_n\|^2) \int_\Omega (\nabla u_n, \nabla\varphi)\,dx + \int_\Omega (u_n^+)^5 \varphi\,dx + \lambda \int_\Omega \frac{(u_n^+)^{q-1}}{|x|^\beta}\varphi\,dx \end{aligned}$$

$$= -(a+b\|u_n\|^2)\int_\Omega(\nabla u_n,\nabla\varphi)\,dx + \int_\Omega (u_n^+)^5\varphi\,dx + \lambda\int_\Omega \frac{(u_n^+)^{q-1}\varphi}{|x|^\beta}\,dx$$
$$= -(a+b\|u_n\|^2)\int_\Omega(\nabla u_n,\nabla\varphi)\,dx + \int_\Omega (u_n^+)^5\varphi\,dx + \lambda\int_\Omega \frac{(u_n^+)^{q-1}\varphi}{|x|^\beta}\,dx.$$

Hence, we deduce that

$$\frac{|f_n'(0)|\,\|u_n\|+\|\varphi\|}{n} \le (a+b\|u_n\|^2)\int_\Omega(\nabla u_n,\nabla\varphi)\,dx - \int_\Omega (u_n^+)^5\varphi\,dx - \lambda\int_\Omega \frac{(u_n^+)^{q-1}\varphi}{|x|^\beta}\,dx = \langle I_\lambda'(u_n),\varphi\rangle \tag{2.10}$$

for any $\varphi\in H_0^1(\Omega)$. As (2.10) also holds for $-\varphi$, we see that (2.6) holds. Moreover, by Lemma 2.4, there exists a constant $C>0$ such that $|f_n'(0)|\le C$ for all $n\in\mathbb{N}$. Therefore, letting $n\to\infty$ in (2.6) we get

$$\Big(a+b\lim_{n\to\infty}\|u_n\|^2\Big)\int_\Omega(\nabla u_*,\nabla\varphi)\,dx = \int_\Omega (u_*^+)^5\varphi\,dx+\lambda\int_\Omega \frac{(u_*^+)^{q-1}\varphi}{|x|^\beta}\,dx \tag{2.11}$$

for all $\varphi\in H_0^1(\Omega)$. This completes the proof of Lemma 2.6. ∎

We define

$$\Lambda = \frac{abS^3}{4}+\frac{b^3S^6}{24}+\frac{(b^2S^4+4aS)^{3/2}}{24}.$$

LEMMA 2.7. *Assume $1<q<2$ and $0\le\beta<2$, and let $\{u_n\}\subset\mathcal{N}_\lambda^-$ be a minimizing sequence for I_λ with*

$$\alpha_\lambda^- < \Lambda - D\lambda^{2/(2-q)} \quad \textit{where} \quad D=\left(\frac{(4-q)}{4q}T\right)^{2/(2-q)}\left(\frac{2q}{a}\right)^{q/(2-q)}.$$

Then there exists $u\in H_0^1(\Omega)$ such that $u_n\to u$ in $L^6(\Omega)$.

Proof. We have

$$I_\lambda(u_n)\to\alpha_\lambda^- \quad \text{as } n\to\infty. \tag{2.12}$$

By Lemma 2.2, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Passing to a subsequence if necessary, there exists $u\in H_0^1(\Omega)$ such that

$$\begin{cases} u_n\rightharpoonup u & \text{weakly in } H_0^1(\Omega),\\ u_n\to u & \text{strongly in } L^p(\Omega)\ (1\le p<6),\\ u_n(x)\to u(x) & \text{a.e. in } \Omega.\end{cases}$$

Furthermore, by the concentration-compactness principle (see [17]), there exists a subsequence, still denoted by $\{u_n\}$, such that

$$|\nabla u_n|^2 \rightharpoonup d\mu \geq \|u\|^2 + \sum_{j\in J} \mu_j \delta_{x_j}, \qquad |u_n|_6^6 \to d\nu = |u|_6^6 + \sum_{j\in J} \nu_j \delta_{x_j},$$

where J is an at most countable index set, δ_{x_j} is the Dirac mass at x_j, and $x_j \in \Omega$ is in the support of μ, ν. Moreover,

$$\mu_j, \nu_j \geq 0, \qquad \mu_j \geq S\nu_j^{1/3}. \tag{2.13}$$

For any $\varepsilon > 0$ small, let $\psi_{\varepsilon,j}(x)$ be a smooth cut-off function centered at x_j such that $0 \leq \psi_{\varepsilon,j}(x) \leq 1$,

$$\psi_{\varepsilon,j}(x) = 1 \text{ in } B(x_j, \varepsilon/2), \qquad \psi_{\varepsilon,j}(x) = 0 \text{ in } B(x_j, \varepsilon), \qquad |\nabla \psi_{\varepsilon,j}(x)| \leq 4/\varepsilon.$$

By (1.4), we have

$$\begin{aligned}
&\left| \int_\Omega \frac{(u_n^+)^{q-1}}{|x|^\beta} \psi_{\varepsilon,j}(x) u_n \, dx \right| \\
&\quad \leq \int_{B(x_j,\varepsilon)} \frac{(u_n^+)^q}{|x|^\beta} \, dx \leq \Big(\int_{B(x_j,\varepsilon)} |u_n|^{q\cdot\frac{6}{q}} \, dx \Big)^{\frac{q}{6}} \Big(\int_{B(x_j,\varepsilon)} \frac{1}{|x|^{\frac{6\beta}{6-q}}} \, dx \Big)^{\frac{6-q}{6}} \\
&\quad \leq S^{-q/2} \|u_n\|^q \Big(\int_{B(x_j,\varepsilon)} \frac{1}{|x - x_j|^{\frac{6\beta}{6-q}}} \, dx \Big)^{\frac{6-q}{6}} \\
&\quad = S^{-q/2} \|u_n\|^q \Big(\int_0^\varepsilon \frac{r^2}{r^{\frac{6\beta}{6-q}}} \, dr \Big)^{\frac{6-q}{6}} \\
&\quad = S^{-q/2} \|u_n\|^q \Big(\int_0^\varepsilon \frac{1}{r^{\frac{6\beta}{6-q}-2}} \, dr \Big)^{\frac{6-q}{6}} \\
&\quad = S^{-q/2} \Big(\frac{6-q}{18-3q-6\beta} \Big)^{\frac{6-q}{6}} \|u_n\|^q \varepsilon^{\frac{6-2\beta-q}{2}}.
\end{aligned}$$

Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, it follows that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_\Omega \frac{(u_n^+)^{q-1}}{|x|^\beta} \psi_{\varepsilon,j}(x) u_n \, dx = 0.$$

By Hölder's inequality, we have

$$\begin{aligned}
\int_\Omega |\nabla(\psi_{\varepsilon,j}(x) u_n)|^2 \, dx &= \int_\Omega |u_n \nabla \psi_{\varepsilon,j}(x) + \psi_{\varepsilon,j}(x) \nabla u_n|^2 \, dx \\
&\leq \frac{16}{\varepsilon^2} \int_{B(x_j,\varepsilon)} |u_n|^2 \, dx + \frac{8}{\varepsilon} \int_{B(x_j,\varepsilon)} u_n |\nabla u_n| \, dx + \|u_n\|^2
\end{aligned}$$

$$\leq \frac{16}{\varepsilon^2}\Big(\int_\Omega |u_n|^6\,dx\Big)^{1/3}\Big(\int_{B(x_j,\varepsilon)} 1\,dx\Big)^{2/3} + \|u_n\|^2 + \frac{8}{\varepsilon}\|u_n\|^2\Big(\int_{B(x_j,\varepsilon)} u_n^2\,dx\Big)^{1/2}$$

$$\leq \frac{16}{\varepsilon^2}C_1\|u_n\|^2\varepsilon^2 + \|u_n\|^2 + \frac{8}{\varepsilon}C_2\|u_n\|^3\varepsilon = (16C_1+1)\|u_n\|^2 + 8C_2\|u_n\|^3,$$

where C_1, C_2 are positive constants. Since $\{f_n'(0)\}$ and $\{u_n\}$ are bounded in $H_0^1(\Omega)$, one gets

$$\lim_{n\to\infty}\frac{|f_n'(0)|\,\|u_n\| + \|\psi_{\varepsilon,j}(x)u_n\|}{n} = 0,$$

so that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{|f_n'(0)|\,\|u_n\| + \|\psi_{\varepsilon,j}(x)u_n\|}{n} = 0.$$

Setting $\varphi = \psi_{\varepsilon,j}(x)u_n$ in (2.6), and taking $\varepsilon\to 0$, one gets

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle I_\lambda'(u_n), \psi_{\varepsilon,j}(x)u_n\rangle \\
&= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\bigg\{(a+b\|u_n\|^2)\int_\Omega (\nabla u_n, \nabla(\psi_{\varepsilon,j}(x)u_n))\,dx \\
&\qquad\qquad - \int_\Omega (u_n^+)^5\psi_{\varepsilon,j}(x)u_n\,dx - \lambda\int_\Omega \frac{(u_n^+)^{q-1}}{|x|^\beta}\psi_{\varepsilon,j}(x)u_n\,dx\bigg\} \\
&= \Big(a + b\int_\Omega d\mu\Big)\int_\Omega \psi_{\varepsilon,j}\,d\mu - \int_\Omega \psi_{\varepsilon,j}\,d\nu,
\end{aligned}$$

so that

$$\nu_j = (a+b\mu_j)\mu_j.$$

By (2.13) we deduce that

$$\nu_j^{2/3} \geq aS + bS^2\nu_j^{1/3}, \quad \text{or} \quad \nu_j = \mu_j = 0. \tag{2.14}$$

Let $X = \nu_j^{1/3}$. It follows from (2.14) that

$$X^2 \geq aS + bS^2X,$$

which means that

$$X \geq \frac{bS^2 + \sqrt{b^2S^4 + 4aS}}{2},$$

so that

$$\mu_j \geq SX \geq \frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2} =: K.$$

Next we show that $\mu_j \geq (bS^3 + \sqrt{b^2S^6 + 4aS^3})/2$ is impossible, therefore the set J is empty. Assume the contrary: there exists some $j_0 \in J$ such that $\mu_{j_0} \geq (bS^3 + \sqrt{b^2S^6 + 4aS^3})/2$. By (2.12), (1.6), (1.5) and Young's inequality, we obtain

