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Holomorphic series expansion of functions of Carleman type

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Abstract. Let f be a holomorphic function of Carleman type in a bounded convex domain D of the plane. We show that f can be expanded in a series $f = \sum_n f_n$, where f_n is a holomorphic function in D_n satisfying $\sup_{z\in D_n} |f_n(z)| \le C\varrho^n$ for some constants $C > 0$ and $0 < \varrho < 1$, and where $(D_n)_n$ is a suitably chosen sequence of decreasing neighborhoods of the closure of D. Conversely, if f admits such an expansion then f is of Carleman type. The decrease of the sequence D_n characterizes the smoothness of f.

1. Introduction. Let $(M_n)_{n\ge 0}$ be an increasing sequence of positive real numbers. There exist two ways to define that a given $\mathcal{C}^\infty$ function, f, on an interval $[a,b] \subset \mathbb{R}$, belongs to the regular Carleman class $\mathcal{C}(M_n)$. First, there exist positive constants C and ϱ, depending on f, such that $|f^{(n)}(x)| \le C\varrho^n M_n$ for all $x \in [a,b]$ and $n \in \mathbb{N}$. (See [Ko], [Ma].) Second, f admits an extension, F, not unique, to the whole complex plane such that $\overline{\partial} F$ decreases rapidly near $[a,b]$; here $\overline{\partial} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ is the Cauchy–Riemann operator. (See [Dy] and below.)

The purpose of this short note is to characterize the holomorphic functions on a bounded convex domain D belonging to a given Carleman class as those functions that can be expanded in a series of functions holomorphic in an appropriate sequence of decreasing neighborhoods of the closure of D, satisfying some growth estimates. The decrease of these neighborhoods is directly linked with the given class.

The first result of this kind was obtained by J. C. Tougeron [To, 2.7–2.9] in the particular case where D is a bounded sector and $M_n = (n!)^k$, with $k > 1/2$, which corresponds to a Gevrey class. Our approach is different and may be extended to sets which are Whitney regular.

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2. The class $\mathcal{H}_M(D)$. Let D be a bounded convex domain in the plane; $\mathcal{O}(D)$ and $\mathcal{O}(\overline{D})$ are spaces of holomorphic functions on D and in a neighborhood of $\overline{D}$ respectively.

Let $M = (M_n)_n$ be an increasing sequence of positive real numbers. Let $\mathcal{H}_M(D)$ be the class of all functions $f \in \mathcal{O}(D)$ such that, for some positive constants C and ϱ,

$$\sup_{z\in D} |f^{(n)}(z)| \le C\varrho^n M_n, \qquad n \gg 0.$$

Note that every function f belonging to $\mathcal{H}_M(D)$ can be extended to a $\mathcal{C}^\infty$ function on $\overline{D}$: if $w \in \partial D$ and if $z_p \in D$ converges to w, the sequence $f^{(n)}(z_p)$ converges (because $f^{(n+1)}$ is bounded on D; apply the mean value theorem). We denote this extension also by f.

In order to get classes of holomorphic functions with structural properties and to have precise computations, we start, following [El], from a sequence $(M_n)_n$ such that $M_n := M(n)$ with $M(t) = e^{m(t)}$ and $m(t) = t\log t + t\mu(t)$. Throughout the paper $\mu(t)$ will be a strictly increasing $\mathcal{C}^\infty$ function defined for $t \gg 0$ such that $\lim_{t\to+\infty}\mu(t) = +\infty$ (so $\mu'(t) > 0$). We also suppose that $\mu(t)$ belongs to a Hardy field (i.e a field of germs of functions at $+\infty$ in $\mathbb{R}$ which is closed under differentiation) and $\mu(t) \le at$, $t \gg 0$, $a > 0$. This ensures that our class is an algebra closed under differentiation (the proof is easy and it is the same as in the real case, see [El]) and strictly contains $\mathcal{O}(\overline{D})$.

Notice finally that the above class does not change if we replace $M(t)$ by $C\varrho^t M(t)$ where $C > 0$ and $\varrho > 0$. Consequently, μ is defined modulo an additive constant.

3. The functions $\Omega(s)$ and $\Gamma(u)$. Set

$$\Omega(s) := \inf_{t\ge t_0} s^{-t}e^{t\mu(t)}, \qquad s \gg 0,$$

where $t_0 > 0$ is fixed. The infimum is attained when $t\mu'(t) + \mu(t) = \log s$. The function $t\mu'(t)+\mu(t)$ tends to infinity as $t \to +\infty$ and so it is strictly increasing ($\mu(t)$ belongs to a Hardy field); so we have a unique value of t where the infimum is attained. Thus, if $\Omega(s) = e^{-\omega(s)}$, then we get the system

$$(1) \qquad s = e^{t\mu'(t)+\mu(t)}, \qquad \omega(s) = t^2\mu'(t).$$

Since $\mu'(t) > 0$, we have $\omega(s) > 0$ and $\lim_{s\to+\infty}\omega(s) = +\infty$. Thus, $\Omega(s)$ is strictly decreasing and $\lim_{s\to\infty}\Omega(s) = 0$.

