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On the Dirichlet problem in the Cegrell classes

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Abstract. Let μ be a non-negative measure with finite mass given by $\varphi(dd^c\psi)^n$, where ψ is a bounded plurisubharmonic function with zero boundary values and $\varphi \in L^q((dd^c\psi)^n)$, $\varphi \geq 0$, $1 \leq q \leq \infty$. The Dirichlet problem for the complex Monge–Ampère operator with the measure μ is studied.

1. Introduction. Let $\Omega \subseteq \mathbb{C}^n$, $n \geq 2$, be a bounded hyperconvex domain, i.e., a bounded, connected and open set that admits a negative plurisubharmonic exhaustion function. A bounded plurisubharmonic function u defined on Ω belongs to the class $\mathcal{E}_0$ if

$$\lim_{\substack{z\to\xi\\ z\in\Omega}} u(z) = 0$$

for every $\xi \in \partial\Omega$, and

$$\int_{\Omega} (dd^c u)^n < \infty,$$

where $(dd^c\,\cdot)^n$ is the complex Monge–Ampère operator. Let the measure μ defined on Ω be given by

$$\mu = \varphi(dd^c\psi)^n,$$

where $\varphi \in L^q\big((dd^c\psi)^n\big)$, $\varphi \geq 0$, $1 \leq q \leq \infty$ and $\psi \in \mathcal{E}_0$. It is proved in Theorem 5.11 of [4] that every non-negative measure $\widetilde{\mu}$ defined on Ω can be decomposed into

$$\widetilde{\mu} = \widetilde{\varphi}(dd^c\widetilde{\psi})^n + \nu, \tag{1.1}$$

where $\widetilde{\psi} \in \mathcal{E}_0$ and $\widetilde{\varphi} \in L^1_{\mathrm{loc}}((dd^c\widetilde{\psi})^n)$, $\widetilde{\varphi} \geq 0$. The non-negative measure ν is such that there exists a pluripolar set $A \subseteq \Omega$ such that $\nu(A) = \nu(\Omega)$. For $q = 1$, the measure μ has finite mass and it puts no mass on pluripolar sets, by (1.1). Lemma 5.14 in [4] implies that there exists a unique function $u \in \mathcal{F}$

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such that $(dd^c u)^n = \mu$ as measures defined on Ω (see Definition 2.1 for the class $\mathcal{F}$).

In this article some results concerning the solution of this Dirichlet problem will be obtained for the case $q > 1$. If $q > 1$ is such that $n(q-1) \geq 1$, then our Theorem 2.3 states that the unique solution $u \in \mathcal{F}$ belongs to $\mathcal{F}_p$, where $p = n(q-1)$ (see Definition 2.1 for the class $\mathcal{F}_p$). On the other hand, if $q > 1$ is such that $n(q-1) < 1$, then it is proved in Theorem 2.4 that the unique solution $u \in \mathcal{F}$ is such that

$$\int_\Omega (-u)^p (dd^c u)^n < \infty,$$

where $p = n(q-1)$. Moreover there exists a decreasing sequence $[u_j]_{j=1}^\infty$, $u_j \in \mathcal{E}_0$, which converges pointwise to u on Ω as j tends to ∞ and satisfies

$$\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < \infty, \qquad \sup_j \int_\Omega (dd^c u_j)^n < \infty.$$

If there can be no misinterpretation a sequence $[\,\cdot\,]_{j=1}^\infty$ will be denoted by $[\,\cdot\,]$. The results of Theorems 2.3 and 2.4 will be extended to the corresponding classes with continuous boundary values. The note ends by recalling a theorem for the case when $q = \infty$ (see Theorem 2.7). For an introduction to pluripotential theory we recommend [7].

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2. Dirichlet problem. A *domain* is an open and connected set. A domain $\Omega \subseteq \mathbb{C}^n$ is called *hyperconvex* if there exists a plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set

$$\{z \in \Omega : \varphi(z) < c\}$$

is compact in Ω for every $c \in (-\infty, 0)$. Throughout this note Ω will be a bounded hyperconvex domain in $\mathbb{C}^n$, $n \geq 2$.