$$(2.15)\qquad \begin{aligned}\alpha_\lambda^- &= \lim_{n\to\infty} I_\lambda(u_n)\\ &= \lim_{n\to\infty}\bigg\{I_\lambda(u_n)-\frac{1}{4}\bigg(a\|u_n\|^2+b\|u_n\|^4-\int_\Omega (u_n^+)^6\,dx-\lambda\int_\Omega\frac{(u_n^+)^q}{|x|^\beta}\,dx\bigg)\bigg\}\\ &\ge \lim_{n\to\infty}\bigg\{\bigg(\frac{1}{2}-\frac{1}{4}\bigg)a\|u_n\|^2+b\bigg(\frac{1}{4}-\frac{1}{4}\bigg)\|u_n\|^4\\ &\qquad\qquad+\bigg(\frac{1}{4}-\frac{1}{6}\bigg)\int_\Omega u_n^6\,dx-\lambda\bigg(\frac{1}{q}-\frac{1}{4}\bigg)\int_\Omega\frac{|u_n|^q}{|x|^\beta}\,dx\bigg\}\\ &\ge \bigg(\frac{1}{2}-\frac{1}{4}\bigg)a\Big(\|u\|^2+\sum_{j\in J}\mu_j\Big)+b\bigg(\frac{1}{4}-\frac{1}{4}\bigg)\Big(\|u\|^2+\sum_{j\in J}\mu_j\Big)^2\\ &\quad+\bigg(\frac{1}{4}-\frac{1}{6}\bigg)\Big(\int_\Omega u^6\,dx+\sum_{j\in J}\nu_j\Big)-\lambda\bigg(\frac{1}{q}-\frac{1}{4}\bigg)\int_\Omega\frac{|u|^q}{|x|^\beta}\,dx\\ &\ge \bigg(\frac{1}{2}-\frac{1}{4}\bigg)a\mu_{j_0}+\bigg(\frac{1}{4}-\frac{1}{4}\bigg)b\mu_{j_0}^2+\bigg(\frac{1}{4}-\frac{1}{6}\bigg)\nu_{j_0}+\frac{a}{4}\|u\|^2-\lambda\bigg(\frac{1}{q}-\frac{1}{4}\bigg)T\|u\|^q\\ &\ge \bigg(\frac{1}{2}-\frac{1}{4}\bigg)aK+\bigg(\frac{1}{4}-\frac{1}{4}\bigg)bK^2+\bigg(\frac{1}{4}-\frac{1}{6}\bigg)\frac{K^3}{S^3}-D\lambda^{\frac{2}{2-q}}\\ &\ge \frac{a}{2}K+\frac{b}{4}K^2-\frac{K^3}{6S^3}-\frac{1}{4}\bigg(aK+bK^2-\frac{K^3}{S^3}\bigg)-D\lambda^{\frac{2}{2-q}},\end{aligned}$$

where $D=\big(\frac{(4-q)}{4q}T\big)^{2/(2-q)}\big(\frac{2q}{a}\big)^{q/(2-q)}$. We claim that

$$(2.16)\qquad \frac{a}{2}K+\frac{b}{4}K^2-\frac{K^3}{6S^3}=\Lambda.$$

Indeed,

$$\frac{a}{2}K=\frac{abS^3+a\sqrt{b^2S^6+4aS^3}}{4}$$

and

$$(bS^3+\sqrt{b^2S^6+4aS^3})^2=2b^2S^6+4aS^3+2bS^3\sqrt{b^2S^6+4aS^3}.$$

So

$$(bS^3+\sqrt{b^2S^6+4aS^3})^3=12abS^6+4b^3S^9+(4b^2S^6+4aS^3)\sqrt{b^2S^6+4aS^3}.$$

Hence

$$\frac{K^3}{6S^3}=\frac{12abS^3+4b^3S^6+(4b^2S^3+4a)\sqrt{b^2S^6+4aS^3}}{48}$$

and

$$\frac{a}{2}K+\frac{b}{4}K^2=\frac{8abS^3+2b^3S^6+(4a+2b^2S^3)\sqrt{b^2S^6+4aS^3}}{16}.$$

Therefore,

$$\frac{a}{2}K + \frac{b}{4}K^2 - \frac{K^3}{6S^3} = \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(4a+b^2S^3)\sqrt{b^2S^6+4aS^3}}{24} = \Lambda.$$

An easy computation yields

$$aK + bK^2 - \frac{K^3}{S^3} = 0. \tag{2.17}$$

Therefore, by (2.15)–(2.17), we get $\Lambda - D\lambda^{2/(2-q)} \le \alpha_\lambda^- < \Lambda - D\lambda^{2/(2-q)}$. This is a contradiction. Hence J is empty, thus $\int_\Omega u_n^6\,dx \to \int_\Omega u^6\,dx$ as $n \to \infty$. This completes the proof of Lemma 2.7. ∎

It is well known that the extremal function

$$U(x) = \frac{3^{1/4}}{(1+|x|^2)^{1/2}}$$

solves

$$-\Delta u = u^5 \quad \text{in } \mathbb{R}^3,$$

and $|\nabla U|^2_{L^2(\mathbb{R}^3)} = |U|^6_{L^6(\mathbb{R}^3)} = S^{3/2}$. Let $\eta \in C_0^\infty(\Omega)$ be a cut-off function such that $0 \le \eta \le 1$, $|\nabla \eta| \le C$ and $\eta(x) = 1$ for $|x| < 2R$, and $\eta(x) = 0$ for $|x| > 3R$. We define

$$u_\varepsilon(x) = \varepsilon^{-1/2}\eta(x)U\left(\frac{x}{\varepsilon}\right) = \frac{(3\varepsilon^2)^{1/4}\eta(x)}{(\varepsilon^2+|x|^2)^{1/2}}.$$

It is known (see [25, Lemma 1.46], [5]) that

$$\begin{cases} |u_\varepsilon|_6^6 = |U|^6_{L^6(\mathbb{R}^3)} + O(\varepsilon^3) = S^{3/2} + O(\varepsilon^3), \\ \|u_\varepsilon\|^2 = |\nabla U|^2_{L^2(\mathbb{R}^3)} + O(\varepsilon) = S^{3/2} + O(\varepsilon). \end{cases} \tag{2.18}$$