Set $\Gamma(u) := e^{-\gamma(u)}$, where u and $\gamma(u)$ are defined by

$$(2) \qquad u = t^2\mu'(t), \qquad \gamma(u) = t\mu'(t) + \mu(t).$$

As $\mu(t)$ is strictly increasing and $\lim_{t\to+\infty}\mu(t) = +\infty$, it follows that $\gamma(u)$ is strictly increasing and $\lim_{u\to+\infty}\gamma(u) = +\infty$. Hence, $\Gamma(u)$ is strictly decreasing and $\lim_{u\to+\infty} \Gamma(u) = 0$. The system (2) gives easily

$$t = 1/\gamma'(u), \qquad \mu(t) = \gamma(u) - u\gamma'(u), \tag{3}$$

which shows that γ' is strictly decreasing, positive and $\lim_{u\to+\infty} \gamma'(u) = 0$. Notice that $\gamma(u)$, just as $\mu(t)$, is defined modulo an additive constant.

4. Main result. Define

$$D^{\Gamma}_{n,R} := \{z \in \mathbb{C};\, d(z, D) < R\Gamma(n)\}.$$

Under the condition $\lim_{t\to+\infty}(\log t)/\mu(t) \neq 0$, we have the following:

THEOREM 1. (a) *Let $f \in \mathcal{H}_M(D)$. Then there exist $R > 0$, $C > 0$, $0 < \varrho < 1$, and a sequence $(f_n)_n$ with $f_n \in \mathcal{O}(D^{\Gamma}_{n,R})$ such that*:

(i) $\|f_n\|_{D^{\Gamma}_{n,R}} \le C\varrho^n$ *for all* n;

(ii) $\sum_n f_n = f$ *uniformly on* $\overline{D}$.

(b) *Conversely, let $R > 0$ and let $f_n \in \mathcal{O}(D^{\Gamma}_{n,R})$ be such that $\|f_n\|_{D^{\Gamma}_{n,R}} \le C\varrho^n$ for some constants $C > 0$ and $0 < \varrho < 1$, and for all $n \ge n_0$ (n_0 fixed). Then $f := \sum_n f_n$ belongs to the class $\mathcal{H}_M(D)$.*

5. Technical lemmas. With the above notations we have

LEMMA 1. *The function $\omega(s)$ is the inverse, under composition, of the function $e^{\gamma(u)} = 1/\Gamma(u)$, i.e. $\omega(s) = \gamma^{-1}(\log s)$, $s \gg 0$.*

Proof. It suffices to compare the systems (1) and (2).

Let us introduce the class $\mathcal{H}_{M_\alpha}(D)$, $\alpha > 0$, which corresponds to $\mu(\alpha t)$, i.e. we replace $\mu(t)$ by $\mu_\alpha(t) = \mu(\alpha t)$. So $M_\alpha(t) = e^{m_\alpha(t)}$, where $m_\alpha(t) = t\log t + t\mu(\alpha t)$. Let $\Omega_\alpha(s) = e^{-\omega_\alpha(s)}$ and $\Gamma_\alpha(u) = e^{-\gamma_\alpha(u)}$ be the corresponding functions of the class $\mathcal{H}_{M_\alpha}(D)$. We have the following:

LEMMA 2. *For all $\alpha > 0$,*

(i) $\omega_\alpha(s) = (1/\alpha)\omega(s)$,

(ii) $\gamma_\alpha(u) = \gamma(\alpha u)$.

Proof. By (2), we have $u = t^2\alpha\mu'(\alpha t)$ and $\gamma_\alpha(u) = t\alpha\mu'(\alpha t) + \mu(\alpha t)$; so if $\widetilde{t} := \alpha t$, then $\alpha u = \widetilde{t}\mu'(\widetilde{t})$ and $\gamma_\alpha(u) = \widetilde{t}\mu'(\widetilde{t}) + \mu(\widetilde{t})$. Thus we have (ii). Using Lemma 1 and (ii) we get

$$\omega_\alpha(s) = \gamma_\alpha^{-1}(\log s) = \frac{1}{\alpha}\,\gamma^{-1}(\log s) = \frac{1}{\alpha}\,\omega(s).$$

LEMMA 3. *If* $\lim_{t\to+\infty}(\log t)/\mu(t) \neq 0$, *then* $\mathcal{H}_{M_\alpha}(D) = \mathcal{H}_M(D)$.