DEFINITION 2.1. Define $\mathcal{F}$ $(= \mathcal{F}(\Omega))$ to be the class of plurisubharmonic functions φ defined on Ω such that there exists a decreasing sequence $[\varphi_j]$, $\varphi_j \in \mathcal{E}_0$, which converges pointwise to φ on Ω as j tends to ∞, and

$$\sup_j \int_\Omega (dd^c \varphi_j)^n < \infty.$$

Let $p \geq 1$. If $[\varphi_j]$ can also be chosen such that

$$\sup_j \int_\Omega (-\varphi_j)^p (dd^c \varphi_j)^n < \infty,$$

then φ is said to be in the class $\mathcal{F}_p$ $(= \mathcal{F}_p(\Omega))$.

The classes $\mathcal{F}_p$ were first introduced in [3] and $\mathcal{F}$ in [4]. These classes are two of the so-called *Cegrell classes*. For further information about the Cegrell classes see, e.g., [3]–[6] and [9].

LEMMA 2.2. *If $\phi, \psi \in \mathcal{E}_0$, then for each $p \geq 0$,*

$$\int_\Omega (-\phi)^{p+n}(dd^c\psi)^n \leq C\int_\Omega (-\phi)^p(dd^c\phi)^n,$$

where $C \geq 0$ is a constant, depending only on n, p and the supremum of ψ.

Proof. Cf. [1] (see also [2]). ∎

THEOREM 2.3. *Let $\psi \in \mathcal{E}_0$ and $\varphi \in L^q((dd^c\psi)^n)$, $\varphi \geq 0$, $n \geq 2$. If $1 < q < \infty$ is such that $n(q-1) \geq 1$, then there exists a unique function $u \in \mathcal{F}_p$ such that*

$$(dd^c u)^n = \varphi(dd^c\psi)^n,$$

where $p = n(q-1)$.

Proof. Let $\phi \in \mathcal{E}_0$. Hölder's inequality implies that

$$\begin{aligned}(2.1)\quad \int_\Omega (-\phi)^p\varphi(dd^c\psi)^n &\leq \Big(\int_\Omega \varphi^q(dd^c\psi)^n\Big)^{1/q}\Big(\int_\Omega (-\phi)^{pq/(q-1)}(dd^c\psi)^n\Big)^{(q-1)/q}\\ &= C_1\Big(\int_\Omega (-\phi)^{p+n}(dd^c\psi)^n\Big)^{p/(n+p)},\end{aligned}$$

where $C_1 \geq 0$ is a constant and $p = n(q-1)$. Since $p \geq 1$ by assumption, it follows from Lemma 2.2 that

$$(2.2)\qquad \int_\Omega (-\phi)^{p+n}(dd^c\psi)^n \leq C_2\int_\Omega (-\phi)^p(dd^c\phi)^n,$$

where $C_2 \geq 0$ is a constant. Inequalities (2.1) and (2.2) imply that there exists a constant A such that

$$\int_\Omega (-\phi)^p\varphi(dd^c\psi)^n \leq A\Big(\int_\Omega (-\phi)^p(dd^c\phi)^n\Big)^{p/(p+n)}$$

for every $\phi \in \mathcal{E}_0$, hence Theorem 5.1 in [3] shows that there exists a unique $u \in \mathcal{F}_p$ such that

$$(dd^c u)^n = \varphi(dd^c\psi)^n. \blacksquare$$

THEOREM 2.4. *Let $\psi \in \mathcal{E}_0$ and $\varphi \in L^q((dd^c\psi)^n)$, $\varphi \geq 0$, $n \geq 2$. If $1 < q < \infty$ is such that $n(q-1) < 1$, then there exists a unique function $u \in \mathcal{F}$ such that $(dd^c u)^n = \varphi(dd^c\psi)^n$ and*

$$\int_\Omega (-u)^p(dd^c u)^n < \infty,$$

where $p = n(q-1)$. *Moreover there exists a decreasing sequence* $[u_j]$, $u_j \in \mathcal{E}_0$, *which converges pointwise to* u *on* Ω *as* j *tends to* ∞ *and satisfies*

$$\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < \infty, \qquad \sup_j \int_\Omega (dd^c u_j)^n < \infty.$$