In much the same way as in [26] we can deduce

$$\begin{cases} \|u_\varepsilon\|^4 = |\nabla U|^4_{L^2(\mathbb{R}^3)} + O(\varepsilon) = S^3 + O(\varepsilon), \\ \|u_\varepsilon\|^6 = |\nabla U|^6_{L^2(\mathbb{R}^3)} + O(\varepsilon) = S^{9/2} + O(\varepsilon), \\ \|u_\varepsilon\|^8 = |\nabla U|^8_{L^2(\mathbb{R}^3)} + O(\varepsilon) = S^6 + O(\varepsilon), \\ \|u_\varepsilon\|^{12} = |\nabla U|^{12}_{L^2(\mathbb{R}^3)} + O(\varepsilon) = S^9 + O(\varepsilon). \end{cases} \tag{2.19}$$

Lemma 2.8. *Assume $1 < q < 2$ and $3 - q \le \beta < 2$. Then there exists $\bar u \in H_0^1(\Omega)$ such that*

$$\sup_{t \ge 0} I_\lambda(t\bar u) < \Lambda - D\lambda^{2/(2-q)},$$

where D is given in Lemma 2.7. In particular,

$$\alpha_\lambda^- < \Lambda - D\lambda^{2/(2-q)}.$$

Proof. We claim that there exist $t_\varepsilon > 0$ and positive constants t_0, T_1, independent of ε, λ, such that $\sup_{t\geq 0} I_\lambda(tu_\varepsilon) = I_\lambda(t_\varepsilon u_\varepsilon)$ and

$$0 < t_0 \leq t_\varepsilon \leq T_1 < \infty. \tag{2.20}$$

In fact, since $\lim_{t\to\infty} I_\lambda(tu_\varepsilon) = -\infty$, there exists $t_\varepsilon > 0$ such that

$$I_\lambda(t_\varepsilon u_\varepsilon) = \sup_{t\geq 0} I_\lambda(t_\varepsilon u_\varepsilon) \quad \text{and} \quad \frac{dI_\lambda(t_\varepsilon u_\varepsilon)}{dt}\bigg|_{t=t_\varepsilon} = 0. \tag{2.21}$$

It follows from (2.21) that

$$t_\varepsilon a\|u_\varepsilon\|^2 + t_\varepsilon^3 b\|u_\varepsilon\|^4 - t_\varepsilon^5 \int_\Omega u_\varepsilon^6\, dx - \lambda t_\varepsilon^{q-1} \int_\Omega \frac{u_\varepsilon^q}{|x|^\beta}\, dx = 0, \tag{2.22}$$

$$a\|u_\varepsilon\|^2 + 3t_\varepsilon^2 b\|u_\varepsilon\|^4 - 5t_\varepsilon^4 \int_\Omega u_\varepsilon^6\, dx - \lambda(q-1) t_\varepsilon^{q-2} \int_\Omega \frac{u_\varepsilon^q}{|x|^\beta}\, dx < 0. \tag{2.23}$$

Combining (2.22) and (2.23) implies that

$$(2-q)t_\varepsilon a\|u_\varepsilon\|^2 + (4-q)t_\varepsilon^3 b\|u_\varepsilon\|^4 < (6-q)t_\varepsilon^5 \int_\Omega u_\varepsilon^6\, dx. \tag{2.24}$$

On the one hand, we can calculate easily from (2.24) that t_ε is bounded below, that is, there exists a positive constant $t_0 > 0$ (independent of ε, λ) such that $0 < t_0 \leq t_\varepsilon$.

On the other hand, it follows from (2.22) that

$$\frac{a\|u_\varepsilon\|^2}{t_\varepsilon^2} + b\|u_\varepsilon\|^2 = t_\varepsilon^2 \int_\Omega u_\varepsilon^6\, dx + \frac{\lambda}{t_\varepsilon^{4-q}} \int_\Omega \frac{u_\varepsilon^q}{|x|^\beta}\, dx,$$

so t_ε is bounded above for all $\varepsilon > 0$ small enough. Thus (2.20) is true.

We set $I_\lambda(t_\varepsilon u_\varepsilon) = A(t_\varepsilon u_\varepsilon) - \lambda B(t_\varepsilon u_\varepsilon)$, where

$$A(t_\varepsilon u_\varepsilon) = \frac{a}{2} t_\varepsilon^2 \|u_\varepsilon\|^2 + \frac{b}{4} t_\varepsilon^4 \|u_\varepsilon\|^4 - \frac{t_\varepsilon^6}{6} \int_\Omega u_\varepsilon^6\, dx, \qquad B(t_\varepsilon u_\varepsilon) = \frac{1}{q} t_\varepsilon^q \int_\Omega \frac{u_\varepsilon^q}{|x|^\beta}\, dx.$$

Firstly, we claim that there exists a positive constant C_3 (independent of ε, λ) such that

$$A(t_\varepsilon u_\varepsilon) \leq \Lambda + C_3 \varepsilon. \tag{2.25}$$

Indeed, let

$$h(t) = \frac{a}{2} t^2 \|u_\varepsilon\|^2 + \frac{b}{4} t^4 \|u_\varepsilon\|^4 - \frac{t^6}{6} \int_\Omega u_\varepsilon^6\, dx.$$

Since $\lim_{t\to\infty} h(t) = -\infty$, $h(0) = 0$ and $\lim_{t\to 0^+} h(t) > 0$, it follows that $\sup_{t\geq 0} h(t)$ is attained at $T_\varepsilon > 0$, that is,

$$h'(t)|_{T_\varepsilon} = aT_\varepsilon \|u_\varepsilon\|^2 + bT_\varepsilon^3 \|u_\varepsilon\|^4 - T_\varepsilon^5 \int_\Omega u_\varepsilon^6\, dx = 0.$$