Proof. By assumption, there exists $A > 0$ such that $\mu(t) \le A\log t$ for $t \gg 0$; then $t\mu'(t) \le B$ for some constant $B > 0$. Hence $|\mu(\alpha t) - \mu(t)| \le$

$|1-\alpha|t\mu'(t) \le B|1-\alpha|$. Thus $\mu(\alpha t)-\mu(t)$ is bounded. But μ is defined modulo an additive constant, so we can choose $\mu_\alpha = \mu$.

6. Proof of Theorem 1. Let $f \in \mathcal{H}_M(D)$. By Lemma 3, $f \in \mathcal{H}_{M_\alpha}(D)$ for every $\alpha > 0$. With the help of Dynkin's scheme (see [Dy, pp. 41–43] adjusted to our situation, f can be extended to a $\mathcal{C}^\infty$ function on the whole plane, say F, with compact support and such that the following estimate holds:

$$\left|\frac{\partial F}{\partial \overline{\zeta}}(\zeta)\right| \le C_2 e^{-\omega_\alpha(1/C_1 d(\zeta,D))} \tag{4}$$

for every $\zeta \in \mathbb{C} - \overline{D}$. In the above C_1 and C_2 are positive constants depending on α and f; $d(\zeta, D)$ is the Euclidean distance from ζ to D; and $\partial/\partial\overline{\zeta}$ is the Cauchy–Riemann operator. Fix such an F with $\operatorname{supp} F \subset \{\zeta \in \mathbb{C};\, d(\zeta, D) < r\}$, where r is a fixed positive number; and let $R = r/\Gamma(n_0)$, where n_0 is an integer to be chosen later. Now, set

$$D_n := D^{\Gamma}_{n,R} = \left\{\zeta \in \mathbb{C};\, d(\zeta, D) < r\,\frac{\Gamma(n)}{\Gamma(n_0)}\right\}, \qquad n \ge n_0. \tag{5}$$

Notice that $\operatorname{supp} F \subset D_{n_0}$; D_n is an open convex neighborhood of $\overline{D}$; $D_{n+1} \subset D_n$ for all n; and $\bigcap_{n\ge n_0} D_n = \overline{D}$.

Let f_n be the $\mathbb{C}$-valued function defined for every $z \in D_n$ by

$$f_n(z) = \frac{-1}{\pi} \int\limits_{D_{n-1}\setminus D_n} \frac{\frac{\partial F}{\partial\overline{\zeta}}(\zeta)}{\zeta - z}\, d\xi\, d\eta, \qquad \zeta = \xi + i\eta,\ n \ge n_0.$$

Clearly $f_n \in \mathcal{O}(D_n)$ and since F $(= f)$ is holomorphic on D, by the Cauchy–Green formula we have

$$\begin{aligned} f(z) &= \frac{-1}{\pi}\int\limits_{C} \frac{\frac{\partial F}{\partial\overline{\zeta}}(\zeta)}{\zeta - z}\, d\xi\, d\eta = \frac{-1}{\pi}\int\limits_{D_{n_0}\setminus\overline{D}} \frac{\frac{\partial F}{\partial\overline{\zeta}}(\zeta)}{\zeta - z}\, d\xi\, d\eta \\ &= \sum_{n\ge n_0+1} \frac{-1}{\pi}\int\limits_{D_{n-1}\setminus D_n} \frac{\frac{\partial F}{\partial\overline{\zeta}}(\zeta)}{\zeta - z}\, d\xi\, d\eta = \sum_{n\ge n_0+1} f_n(z), \qquad z \in D. \end{aligned}$$

Next, by the estimate (4) on $\partial F/\partial\overline{\xi}$, by Lemmas 1–3, choosing n_0 equal to the integer part of $\Gamma^{-1}(rC_1)$ where Γ^{-1} is the inverse of the function Γ, we have

$$\begin{aligned} \left|\frac{\partial F}{\partial\overline{\zeta}}(\zeta)\right| &\le C_2 \exp\biggl(-\frac{1}{\alpha}\omega\biggl(\frac{1}{C_1 d(\zeta, D)}\biggr)\biggr) \le C_2 \exp\biggl(-\frac{1}{\alpha}\omega\biggl(\frac{\Gamma(n_0)}{rC_1\Gamma(n-1)}\biggr)\biggr) \\ &\le C_2 \exp\biggl(-\frac{1}{\alpha}\omega\biggl(\frac{1}{\Gamma(n-1)}\biggr)\biggr) = C_2 e^{-(n-1)/\alpha} \end{aligned}$$

for every $\zeta \in D_{n-1} \setminus D_n$. Otherwise, a proof similar to that of the Ahlfors–Beurling inequality (see [Ra, pp. 141–142]) gives the estimates

$$\int\limits_{D_{n-1}\setminus D_n} \frac{1}{|\zeta - z|}\, d\xi\, d\eta \le \sqrt{\pi \operatorname{area}(D_{n_0})}.$$

Now, taking $C = C_2 e^{1/\alpha}\sqrt{\pi \operatorname{area}(D_{n_0})}$ and $\varrho > e^{-1/\alpha}$ we get $\|f_n\|_{D_n} \le C\varrho^n$; and the proof of Theorem 1(a) is finished.