Proof. Let $\phi \in \mathcal{E}_0$. By using the same technique as in the proof of Theorem 2.3 it follows that there exists a constant A not depending on ϕ such that

$$(2.3) \qquad \int_\Omega (-\phi)^p \varphi (dd^c\psi)^n \le A\Big(\int_\Omega (-\phi)^p (dd^c\phi)^n\Big)^{p/(n+p)}.$$

In particular, this shows that the measure $\varphi(dd^c\psi)^n$ vanishes on pluripolar sets. Lemma 5.14 in [4] implies that there exists a unique $u \in \mathcal{F}$ such that

$$(dd^c u)^n = \varphi(dd^c\psi)^n,$$

since the given measure has finite total mass. Let $u_j \in \mathcal{E}_0$ be such that

$$(dd^c u_j)^n = \min(j, \varphi)(dd^c\psi)^n.$$

The comparison principle shows that the sequence $[u_j]$ is decreasing and converges pointwise to u on Ω as j tends to ∞. Inequality (2.3) implies that

$$\begin{aligned}\int_\Omega (-u_j)^p (dd^c u_j)^n &= \int_\Omega (-u_j)^p \min(j,\varphi)(dd^c\psi)^n \\ &\le \int_\Omega (-u_j)^p (dd^c u)^n \le A\Big(\int_\Omega (-u_j)^p (dd^c u_j)^n\Big)^{p/(n+p)}.\end{aligned}$$

This yields

$$\int_\Omega (-u_j)^p (dd^c u_j)^n \le A^{(p+n)/n},$$

hence

$$\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n \le A^{(p+n)/n}.$$

Inequality (2.3) implies that

$$(2.4) \qquad \int_\Omega (-u_j)^p (dd^c u)^n \le A\Big(\int_\Omega (-u_j)^p (dd^c u_j)^n\Big)^{p/(n+p)} \le A^{(p+n)/n}.$$

By the monotone convergence theorem,

$$(2.5) \qquad \lim_{j\to\infty} \int_\Omega (-u_j)^p (dd^c u)^n = \int_\Omega (-u)^p (dd^c u)^n.$$

Combining (2.4) and (2.5) yields

$$\int_\Omega (-u)^p (dd^c u)^n \le A^{(p+n)/n}. \ \blacksquare$$

The next step is to generalize the results in Theorems 2.3 and 2.4 to some more general classes of bounded plurisubharmonic functions. Recall that the *Perron–Bremermann envelope* is defined by

$$\mathrm{PB}_f(z) := \sup\{w(z) : w \in \mathcal{PSH}(\Omega),\ \limsup_{\substack{\zeta\to\xi\\ \zeta\in\Omega}} w(\zeta) \le f(\xi)\ \forall \xi \in \partial\Omega\},$$

where $\mathcal{PSH}(\Omega)$ denotes the class of plurisubharmonic functions defined on Ω and $f : \partial\Omega \to \mathbb{R}$ is a given function. Recall that if Ω is a bounded hyperconvex domain and $f : \partial\Omega \to \mathbb{R}$ is a continuous function, then $\mathrm{PB}_f \in \mathcal{PSH}(\Omega)$.

Definition 2.5. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{F}\}$ and $f : \partial\Omega \to \mathbb{R}$ be a continuous function such that

$$\lim_{\substack{z\to\xi\\ z\in\Omega}} \mathrm{PB}_f(z) = f(\xi),$$

for every $\xi \in \partial\Omega$. A plurisubharmonic function u defined on Ω belongs to the class $\mathcal{K}(f)$ $(= \mathcal{K}(\Omega, f))$ if there exists a function $\varphi \in \mathcal{K}$ such that

$$\mathrm{PB}_f \ge u \ge \varphi + \mathrm{PB}_f.$$

Remark. If $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{F}\}$, then $\mathcal{K}(0) = \mathcal{K}$.

Remark. Theorem 2.3 is also valid for the Cegrell class $\mathcal{F}_p(f)$.