Observe that

$$T_\varepsilon^4 \int_\Omega u_\varepsilon^6\,dx - a\|u_\varepsilon\|^2 - bT_\varepsilon^2\|u_\varepsilon\|^4 = 0,$$

so

$$T_\varepsilon = \left(\frac{b\|u_\varepsilon\|^4 + \sqrt{b^2\|u_\varepsilon\|^8 + 4a\|u_\varepsilon\|^2\int_\Omega u_\varepsilon^6\,dx}}{2\int_\Omega u_\varepsilon^6\,dx}\right)^{1/2}.$$

Since $h(t)$ is increasing in $[0, T_\varepsilon]$, by (2.18) and (2.19) we get

$$\begin{aligned}
A(t_\varepsilon u_\varepsilon) &\le h(T_\varepsilon) \\
&= \frac{ab\|u_\varepsilon\|^6}{4\int_\Omega u_\varepsilon^6\,dx} + \frac{b^3\|u_\varepsilon\|^{12}}{24(\int_\Omega u_\varepsilon^6\,dx)^2} + \frac{(b^2\|u_\varepsilon\|^8 + 4a\|u_\varepsilon\|^2\int_\Omega u_\varepsilon^6\,dx)^{3/2}}{24(\int_\Omega u_\varepsilon^6\,dx)^2} \\
&= \frac{ab(S^{9/2} + O(\varepsilon))}{4(S^{3/2} + O(\varepsilon^3))} + \frac{b^3(S^9 + O(\varepsilon))}{24(S^{3/2} + O(\varepsilon^3))^2} \\
&\quad + \frac{[b^2(S^6 + O(\varepsilon)) + 4a(S^{3/2} + O(\varepsilon))(S^{3/2} + O(\varepsilon^3))]^{3/2}}{24(S^{3/2} + O(\varepsilon^3))^2} \\
&= \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^6 + 4aS^3)^{3/2}}{24S^3} + O(\varepsilon) \\
&= \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{3/2}}{24} + O(\varepsilon) = \Lambda + O(\varepsilon).
\end{aligned}$$

Therefore, there exists $C_3 > 0$ (independent of ε, λ) such that (2.25) holds.

We now estimate $B(t_\varepsilon u_\varepsilon)$. By the definition of u_ε and (2.20), in addition, let $0 < \varepsilon < \rho_1 < 2R$. We have

$$\begin{aligned}
B(t_\varepsilon u_\varepsilon) &= t_\varepsilon^q \int_\Omega \frac{|u_\varepsilon|^q}{|x|^\beta}\,dx \\
&\ge t_0^q(3\varepsilon^2)^{q/4} \int_{|x|<\rho_1} \frac{|x|^{-\beta}}{(\varepsilon^2 + |x|^2)^{q/2}}\,dx + t_0^q \int_{|x|\ge\rho_1} \frac{|v_\varepsilon|^q}{|x|^\beta}\,dx \\
&\ge t_0^q 3^{q/4}\varepsilon^{q/2} \int_0^{\rho_1} \frac{r^2}{r^\beta(\varepsilon^2 + r^2)^{q/2}}\,dr \\
&= t_0^q 3^{q/4}\varepsilon^{(6-q-2\beta)/2} \int_0^{\rho_1\varepsilon^{-1}} \frac{r^2}{r^\beta(1 + r^2)^{q/2}}\,dr \\
&= t_0^q 3^{q/4}\varepsilon^{(6-q-2\beta)/2} \int_0^1 \frac{r^2}{r^\beta(1 + r^2)^{q/2}}\,dr \\
&\quad + t_0^q 3^{q/4}\varepsilon^{(6-q-2\beta)/2} \int_1^{\rho_1\varepsilon^{-1}} \frac{r^2}{r^\beta(1 + r^2)^{q/2}}\,dr.
\end{aligned} \tag{2.26}$$

From (2.26), we get

$$\int_\Omega \frac{|u_\varepsilon|^q}{|x|^\beta}\,dx \ge \begin{cases} C\varepsilon^{(6-q-2\beta)/2}, & q > 3-\beta, \\ C\varepsilon^{(6-q-2\beta)/2}|\ln\varepsilon|, & q = 3-\beta, \\ C\varepsilon^{q/2}, & q < 3-\beta. \end{cases}$$

Case $\beta > 3-q$. Then $q > 3-\beta$, so there exists a constant $C_4 > 0$ (independent of ε, λ) such that

$$B(t_\varepsilon u_\varepsilon) \ge C_4\varepsilon^{(6-q-2\beta)/2}. \tag{2.27}$$

Noting that $1 < q < 2$ and $3-q < \beta < 2$, it follows that $(6-q-2\beta)/2 < 1$ and $\frac{6-2q-2\beta}{2-q} < 0$. Let $\varepsilon = \lambda^{2/(2-q)}$ and $\lambda < \lambda_0 = \big(\frac{C_3}{C_4+D}\big)^{\frac{2-q}{2q+2\beta-6}}$. Then

$$\begin{aligned} C_3\varepsilon - C_4\lambda\varepsilon^{(6-q-2\beta)/2} &= C_3\lambda^{2/(2-q)} - C_4\lambda^{\frac{8-2q-2\beta}{2-q}} \\ &= \lambda^{2/(2-q)}(C_3 - C_4\lambda^{\frac{6-2q-2\beta}{2-q}}) \\ &< -D\lambda^{2/(2-q)}. \end{aligned}$$

Therefore, the combination of (2.25) and (2.27) implies that

$$\begin{aligned} I_\lambda(t_\varepsilon u_\varepsilon) &= A(t_\varepsilon u_\varepsilon) - \lambda B(t_\varepsilon u_\varepsilon) \\ &\le \Lambda + C_3\varepsilon - C_4\lambda\varepsilon^{(6-q-2\beta)/2} \\ &\le \Lambda - D\lambda^{2/(2-q)}. \end{aligned}$$