To prove the converse, let $z \in D$. Since the closed disc $\overline{D}(z, (R/2)\Gamma(n))$ is contained in $D_n := D^{\Gamma}_{n,R}$ ($R > 0$ is given), we use Cauchy's inequalities to get

$$\frac{|f_n^{(p)}(z)|}{p!} \le C\varrho^n \left(\frac{2}{R\Gamma(n)}\right)^p, \quad p = 0, 1, \ldots,\ n \ge 0.$$

Choose ϱ' such that $\varrho < \varrho' < 1$; then

$$\sup_{z\in D} \frac{|f_n^{(p)}(z)|}{p!} \le C(2R^{-1})^p \left(\frac{\varrho}{\varrho'}\right)^n \varrho'^n \sup_{u>0} \frac{\varrho'^u}{(\Gamma(u))^p}, \quad p = 0, 1, \ldots,\ n \ge 0.$$

By summing the preceding inequalities over n we get

$$\begin{aligned}\sup_{z\in D} \frac{|f^{(p)}(z)|}{p!} &\le C(2R^{-1})^p \left(\frac{\varrho}{\varrho'}\right)^{n_0} \frac{\varrho'}{\varrho' - \varrho} \sup_{u>0} \frac{\varrho'^u}{(\Gamma(u))^p} \\ &= C(2R^{-1})^p \left(\frac{\varrho}{\varrho'}\right)^{n_0} \frac{\varrho'}{\varrho' - \varrho} \sup_{u>0} e^{\{u \log \varrho' + p\gamma(u)\}}, \quad p = 0, 1, \ldots.\end{aligned}$$

Furthermore the supremum is reached when $\gamma'(u) = -(\log \varrho')/p$ and it is equal to $\exp\{p(\gamma(u) - u\gamma'(u))\}$. So, f belongs to the class such that $\mu(p) = \gamma(u) - u\gamma'(u)$. Thus, by (3), $f \in \mathcal{H}_{M_\alpha}(D)$ with $\alpha = -1/\log \varrho'$. Consequently, by Lemma 3, $f \in \mathcal{H}_M(D)$ and the proof of Theorem 1 is complete.

Theorem 1 for $\mathcal{C}(M(n))$. *Let $f \in \mathcal{C}^\infty([a,b])$. Then $f \in \mathcal{C}(M(n))$ if and only if there exist constants $C > 0$, $0 < \varrho < 1$, $R > 0$ and a sequence of functions f_n, holomorphic in $E_n := \{z \in \mathbb{C};\ d(z, [a,b]) < R\Gamma(n)\}$ such that $\|f_n\|_{E_n} \le C\varrho^n$ and $\sum_n f_n = f$ uniformly on $[a,b]$.*

Note that our result is valid for the class $\mathcal{C}(M(n))$ whether or not the class is quasianalytic.

Examples. 1. $\mu(t) = \frac{1}{k}\log t$, $k > 0$, which corresponds to the Gevrey class of order k. From (3) we obtain $u = \frac{1}{k}t$ and

$$\gamma(u) = \frac{1}{k}\log t + \frac{1}{k} = \frac{1}{k}(\log u + \log k) + \frac{1}{k};$$

so we can choose $\gamma(u) = \frac{1}{k}\log u$.

2. $\mu(t) = \beta \log\log t$ ($\beta > 0$). We obtain $u = \beta t/\log t$, so $\log u \sim \log t$, and

$$\gamma(u) = \beta \log\log t + \frac{\beta}{\log t};$$

so we can choose $\gamma(u) = \beta \log\log u$.

3. $\mu(t) = at$, $a > 0$ (extreme case); $\gamma(u) = 2\sqrt{au}$.

REMARKS. The condition $\mu(t) \le at$ implies that every function $\Gamma(u)$ is lower bounded by $e^{-2\sqrt{au}}$ at infinity, for some $a \gg 0$. Consequently, $\Gamma(u)$ is always subexponentially decreasing.

2. We can say more on the link between the function $M(t) = t^t e^{t\mu(t)}$, which ensures the growth of the derivatives, and the function $\Gamma(u) = e^{-\gamma(u)}$, which ensures the decrease of the neighborhoods D_n: if $\lim_{t\to+\infty} (\log t)/\mu(t) \neq 0$ we can choose $\gamma = \mu$ as in Examples 1 and 2.

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