The classes $\mathcal{E}_0(f)$ and $\mathcal{F}_p(f)$ were first introduced in [3] and the class $\mathcal{F}(f)$ in [9].

Theorem 2.6. *Assume that $f : \partial\Omega \to \mathbb{R}$ is a continuous function such that*

$$\lim_{\substack{z\to\xi\\ z\in\Omega}} \mathrm{PB}_f(z) = f(\xi)$$

for every $\xi \in \partial\Omega$, and $\mathrm{PB}_f + \mathrm{PB}_{-f} \in \mathcal{E}_0$. Let $\psi \in \mathcal{E}_0$ and $\varphi \in L^q((dd^c\psi)^n)$, $\varphi \ge 0$, $n \ge 2$. If $1 < q < \infty$ is such that $n(q-1) < 1$, then there exists a unique function $u \in \mathcal{F}(f)$ such that $(dd^cu)^n = \varphi(dd^c\psi)^n$ and

$$\int_\Omega (-u - \mathrm{PB}_{-f})^p (dd^cu)^n < \infty,$$

where $p = n(q-1)$. Moreover there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0(f)$, which converges pointwise to u on Ω as j tends to ∞ and satisfies

$$\sup_j \int_\Omega (-u_j - \mathrm{PB}_{-f})^p (dd^cu_j)^n < \infty, \qquad \sup_j \int_\Omega (dd^cu_j)^n < \infty.$$

Proof. From Theorem 2.4 it follows that there exists $v \in \mathcal{F}$ such that

$$(dd^cv)^n = \varphi(dd^c\psi)^n, \quad \int_\Omega (-v)^p (dd^cv)^n < \infty,$$

where $p = n(q-1)$. Theorem 7.4 in [9] implies that there exists $u \in \mathcal{F}(f)$ such that $(dd^c u)^n = \varphi(dd^c\psi)^n$. The function $v + \mathrm{PB}_f$ belongs to $\mathcal{F}(f)$ and

$$(dd^c u)^n = (dd^c v)^n \le (dd^c(v + \mathrm{PB}_f))^n,$$

hence

$$v + \mathrm{PB}_f \le u,$$

by Corollary 7.7 in [9]. Thus

$$\int_\Omega (-u - \mathrm{PB}_{-f})^p (dd^c u)^n \le \int_\Omega (-v - \mathrm{PB}_f - \mathrm{PB}_{-f})^p (dd^c v)^n < \infty.$$

Now for the second part of the theorem. Theorem 4.10 in [9] together with the assumption that $\mathrm{PB}_f + \mathrm{PB}_{-f} \in \mathcal{E}_0$ implies that there exists $u_j \in \mathcal{E}_0(f)$ such that

$$(dd^c u_j)^n = \min(j, \varphi)(dd^c\psi)^n.$$

Moreover $[u_j]$ is a decreasing sequence which converges pointwise to u on Ω as $j \to \infty$, and

$$\sup_j \int_\Omega (dd^c u_j)^n < \infty.$$

Furthermore, $u_j + \mathrm{PB}_{-f} \in \mathcal{E}_0$. The assertion then follows by repeating the argument in the proof of Theorem 2.4. ∎

THEOREM 2.7. *Let $\psi \in \mathcal{E}_0$ and $\varphi \in L^q((dd^c\psi)^n)$, $\varphi \ge 0$, $n \ge 2$. If $q = \infty$, then there exists a unique $u \in \mathcal{E}_0(f)$ such that $(dd^c u)^n = \varphi(dd^c\psi)^n$. In particular, if $f = 0$, then there exists a unique $u \in \mathcal{E}_0$ such that $(dd^c u)^n = \varphi(dd^c\psi)^n$.*

Proof. There exists a constant $c \ge 0$ such that

$$\varphi(dd^c\psi)^n \le c(dd^c\psi)^n = (dd^c(c^{1/n}\psi))^n, \tag{2.6}$$

hence there exists a unique $u \in \mathcal{E}_0(f)$ such that $(dd^c u)^n = \varphi(dd^c\psi)^n$, by the proof of Theorem 7.4 in [9]. ∎

REMARK. Theorem 2.7 is a special case of Theorem A in [8].

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