Case $\beta = 3-q$. Then there exists a constant $C_5 > 0$ (independent of ε, λ) such that

$$B(t_\varepsilon u_\varepsilon) \ge C_5\varepsilon^{(6-q-2\beta)/2}|\ln\varepsilon|. \tag{2.28}$$

Let $\varepsilon = \lambda^{2/(2-q)}$, $\lambda < \tilde\lambda_0 = \min\{1, e^{-(C_3+D)/C_6}\}$, where $C_6 = \frac{2C_5}{2-q}$, so that

$$\begin{aligned} C_3\varepsilon - C_5\lambda\varepsilon^{(6-q-2\beta)/2}|\ln\varepsilon| &= C_3\lambda^{2/(2-q)} - \frac{2C_5}{2-q}\lambda^{\frac{8-2q-2\beta}{2-q}}|\ln\lambda| \\ &= \lambda^{2/(2-q)}(C_3 - C_6\lambda^{\frac{6-2q-2\beta}{2-q}}|\ln\lambda|) \\ &= \lambda^{2/(2-q)}(C_3 - C_6|\ln\lambda|) < -D\lambda^{2/(2-q)}. \end{aligned}$$

It follows from (2.25) and (2.28) that

$$\begin{aligned} I_\lambda(t_\varepsilon u_\varepsilon) &= A(t_\varepsilon u_\varepsilon) - \lambda B(t_\varepsilon u_\varepsilon) \le \Lambda + C_3\varepsilon - C_5\lambda\varepsilon^{(6-q-2\beta)/2}|\ln\varepsilon| \\ &\le \Lambda - D\lambda^{2/(2-q)}. \end{aligned}$$

This completes the proof of Lemma 2.8. ∎

3. Proofs of the theorems

Proof of Theorem 1.1. There exists a constant $\delta > 0$ such that $\Lambda - D\lambda^{2/(2-q)} > 0$ for $\lambda < \delta$. Set $\lambda_* = \min\big\{\frac{6-q}{2(4-q)}T_1, T_2, \delta\big\}$. Then Lemmas 2.1–2.4, 2.6, 2.7 hold for all $0 < \lambda < \lambda_*$. By Lemma 2.6, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ of I_λ, obviously bounded in $H_0^1(\Omega)$; going if necessary to a subsequence, still denoted by $\{u_n\}$, there exists $u_\lambda \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_\lambda & \text{weakly in } H_0^1(\Omega), \\ u_n \to u_\lambda & \text{strongly in } L^s(\Omega),\ 1 \le s < 6, \\ u_n(x) \to u_\lambda(x) & \text{a.e. in } \Omega, \end{cases}$$

as $n \to \infty$. Now we will prove that u_λ is a positive ground state solution of problem (1.1).

First, we prove that u_λ is a positive solution of (1.1). Indeed, by Lemma 2.6, for all $\varphi \in H_0^1(\Omega)$,

$$\Big(a + b \lim_{n\to\infty} \|u_n\|^2\Big) \int_\Omega (\nabla u_\lambda, \nabla\varphi)\, dx - \int_\Omega (u_\lambda^+)^5 \varphi\, dx - \lambda \int_\Omega \frac{(u_\lambda^+)^{q-1}\varphi}{|x|^\beta}\, dx = 0.$$

Set $\lim_{n\to\infty} \|u_n\| = l$. Then

$$(a + bl^2) \int_\Omega (\nabla u_\lambda, \nabla\varphi)\, dx = \int_\Omega (u_\lambda^+)^5 \varphi\, dx + \lambda \int_\Omega \frac{(u_\lambda^+)^{q-1}\varphi}{|x|^\beta}\, dx. \tag{3.1}$$

Taking the test function $\varphi = u_\lambda$ in (3.1) yields

$$(a + bl^2)\|u_\lambda\|^2 - \int_\Omega (u_\lambda^+)^6\, dx - \lambda \int_\Omega \frac{(u_\lambda^+)^q}{|x|^\beta}\, dx = 0. \tag{3.2}$$

The fact that $u_n \in \mathcal{N}_\lambda$ implies that

$$(a + b\|u_n\|^2)\|u_n\|^2 - \int_\Omega (u_n^+)^6\, dx - \lambda \int_\Omega \frac{(u_n^+)^q}{|x|^\beta}\, dx = 0.$$

As $\alpha_\lambda < 0 < \Lambda - D\lambda^{2/(2-q)}$, by Lemma 2.7 and (1.6) one has

$$(a + bl^2)l^2 - \int_\Omega (u_\lambda^+)^6\, dx - \lambda \int_\Omega \frac{(u_\lambda^+)^q}{|x|^\beta}\, dx = 0. \tag{3.3}$$

It follows from (3.2) and (3.3) that $\|u_\lambda\| = l$, so $u_n \to u_\lambda$ in $H_0^1(\Omega)$, and u_λ is a solution of problem (1.1), that is,

$$(a + b\|u_\lambda\|^2) \int_\Omega (\nabla u_\lambda, \nabla\varphi)\, dx = \int_\Omega (u_\lambda^+)^5 \varphi\, dx + \lambda \int_\Omega \frac{(u_\lambda^+)^{q-1}\varphi}{|x|^\beta}\, dx \tag{3.4}$$

for all $\varphi \in H_0^1(\Omega)$. Taking the test function $\varphi = u_\lambda^-$ in (3.4), we get $\|u_\lambda^-\| = 0$, so $u_\lambda \ge 0$. Furthermore, note that $u_\lambda \in \mathcal{N}_\lambda$ (u_λ is a nontrivial solution of problem (1.1)) and $\alpha_\lambda < 0$ (by Lemma 2.3), so

$$\left(\frac{1}{q}-\frac{1}{6}\right)\int_\Omega \frac{u_\lambda^q}{|x|^\beta}\,dx = \frac{a}{3}\|u_\lambda\|^2 + \frac{b}{12}\|u_\lambda\|^4 - I_\lambda(u_\lambda)$$
$$\geq \frac{a}{3}\|u_\lambda\|^2 + \frac{b}{12}\|u_\lambda\|^4 - \alpha_\lambda > 0,$$

which implies that $u_\lambda \not\equiv 0$. Therefore, by the strong maximum principle, $u_\lambda > 0$ in Ω. Furthermore, by Lemma 2.7 and (1.6), we have

$$\alpha_\lambda = \lim_{n\to\infty} I_\lambda(u_n) = I_\lambda(u_\lambda). \tag{3.5}$$

Next, we want to show that $u_\lambda \in \mathcal{N}_\lambda^+$ and $I_\lambda(u_\lambda) = \alpha_\lambda^+$. We first prove that $u_\lambda \in \mathcal{N}_\lambda^+$. On the contrary, assume that $u_\lambda \in \mathcal{N}_\lambda^-$ ($\mathcal{N}_\lambda^0 = \emptyset$ for $\lambda \in (0, \frac{6-q}{2(4-q)}T_1)$). By Lemma 2.1, there exist $0 < t^+ < t_{\max} < t^- = 1$ such that $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda^-$ and

$$\alpha_\lambda < I_\lambda(t^+u_\lambda) < I_\lambda(t^-u_\lambda) = I_\lambda(u_\lambda) = \alpha_\lambda,$$

which is a contradiction. Hence, $u_\lambda \in \mathcal{N}_\lambda^+$. By the definition of α_λ^+, we obtain $\alpha_\lambda^+ \leq I_\lambda(u_\lambda)$. It follows from Lemma 2.3(i) and (3.5) that

$$I_\lambda(u_\lambda) = \alpha_\lambda^+ = \alpha_\lambda < 0.$$

From the above arguments, u_λ is a positive ground state solution of problem (1.1). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Set $\lambda_{**} = \min\{\lambda_*, \lambda_0, \tilde{\lambda}_0\}$. Then Lemmas 2.1–2.8 hold for all $0 < \lambda < \lambda_*$. By Theorem 1.1, $u_\lambda \in \mathcal{N}_\lambda^+$ is a positive ground state solution of (1.1). Now, we shall verify that (1.1) has another solution v_λ, and $v_\lambda \in \mathcal{N}_\lambda^-$ with $I_\lambda(v_\lambda) > 0$.

Since I_λ is also coercive on $\mathcal{N}_\lambda^-$, Ekeland's variational principle applied to the minimization problem $\alpha_\lambda^- = \inf_{v\in\mathcal{N}_\lambda^-} I_\lambda(v)$ yields a minimizing sequence $\{v_n\} \subset \mathcal{N}_\lambda^-$ for I_λ with the following properties:

(i) $I_\lambda(v_n) < \alpha_\lambda^- + 1/n$,
(ii) $I_\lambda(u) \geq I_\lambda(v_n) - (1/n)\|u - v_n\|$ for all $u \in \mathcal{N}_\lambda^-$.

Since $\{v_n\}$ is bounded in $H_0^1(\Omega)$, passing to a subsequence if necessary, there exists $v_\lambda \in H_0^1(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v_\lambda & \text{weakly in } H_0^1(\Omega),\\ v_n \to v_\lambda & \text{strongly in } L^s(\Omega),\ 1 \leq s \leq 6,\\ v_n(x) \to v_\lambda(x) & \text{a.e. in } \Omega,\end{cases}$$

as $n \to \infty$. Now we will prove that v_λ is a positive solution of (1.1). As in the proof of Theorem 1.1, we get $v_n \to v_\lambda$ in $H_0^1(\Omega)$, and v_λ is a nonnegative solution of (1.1).

Now, we prove that $v_\lambda > 0$ in Ω. Since $v_n \in \mathcal{N}_\lambda^-$, we have

$$\begin{aligned} a(2-q)\|v_n\|^2 &\le (6-q)\int_\Omega (v_n^+)^6\,dx - b(4-q)\|v_n\|^4 \\ &\le (6-q)\int_\Omega |v_n|^6\,dx < (6-q)S^{-3}\|v_n\|^6, \end{aligned}$$

so that

$$\|v_n\| > \left(\frac{a(2-q)S^3}{6-q}\right)^{1/4}. \tag{3.6}$$

As $v_n \to v_\lambda$ in $H_0^1(\Omega)$, (3.6) implies that $v_\lambda \not\equiv 0$. Therefore, the strong maximum principle implies that $v_\lambda > 0$ in Ω.

Next, we prove that $v_\lambda \in \mathcal{N}_\lambda^-$; it suffices to show that $\mathcal{N}_\lambda^-$ is closed.

Indeed, by Lemmas 2.7 and 2.8, for $\{v_n\} \subset \mathcal{N}_\lambda^-$, we have

$$\lim_{n\to\infty}\int_\Omega (v_n^+)^6\,dx = \int_\Omega v_\lambda^6\,dx.$$

By the definition of $\mathcal{N}_\lambda^-$,

$$(2-q)a\|v_n\|^2 + (4-q)b\|v_n\|^4 - (6-q)\int_\Omega (v_n^+)^6\,dx < 0,$$

thus

$$(2-q)a\|v_\lambda\|^2 + (4-q)b\|v_\lambda\|^4 - (6-q)\int_\Omega v_\lambda^6\,dx \le 0,$$

which implies that $v_\lambda \in \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^-$. If $\mathcal{N}_\lambda^-$ is not closed, then $v_\lambda \in \mathcal{N}_\lambda^0$, and by Lemma 2.1 it follows that $v_\lambda = 0$, which contradicts $v_\lambda > 0$. Consequently, $v_\lambda \in \mathcal{N}_\lambda^-$. Furthermore, by Lemma 2.3,

$$I_\lambda(v_\lambda) = \lim_{n\to\infty} I_\lambda(v_n) = \alpha_\lambda^- > 0.$$

This completes the proof of Theorem 1.2.

Acknowledgments. The authors would like to thank the anonymous referees and the editors for valuable comments.

This research was partly supported by the National Natural Science Foundation of China (No. 11471267) and the Science and Technology Foundation of Guizhou Province (No. LKM[2011]31; No. J[2013]2141).

References

[1] C. O. Alves, F. J. S. A. Corrêa and G. M. Figueiredo, *On a class of nonlocal elliptic problems with critical growth*, Differential Equations Appl. 23 (2010), 409–417.

[2] C. O. Alves, F. J. S. A. Corrêa and T. F. Ma, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl. 49 (2005), 85–93.

[3] A. Ambrosetti, H. Brézis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. 122 (1994), 519–477.

[4] G. Anello, *A uniqueness result for a nonlocal equation of Kirchhoff type and some related open problem*, J. Math. Anal. Appl. 373 (2011), 248–251.

[5] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36 (1983), 437–477.

[6] C. Chen, Y. Kuo and T. Wu, *The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions*, J. Differential Equations 250 (2011), 1876–1908.

[7] B. T. Cheng, *New existence and multiplicity of nontrivial solutions for nonlocal elliptic Kirchhoff type problems*, J. Math. Anal. Appl. 394 (2012), 488–495.

[8] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (N.S.) 1 (1979), 443–474.

[9] G. M. Figueiredo, *Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument*, J. Math. Anal. Appl. 401 (2013), 706–713.

[10] G. M. Figueiredo and J. R. Santos, Jr., *Multiplicity of solutions for a Kirchhoff equation with subcritical or critical growth*, Differential Integral Equations 25 (2012), 853–868.

[11] A. Hamydy, M. Massar and N. Tsouli, *Existence of solution for p-Kirchhoff type problems with critical exponent*, Electron. J. Differential Equations 2011, no. 105, 8 pp.

[12] J. H. Jin and X. Wu, *Infinitely many radial solutions for Kirchhoff-type problems in* $\mathbb{R}^N$, J. Math. Anal. Appl. 369 (2010), 564–574.

[13] G. Kirchhoff, *Vorlesungen über mathematische Physik: Mechanik*, Teubner, Leipzig, 1876.

[14] Y. H. Li, F. Y. Li and J. P. Shi, *Existence of a positive solution to Kirchhoff type problems without compactness conditions*, J. Differential Equations 253 (2012), 2285–2294.

[15] G. B. Li and H. Y. Ye, *Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in* $\mathbb{R}^3$, J. Differential Equations 257 (2014), 566–600.

[16] S. H. Liang and S. Y. Shi, *Soliton solutions to Kirchhoff type problems involving the critical growth in* $\mathbb{R}^N$, Nonlinear Anal. 81 (2013), 31–41.

[17] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case*, Rev. Mat. Iberoamer. 1 (1985), 145–201.

[18] T. F. Ma, *Remarks on an elliptic equation of Kirchhoff type*, Nonlinear Anal. 63 (2005), 1967–1977.

[19] T. F. Ma and J. E. Muñoz Rivera, *Positive solutions for a nonlinear nonlocal elliptic transmission problem*, Appl. Math. Lett. 16 (2003), 243–248.

[20] A. M. Mao and S. X. Luan, *Sign-changing solutions of a class of nonlocal quasilinear elliptic boundary value problems*, J. Math. Anal. Appl. 383 (2011), 239–243.

[21] A. M. Mao and Z. T. Zhang, *Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition*, Nonlinear Anal. 70 (2009), 1275–1287.

[22] D. Naimen, *The critical problem of Kirchhoff type elliptic equations in dimension four*, J. Differential Equations 257 (2014), 1168–1193.

[23] J. J. Nie and X. Wu, *Existence and multiplicity of non-trivial solutions for Schrödinger–Kirchhoff-type equations with radial potential*, Nonlinear Anal. 75 (2012), 3470–3479.

[24] L. Wei and X. M. He, *Multiplicity of high energy solutions for superlinear Kirchhoff equations*, J. Appl. Math. Comput. 39 (2012), 473–487.

[25] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[26] Q. L. Xie, X. P. Wu and C. L. Tang, *Existence and multiplicity of solutions for Kirchhoff type problem with critical exponent*, Comm. Pure Appl. Anal. 12 (2013), 2773–2786.

[27] Y. Yang and J. H. Zhang, *Positive and negative solutions of a class of nonlocal problems*, Nonlinear Anal. 73 (2010), 25–30.

[28] Z. T. Zhang and K. Perera, *Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow*, J. Math. Anal. Appl. 317 (2006), 456–463.

Chun-Yu Lei, Chang-Mu Chu, Hong-Min Suo
School of Sciences
GuiZhou Minzu University
550025 Guiyang, China
E-mail: 969290985@qq.com

Chun-Lei Tang (corresponding author)
School of Mathematics and Statistics
Southwest University
400715 Chongqing, China
E-mail: tangcl@swu.edu.cn

Received 2.9.2014
and in final form 17.3.2015 (3481)